CHAPTER- VII

ON WEAKLY $\varphi$-RICCI SYMMETRIC TRANS-SASAKIAN MANIFOLDS

In this chapter we introduced the notion of weakly $\varphi$-Ricci-symmetric trans -Sasakian manifolds of dimension $(M^{2n+1}, g)$ $(n>1)$ and studied the various properties. Finally the existence of weakly $\varphi$-Ricci-symmetric trans -Sasakian manifold is ensured by an example.

7.1. Introduction: In this chapter we considered the $2n+1$ dimensional trans-Sasakian manifold of type $(\alpha, \beta)$ and introduced a notion of weakly $\varphi$-Ricci-symmetric trans -Sasakian manifolds defined as, a Trans -Sasakian manifold $(M^{2n+1}, g)$ $(n>1)$ is said to be weakly $\varphi$-Ricci Symmetric if its Ricci tensor $Q$ of type $(1, 1)$ is not identically zero and satisfies the condition [8]

$$\varphi^2 (\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX, Y)\rho ,$$

where $g(X, \rho) = C(X)$, $Q$ is the Ricci operator such that

$$g(QX, Y) = S(X, Y),$$

for all vector fields $X, Y$ in $M^{2n+1}$, $A, B$ and $C$ are non zero 1-forms, called the associated 1-forms of the manifold and $\nabla$ denotes the operator of the covariant differentiation with respect to the metric tensor $g$, $\varphi$ is a tensor field of type(1,1) and $S$ is the Ricci curvature tensor of type $(0, 2)$.

The present paper is divided into four sections, in section 1 we have given introduction and defined a new Tensor $Q$ of type $(1, 1)$. Section 2 is devoted to the preliminary results of Trans-Sasakian Manifolds, and in Section 3 we have found expressions for the
associated 1-forms A, B and C along with some corollaries and Section 4 is the concluding section of the paper wherein we have provided an concrete example for the existence of weakly $\phi$- Ricci-symmetric trans -Sasakian manifold of type $(\alpha, \beta)$.

**7.2. Trans-Sasakian Manifolds.**

This section is devoted to preliminary results on almost contact metric manifolds [1], [3], [6] and trans–Sasakian manifolds by [4], [9], and [10].

A $(2n+1)$ - dimensional smooth manifold $M$ is said to be an almost contact metric manifold if it admits a $(1, 1)$ tensor field $\phi$, an associated vector field $\xi$, a 1-form $\eta$ and a Riemannian metric ‘$g$’ which satisfy

\begin{align*}
\phi(\xi) &= 0, \quad \eta(\phi(X)) = 0, \quad \phi^2(X) = -X + \eta(X)\xi, \\
(\phi X, Y) &= -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, X) = 0,
\end{align*}

for all vector fields $X, Y$ on $M$.

An almost contact metric manifold is said to be Trans–Sasakian manifold [5], [2], [11] if $(M \times R, J, G)$ belongs to the class $W_4$ of the Hermitian manifolds, where $J$ is the almost complex structure on $M \times R$ defined by

$$J(Z,f\frac{d}{dt}) = (\phi Z - f \xi, \eta(Z)\frac{d}{dt}),$$

For any vector field $Z$ on $M$ and a smooth function $f$ on $M \times R$, and $G$ is the product matrix on $M \times R$, this may be stated by the condition [3]

$$\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where $\alpha$ and $\beta$ are smooth functions on $M$, and $X, Y$ are smooth vector
Fields on $M$. The manifold with the above trans-Sasakian structure is called a trans-Sasakian manifold of type $(\alpha, \beta)$. From (2.4) it is easy to see that

\begin{align}
(7.2.5) & \quad \nabla_X \xi = -\alpha \varphi X + \beta \{X - \eta(X)\xi\}, \\
(7.2.6) & \quad (\nabla_X \eta) = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y) \\
\end{align}

For the Trans – Sasakian manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ the following results hold [5], [10].

\begin{align}
(7.2.7) & \quad R(X, Y)\xi = (\alpha^2 - \beta^2) [\eta (Y) X - \eta (X) Y] - (X \alpha \varphi Y - (X \beta) \varphi^2 Y \\
& \quad + 2 \alpha \beta [\eta (Y) \varphi X - \eta (X) \varphi Y] + (Y \alpha \varphi X + (Y \beta) \varphi^2 X, \\
(7.2.8) & \quad \eta (R (X, Y) Z) = (\alpha^2 - \beta^2) [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] \\
& \quad - 2 \alpha \beta [g(\varphi X, Z) \eta(Y) - g(\varphi Y, Z) \eta(X)] \\
& \quad - (Y \alpha) g(\varphi X, Z) - (X \beta) [g(Y, Z) - \eta(Y) \eta(Z)] \\
& \quad + (X \alpha) g(\varphi Y, Z) - (Y \beta) [g(X, Z) - \eta(X) \eta(Z)] \\
(7.2.9) & \quad S(X, \xi) = [2n(\alpha^2 - \beta^2) - (\xi \beta)] \eta (X) - (\varphi X) \alpha - (2n - 1)(X \beta), \\
(7.2.10) & \quad R(\xi, X)\xi = [(\alpha^2 - \beta^2) - (\xi \beta)] \eta (X) \xi - X, \\
(7.2.11) & \quad S(\xi, \xi) = [2n(\alpha^2 - \beta^2) - (\xi \beta)], \\
(7.2.12) & \quad (\xi \alpha) + 2 \alpha \beta = 0 \\
(7.2.13) & \quad Q \xi = [2n(\alpha^2 - \beta^2) - (\xi \beta)] \xi + \varphi \text{grad} \alpha - (2n - 1)(\text{grad} \beta),
\end{align}

where $R$ is the curvature of type $(1,3)$ of the manifold and $Q$ is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor $S$ and $g(QX, Y) = S(X, Y)$ for all vector fields $X$ and $Y$ on $M$.  

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(7.2.14)
\[ R(\xi, Y)X = (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(X)Y] + 2\alpha \beta [g(\varphi X, Y)\xi + \eta(X)\varphi Y] + (X\alpha)\varphi Y + g\{ (\varphi X), Y\} (\text{grad}\alpha) + (X\beta)(Y - \eta(Y)\xi) - g(\varphi X, \varphi Y)(\text{grad}\beta). \]

Since \( R(\xi, Y) = -R(Y, \xi) \), we can give the expression for \( R(Y, \xi)X \), and simply by changing the sign in each term of (7.2.14).

**Definition 7.2.1** A Trans -Sasakian manifold \((M^{2n+1}, g)\) \((n > 1)\) is called **Weakly Locally \(\varphi\)-Ricci Symmetric** if its Ricci tensor \(Q\) is not identically zero and satisfies the condition:

(7.2.15) \[ \varphi^2 (\nabla_\xi Q)(Y) = 0, \]
and if \(B=C=0\), then weakly \(\varphi\)-Ricci recurrent trans-Sasakian reduces to locally \(\varphi\)-Ricci recurrent.

**Definition 7.2.2** A Trans -Sasakian manifold \((M^{2n+1}, g)\) \((n > 1)\) is called **weakly \(\varphi\)-Ricci Recurrent** if its Ricci tensor \(Q\) is not identically zero and satisfies the condition:

(7.2.16) \[ \varphi^2 (\nabla_\xi Q)(Y) = A(X)Q(Y), \]

\(\xi\)-**Sectional Curvature.** The \(\xi\)-**Sectional Curvature** \(K(\xi, X)\) of an

Trans –Sasakian manifold of type \((\alpha, \beta)\) for a unit vector field \(X\) orthogonal to \(\xi\) is given by

\[ K(\xi, X) = R(\xi, X, \xi, X) \]

From (2.10) replacing \(Y=X\) we have,

(7.2.17) \[ R(\xi, X)\xi = \{\alpha^2 - \beta^2 - \xi\beta\} \{-X - \eta(X)\xi\} - \{2\alpha\beta + \xi\alpha\}gX \]

\[ g(R(\xi, X)\xi, X) = \{\alpha^2 - \beta^2 - \xi\beta\} \{-g(X, X) - \eta(X)g(\xi, X)\} - \{2\alpha\beta + \xi\alpha\}g(\varphi X, X) \]

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This under the above conditions simplifies to,

\[ K(\xi, X) = R(\xi, X, \xi, X) = \alpha^2 - \beta^2 - \xi \beta \]

which is an expression for \( \xi \)-Sectional Curvature \( K(\xi, X) \) of an Trans–Sasakian manifold of type \((\alpha, \beta)\).

If \( \{\alpha^2 - \beta^2 - \xi \beta\} \neq 0 \), then Trans–Sasakian manifold is of non zero \( \xi \)-Sectional Curvature. Further if \( \alpha^2 - \beta^2 - \xi \beta = 0 \), then \( \varepsilon \)-Trans–Sasakian manifold is of zero \( \xi \)-Sectional Curvature. Thus,

**Theorem 7.2.1.** In a Trans–Sasakian manifold \( M \),the \( \xi \)-Sectional Curvature is given by (7.2.18).

From (7.2.18) we have the following remarks.

**Remarks.**

i) In an \( \alpha \)-Sasakian Manifold the \( \xi \)-Sectional Curvature

\[ K(\xi, X) = \alpha^2 \] so that for an \( \alpha \)-Sasakian Manifold the \( \xi \)-Sectional Curvature is 1.

ii) In an \( \beta \)-Kenmotsu Manifold the \( \xi \)-Sectional Curvature

\[ K(\xi, X) = \beta^2 \] so that for an \( \beta \)-Kenmotsu Manifold the \( \xi \)-Sectional Curvature is 1.

iii) In an cosymplectic Manifold the \( \xi \)-Sectional Curvature

\[ K(\xi, X) = 0. \]

**7.3. Weakly \( \varphi \)-Ricci Symmetric Trans-Sasakian Manifolds**

In this section some Theorems on Weakly \( \varphi \)-Ricci Symmetric Trans-Sasakian manifolds are studied.

**Theorem 7.3.1.** In a Weakly \( \varphi \)-Ricci Symmetric Trans-Sasakian Manifold
\((M^{2n+1}, g) (n > 1)\) of non vanishing \(\xi\)-sectional curvature, the relation (7.3.1)

\[
A(\xi) + B(\xi) + C(\xi) = \frac{1}{2n} \left[ \frac{2n\{2\beta(\xi\beta) - 2\alpha(\xi\alpha) \} - \xi(\xi\beta)}{\alpha^2 - (\xi\beta) - \beta^2} \right] \\
+ \frac{(2n - 1)\eta(\xi(\text{grad}\beta))}{2n\{\alpha^2 - (\xi\beta) - \beta^2\}}
\]

holds.

**Proof:** Using (7.2.1), in (7.1.1), we get (7.3.2)

\[-(\nabla_X Q)(Y) + \eta(\nabla_X Q)(Y))\xi = A(X)Q(Y) + B(Y)Q(X) + g(QY, X)\rho\]

Taking inner product on both sides with respect to the vector field \(V\), we find

\[-g((\nabla_X Q)(Y), V) + \eta(\nabla_X Q)(Y))g(\xi, V)\]

(7.3.3)

\[= A(X)g(Q(Y), V) + B(Y)g(Q(X), V) + g(QY, X)g(\rho, V)\]

Taking \(V\) to be orthogonal to \(\xi\) in (3.3) and using the fact that \(g(QX, V) = S(X, V)\) and \(g(\rho, V) = C(V)\), we have (7.3.4)

\[-g((\nabla_X Q)(Y), V) = A(X)S(Y, V) + B(Y)S(X, V) + C(V)S(Y, X)\]

Substituting \(X = Y = V = \xi\) in (7.3.4) we get (7.3.5)

\[-g((\nabla_\xi Q)(\xi), \xi) = A(\xi)S(\xi, \xi) + B(\xi)S(\xi, \xi) + C(\xi)S(\xi, \xi)\]

\[= \{A(\xi) + B(\xi) + C(\xi)\}S(\xi, \xi)\]

\[= \{A(\xi) + B(\xi) + C(\xi)\}(2n(\alpha^2 - (\xi\beta) - \beta^2))\]

The left hand side of (7.3.5) is given by
(7.3.6)  
\[-g(\nabla_\xi Q(\xi) - Q\nabla_\xi \xi, \xi)\]

\[= -g(\nabla_\xi (Q\xi, \xi))\]

\[= -\nabla_\xi (2n(\alpha^2 - \beta^2) - \xi^2))g(\xi, \xi) - g(\nabla_\xi \varphi \text{grad} \alpha, \xi) + (2n - 1)g(\nabla_\xi \text{grad} \beta, \xi)\]

\[= -2n\{(2\alpha(\xi\alpha) - 2\beta(\xi\beta)) - \xi(\xi\beta)\} - g(\nabla_\xi \varphi \text{grad} \alpha, \xi) + (2n - 1)g(\nabla_\xi \text{grad} \beta, \xi)\]

Substituting this on left hand side of (7.3.5), we get

\[\{A(\xi) + B(\xi) + C(\xi)\}(2n(\alpha^2 - (\xi\beta) - \beta^2)\]

(7.3.7)

Further simplifying (7.3.7) , we get (7.3.1) of Theorem 3.1, using the hypothesis of the theorem i.e. the manifold \(M^{2n+1}\) is of non vanishing \(\xi\)-sectional curvature i.e. \(\alpha^2 - \beta^2 - \xi\beta \neq 0\) , the proof of the Theorem 3.1 completes .

**Corollary 7.3.1.** In a Weakly \(\varphi\)-Ricci Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \(\xi\)-sectional curvature with \(\alpha\), and \(\beta\) constants, we have

\[A(\xi) + B(\xi) + C(\xi) = 0\]

**Proof:** Follows from Theorem 3.1.

**Theorem 7.3.2.** In a Weakly \(\varphi\)-Ricci Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \(\xi\)-sectional curvature, we have
(7.3.8)

\[
A(X) = \frac{-2n\{2\alpha(X\alpha) - 2\beta(X\beta)\} - X(\xi\beta) - \eta(X(\varphi\text{grad}\alpha)) + (2n - 1)\eta(X(\text{grad}\beta))}{2n(\alpha^2 - \beta^2 - \xi\beta)} \\
+ \alpha\frac{\{(\varphi^2 X)\alpha\} + (2n - 1)(\varphi X\beta)}{2n(\alpha^2 - \beta^2 - \xi\beta)} \\
+ \frac{\beta[2n(\alpha^2 - \beta^2) - (\xi\beta)](\eta(X) - (\varphi X)\alpha) - (2n - 1)(X\beta)}{2n(\alpha^2 - \beta^2 - \xi\beta)} \\
- \frac{2n\beta\eta(X)(\alpha^2 - \beta^2 - \xi\beta)}{2n(\alpha^2 - \beta^2 - \xi\beta)} - \frac{\{2n\{2\beta(\xi\beta) - 2\alpha(\xi\alpha)\} - \xi(\xi\beta)\}}{\{\alpha^2 - (\xi\beta) - \beta^2\}^2} \\
- \frac{\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}^2}{\{2n(\alpha^2 - \beta^2 - \xi\beta)\}^2} - \frac{A(\xi)}{2n(\alpha^2 - \beta^2 - \xi\beta)} \\
\frac{[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - (\varphi X)\alpha - (2n - 1)(X\beta)]}{2n(\alpha^2 - \beta^2 - \xi\beta)}
\]

**Proof:** Substituting \(Y=V=\xi\) in (7.3.4) we get

(7.3.9) - \(g((\nabla_X Q)(\xi), \xi) = A(X)S(\xi, \xi) + B(\xi)S(X, \xi) + C(\xi)S(\xi, X)\)

Now consider by the linearity property of Ricci tensor \(S\)

\(S(\alpha\varphi X + \beta\{X - \eta(X)\xi, \xi\}) = -\alpha S(\varphi X, \xi) + \beta S(X, \xi) - \beta\eta(X)S(\xi, \xi)\)

(7.3.10)

\(S(\alpha\varphi X + \beta\{X - \eta(X)\xi, \xi\})\)

\(= -\alpha[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(\varphi X) - (\varphi^2 X)\alpha - (2n - 1)(\varphi X\beta)]\)

\(+ \beta[2n(\alpha^2 - \beta^2) - (\xi\beta)](\eta(X) - (\varphi X)\alpha) - (2n - 1)(X\beta)]\)

\(- 2n\beta\eta(X)(\alpha^2 - \beta^2 - \xi\beta)\)

Hence left hand side of (7.3.9) reduces to
(7.3.11)
\[-g(V_X Q(\xi) - QV_X \xi, \xi)) = -2n\{2\alpha(X\alpha) - 2\beta(X\beta)\} - X(\xi\beta) - g(V_X \varphi \text{grad} \alpha, \xi) + (2n - 1)g(V_X \text{grad} \beta, \xi)\]
\[-\alpha[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (\varphi X) - ((\varphi^2 X)\alpha) - (2n - 1)(\varphi X\beta)]
\[+ \beta[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)]
\[-2n\beta\eta(X)(\alpha^2 - \beta^2 - \xi\beta)\]

Also the right hand side of (7.3.9) is
\[A(X)S(\xi, \xi) + B(\xi)S(X, \xi) + C(\xi)S(\xi, X)\]
\[= A(X)2n(\alpha^2 - \beta^2 - \xi\beta) + \frac{[B(\xi) + C(\xi)][2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (X) - (\varphi X)\alpha - (2n - 1)(X\beta)]}{2n[\alpha^2 - (\xi\beta) - \beta^2]}\]

Also from (7.3.3)
(7.3.12)
\[A(X)S(\xi, \xi) + B(\xi)S(X, \xi) + C(\xi)S(\xi, X)\]
\[= A(X)2n(\alpha^2 - \beta^2 - \xi\beta) + \frac{\frac{1}{2n} \{2n[2\beta(\xi\beta) - 2\alpha(\xi\alpha)] - \xi(\xi\beta)\}}{\alpha^2 - (\xi\beta) - \beta^2} + \frac{(2n - 1)\eta(\xi(\text{grad} \beta)) - A(\xi)[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (X) - (\varphi X)\alpha - (2n - 1)(X\beta)]}{2n[\alpha^2 - (\xi\beta) - \beta^2]}\]

From (7.3.11) and (7.3.12) it follows that
(7.3.13)
\[-2n\{2\alpha(X\alpha) - 2\beta(X\beta)\} - X(\xi\beta) - g(V_X \varphi \text{grad} \alpha, \xi) + (2n - 1)g(V_X \text{grad} \beta, \xi)\]
\[-\alpha[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (\varphi X) - ((\varphi^2 X)\alpha) - (2n - 1)(\varphi X\beta)]
\[+ \beta[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)] - 2n\beta\eta(X)(\alpha^2 - \beta^2 - \xi\beta)\]
\[= A(X)2n(\alpha^2 - \beta^2 - \xi\beta) + \frac{[\frac{1}{2n} \{2n[2\beta(\xi\beta) - 2\alpha(\xi\alpha)] - \xi(\xi\beta)\}}{\alpha^2 - (\xi\beta) - \beta^2} + \frac{(2n - 1)\eta(\xi(\text{grad} \beta)) - A(\xi)[2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (X) - (\varphi X)\alpha - (2n - 1)(X\beta)]}{2n[\alpha^2 - (\xi\beta) - \beta^2]}\]
Simplifying (7.3.13) for $A(X)$, this completes the proof of the Theorem 3.2

**Theorem 7.3.3.** In a Weakly $\phi$-Ricci Symmetric Trans-Sasakian Manifold $(M^{2n+1}, g)$ $(n > 1)$ of non vanishing $\xi$-sectional curvature, we have

(7.3.14)

$$B(Y) = \frac{-g(\nabla_\xi Q(Y), \xi) + [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (\nabla_\xi Y) - (\varphi\nabla_\xi Y)\alpha - (2n-1)(\nabla_\xi Y)\beta}{2n(\alpha^2 - (\xi\beta) - \beta^2)}$$

$$- \frac{[2n(\beta(\xi\beta) - 2\alpha(\xi\alpha)) - \xi(\xi\beta)] - (2n-1)\eta(\xi(\text{grad}\beta)) - B(\xi)}{2n(\alpha^2 - (\xi\beta) - \beta^2)^2}$$

$$\cdot [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (Y) - (\varphi Y)\alpha - (2n-1)(Y\beta)$$

**Proof:** Substituting $X=V=\xi$ in (7.3.4) we get

(7.3.15)

$$-g((\nabla_\xi Q)(Y), \xi) = A(\xi)S(Y, \xi) + B(\xi)S(\xi, \xi) + C(\xi)S(Y, \xi)$$

LHS = $-g(\nabla_\xi Q(Y) - Q\nabla_\xi Y, \xi)$

$$= -g(\nabla_\xi (QY), \xi) + S(\nabla_\xi Y, \xi)$$

$$= -g(\nabla_\xi (QY), \xi) + [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (\nabla_\xi Y)$$

$$- (\varphi\nabla_\xi Y)\alpha - (2n-1)(\nabla_\xi Y)\beta$$

(7.3.16)

RHS = $\{A(\xi) + C(\xi)\}S(Y, \xi) + B(\xi)S(\xi, \xi)$

$$= \frac{[2n(\beta(\xi\beta) - 2\alpha(\xi\alpha)) - \xi(\xi\beta)] - (2n-1)\eta(\xi(\text{grad}\beta)) - B(\xi)}{2n(\alpha^2 - (\xi\beta) - \beta^2)^2}$$

$$\cdot [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta (Y) - (\varphi Y)\alpha - (2n-1)(Y\beta)$$

$$+ B(\xi)2n(\alpha^2 - (\xi\beta) - \beta^2)$$

From (7.3.15) equating LHS and RHS, we get

(7.3.17)
\[ B(Y) = \frac{-g(\nabla_{\xi}Q(Y), \xi) + [2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta \cdot (\nabla_{\xi}Y) - (\phi \nabla_{\xi}Y)\alpha - (2n-1)(\nabla_{\xi}Y)\beta}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}} \]
\[ - \frac{[2n\{2\beta(\xi \beta) - 2\alpha(\xi \alpha)\} - \xi(\xi \beta)) + (2n - 1)\eta(\xi(\text{grad} \beta))}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}^2} \cdot \frac{B(\xi)}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}} \]
\[ .[2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta \cdot (Y) - (\phi Y)\alpha - (2n - 1)(Y\beta) \]

Hence the proof of the Theorem 3.3 follows.

**Theorem 7.3.4.** In a Weakly \( \varphi \)-Ricci Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \(\xi\) - sectional curvature, we have

(7.3.18)
\[ C(V) = \frac{-g(\nabla_{\xi}(QV), \xi) + [2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta \cdot (\nabla_{\xi}V) - (\phi \nabla_{\xi}V)\alpha - (2n-1)(\nabla_{\xi}V)\beta}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}} \]
\[ - \frac{[2n\{2\beta(\xi \beta) - 2\alpha(\xi \alpha)\} - \xi(\xi \beta)) + (2n - 1)\eta(\xi(\text{grad} \beta))}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}^2} \cdot \frac{C(\xi)}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}} \]
\[ .[2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta \cdot (V) - (\phi V)\alpha - (2n - 1)(V\beta) \]

**Proof:** On the similar arguments as in the above Theorem 3.3 the proof of the Theorem (3.4) follows.

**Theorem 7.3.5.** In a Weakly \( \varphi \)-Ricci Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \(\xi\) - sectional curvature, we have

(7.3.19)
\[ B(X) - C(X) = [B(\xi) - C(\xi)] \frac{[2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta \cdot (X) - (\phi X)\alpha - (2n-1)(X\beta)}{2n\{\alpha^2 - (\xi \beta) - \beta^2\}} \]

**Proof:** Replacing \( Y \) and \( V \) by \( X \) in (7.3.15) and (7.3.19) then taking the difference and simplifying we get,

(7.3.20)
Now simplifying further (7.3.20) the proof of Theorem 3.5 follows.

**Corollary 7.3.2.** In a Weakly $\varphi$-Ricci Symmetric Trans-Sasakian Manifold $(M^{2n+1}, g)$ $(n > 1)$ of non vanishing $\xi$ - sectional curvature with $\alpha$ and $\beta$ constants, we have

\begin{equation}
B(X) - C(X) = [B(\xi) - C(\xi)] \eta (X)
\end{equation}

**Theorem 7.3.6.** In a Weakly $\varphi$-Ricci Symmetric Trans-Sasakian Manifold $(M^{2n+1}, g)$ $(n > 1)$ of non vanishing $\xi$ - sectional curvature the following relation

\begin{equation}
A(X) + B(X) - C(X) = \frac{[2n(2\alpha(\alpha + (2n-1)\eta(X(\varphi X)) + (2n-1)\eta(X(\varphi X)))]}{2n(\alpha^2 - \beta^2 - \xi \beta)}
\end{equation}
holds.

**Proof:** Follows by taking the addition of (7.3.20) and (7.3.21)

**Corollary 7.3.1:** If a weakly $\phi$-Ricci symmetric non co-symplectic trans-Sasakian manifold $(M^{2n+1}, g)$ $(n > 1)$ satisfies the condition $\phi \text{ grad } \alpha = \text{ grad } \beta$ and is of non vanishing $\xi$-sectional curvature, then the following result holds

(7.3.23)

$$A(X) + B(X) - C(X) = \frac{\beta(\beta) - \alpha(\alpha \alpha)}{n(\alpha^2 - \beta^2)} + \frac{\alpha[((\varphi^2 X)\alpha) + (2n - 1)(\varphi X\beta)]}{2n(\alpha^2 - \beta^2)} + \frac{\alpha(\varphi X\alpha) - (2n - 1)(X\beta)}{2n(\alpha^2 - \beta^2)}$$

for any vector field $X$ provided $\alpha^2 - \beta^2 \neq 0$.

**Proof:** If $\phi \text{ grad } \alpha = \text{ grad } \beta$, then

$$\xi \beta = g(\xi, \text{ grad } \beta) = g(\xi, \varphi \text{ grad } \alpha) = \eta(\varphi \text{ grad } \alpha) = 0$$

from (7.3.20) and simplification gives (7.3.23)

**Corollary 3.2:** If an $\alpha$-Sasakian manifold is weakly $\phi$-Ricci symmetric with non vanishing $\xi$-sectional curvature then the following relation

$$A(X) + B(X) - C(X) = -\frac{2n(2\alpha(\alpha \alpha) + \eta(X(\varphi \text{ grad } \alpha)))}{2n(\alpha^2)} + \frac{(\varphi^2 X)\alpha}{2n\alpha} + \frac{2(\xi\alpha)}{\alpha^3} \{2n(\alpha^2 - \alpha \alpha \alpha \alpha) - (\varphi X)\alpha\}$$

$$+ \frac{A(\xi) + B(\xi) - C(\xi)}{2n(\alpha^2)} [2n(\alpha^2 - \alpha \alpha \alpha \alpha) - (\varphi X)\alpha]$$

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**Proof:** Follows from Theorem 3.6

**Corollary 7.3.3.** If an $\beta$-Kenmotsu manifold is weakly $\varphi$-Ricci symmetric with non vanishing $\xi$-sectional curvature then the following relation

$$A(X) + B(X) - C(X)$$

$$= \frac{[2\beta(X\beta)] - X(\xi\beta) + (2n - 1)\eta(X(\text{grad}\beta))]}{2n(\beta^2 + \xi\beta)} + \frac{\beta[(2n(\beta^2) + (\xi\beta)](\eta (X) + (2n - 1)(X\beta)]}{2n(\beta^2 + \xi\beta)}$$

$$- \frac{2n\beta\eta(X)(\beta^2 + \xi\beta)}{2n(\beta^2 + \xi\beta)} \left[ \frac{2n(\beta(\xi\beta)) + (\xi\beta)]}{\{(\xi\beta) + \beta^2}\}^2 \right]$$

$$+ \frac{(2n - 1)\eta(\xi(\text{grad}\beta))}{{(2n(\xi\beta) + (\xi\beta)]^2} \{2n(\beta^2) + (\xi\beta)]\eta (X) + (2n - 1)(X\beta)}$$

$$+ \frac{A(\xi) + B(\xi) - C(\xi)}{2n(\beta^2 + \xi\beta)} \left[ 2n(\beta^2) + (\xi\beta)]\eta (X) + (2n - 1)(X\beta) \right]$$

holds for any vector field $X$ on $M^{2n+1}$.

**Proof:** Follows from Theorem 3.6

**Corollary 73.4.** There is no weakly symmetric Sasakian or Kenmotsu manifold unless the relation $A - A(\xi)\eta = 0$ holds.

**7.4. Example.**

Let us consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let

$$e_1 = e^z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right), \quad e_2 = e^z \frac{\partial}{\partial y}, \quad \text{and} \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent vector fields at each point of the manifold $M$.

Let $g$ be Riemannian metric defined by,
From above equations the metric \( g \) is given by,
\[
g = (e^{-2z} + y^2)(dx)^2 + e^{-2z}(dy)^2 + (dz)^2.
\]
Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \), for any \( Z \) in \( M \) and \( \varphi \) be the tensor field of the type \((1,1)\) defined by \( \varphi e_i = e_2 \), \( \varphi e_2 = -e_i \), \( \varphi e_3 = 0 \), then applying linearity property of \( \varphi \) and \( g \), we have
\[
\varphi^2 Z = -Z + \eta(Z)e_3, \quad g(\varphi Z, \varphi U) = g(Z, U) - \eta(Z)\eta(U)
\]
for any \( Z, U \) in \( M \), and also we have
\[
\eta(e_i) = 0, \quad \eta(e_2) = 0, \quad \eta(e_3) = 1
\]
Hence for \( e_3 = \xi \), \( (\varphi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \). Let \( \nabla \) be the Levi Civita connection with respect to \( g \). Further
\[
[e_1, e_2] = (ye^e_2 - e^{2z}e_3), \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.
\]
By Using Koszul’s formula for the Levi-Civita connection with respect to \( \text{g} \) one obtains,
\[
\nabla_{e_1} e_3 = -e_1 + \frac{1}{2} e^{2z} e_2, \quad \nabla_{e_2} e_3 = -e_2 - \frac{1}{2} e^{2z} e_1, \quad \nabla_{e_3} e_3 = 0,
\]
\[
\nabla_{e_1} e_2 = -\frac{1}{2} e^{2z} e_3, \quad \nabla_{e_2} e_2 = e_3 + ye^e_1, \quad \nabla_{e_3} e_2 = -\frac{1}{2} e^{2z} e_1,
\]
\[
\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_2} e_1 = -ye^e_2 + \frac{1}{2} e^{2z} e_3, \quad \nabla_{e_3} e_1 = \frac{1}{2} e^{2z} e_2,
\]
Now, for \( \xi = e_1 \), the above results satisfy
\[
\nabla_X \xi = -\alpha \varphi X + \beta (X - \eta(X)\xi),
\]
with \( \alpha = -\frac{1}{2} e^{2z} \) and \( \beta = -1 \), consequently \( M(\varphi, \xi, \eta, g) \) is a 3-dimensional Trans-Sasakian manifold.

Let \( R \) be the curvature tensor of \( g \) of type \((1, 3)\); by using the above relations, one can easily calculate the non vanishing components of the curvature tensor as follows.
\[ R(e_1,e_2)e_1 = ye^{3z}e_3 + \left(\frac{3}{4}e^{4z} + 1\right)e_2, \quad R(e_1,e_2)e_2 = -\left(\frac{3}{4}e^{4z} + 1\right)e_1, \]

\[ R(e_1,e_3)e_3 = ye^{3z}e_1, \quad R(e_1,e_3)e_1 = \left(1 - \frac{e^{4z}}{4}\right)e_3 + ye^{3z}e_2 \]

\[ R(e_1,e_3)e_2 = -ye^{3z}e_1, \quad R(e_1,e_3)e_3 = \left(-\frac{e^{4z}}{4} - 1\right)e_1, \]

\[ R(e_2,e_3)e_2 = \left(1 - \frac{e^{4z}}{4}\right)e_3, \quad R(e_2,e_3)e_3 = \left(\frac{1}{4}e^{4z} - 1\right)e_2, \]

and the components, which can be obtained from these by the symmetry properties.

Using the components of the curvature tensor one can easily obtain the non vanishing components of the Ricci tensor \( S \).

\[ S(e_1,e_1) = -2\left(1 + \frac{1}{4}e^{4z}\right), \quad S(e_2,e_2) = -2\left(1 + \frac{1}{4}e^{4z}\right), \]

\[ S(e_1,e_2) = 2\left(\frac{1}{4}e^{4z} - 1\right), \quad S(e_2,e_3) = -ye^{3z}, \]

Since \( \{e_1,e_2,e_3\} \) is an orthonormal basis of \((M^3, g)\) any vector \( X \) and \( Y \) can be written as \( X = a_1e_1 + a_2e_2 + a_3e_3 \), \( Y = b_1e_1 + b_2e_2 + b_3e_3 \),

where \( a_i, b_i (i = 1,2,3) \) are positive real numbers. Now?

(7.4.1)

\[ S(X,Y) = a_1b_1S(e_1,e_1) + a_2b_2S(e_2,e_2) + a_3b_3S(e_1,e_3) \]

\[ + (a_1b_2 + b_1a_2)S(e_1,e_2) + (a_1b_3 + b_1a_3)S(e_1,e_3) \]

\[ + (a_2b_3 + b_2a_3)S(e_2,e_3) \]

\[ = \frac{1}{2}(a_3b_3 - a_1b_1 - a_2b_2)e^{4z} - 2(a_1b_1 + a_2b_2 + a_3b_3) - (a_2b_3 + a_3b_2)ye^{3z} \]

\[ = \lambda_1. \]

We choose \( a_i, b_i (i = 1,2,3) \) in such a way that \( S(X,Y) = \lambda_1 \neq 0 \).

We know that,
\[ QX = \sum_{i=1}^{3} R(X,e_i)e_i, \]

From which
\[ Qe_1 = -(2 + \frac{1}{2} e^{4x})e_1, \quad Qe_2 = -(2 + \frac{1}{2} e^{4x})e_2 - ye^{3x}e_3, \]
\[ Qe_3 = -(2 + \frac{1}{2} e^{4x})e_3 - ye^{5x}e_2, \]

From the known formula,
\[ (\nabla_X Q)(Y) = \nabla_X (QY) - Q(\nabla_X Y) \]

and lengthy calculations one obtains,
\[ g(\phi^2(\nabla_{e_1}Q)(X), Y) = 2a_1b_1ye^{3x} + 3a_2b_2ye^{4x} + \frac{1}{2} b_1a_3e^{2x} \]
\[ + b_1a_3e^{2x} + \frac{3}{8} b_1a_3e^{6x} + \frac{3}{2} b_2a_3e^{5x} - b_3a_1ye^{3x} \]
\[ + \frac{1}{2} b_2a_3e^{2x} + 3b_2a_3ye^{4x} - \frac{1}{8} b_2a_3e^{6x} \]
\[ = \lambda_2. \]

\[ g(\phi^2(\nabla_{e_1}Q)(X), Y) = 2a_1b_1(\frac{1}{4} e^{4x} - 1)e^{2x} - a_2b_1y(\frac{1}{4} e^{4x} - 1)e^{x} + \frac{1}{2} b_1a_3e^{2x} \]
\[ + b_1a_3e^{2x} - b_1a_3ye^{\frac{3}{4} e^{4x} + 1} - b_2a_3ye^{(2 + \frac{1}{2} e^{4x})} \]
\[ - b_1a_3ye^{\frac{3}{4} e^{4x} + 1} - \frac{1}{2} b_1a_3ye^{2x} + \frac{1}{2} e^{4x} \]
\[ + b_2a_3ye^{\frac{3}{4} e^{4x} + 1} - b_2a_3ye^{(2 + \frac{1}{2} e^{4x})} \]
\[ + 2b_2a_3ye^{\frac{1}{4} e^{4x} - 1} + b_2a_3e^{3x} - \frac{1}{2} b_2a_1 ye^{5x} \]
\[ - b_2a_1 ye^{\frac{1}{4} e^{4x} - 1} - b_2a_2 ye^{3x} \]

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\[-b_2a_3y e^x \left(\frac{1}{4} e^{4x} - 1\right)\]

\[= \lambda_3.\]

(7.4.4) \(g(\phi^2(\nabla_e, Q)(X), Y) = 2a_1b_1e^{4x} - 3a_2b_1 + \frac{1}{2} b_2a_2 \left(\frac{1}{4} e^{4x} - 1\right)\]

\[-\frac{1}{2} b_2a_3y e^{5x} - \frac{1}{2} b_1a_1e^{3x} \left(\frac{3}{4} e^{4x} + 1\right)\]

\[+ \frac{1}{2} b_2a_2e^{2x} \left(2 + \frac{1}{2} e^{4x}\right)\]

\[+ \frac{1}{2} b_2a_1e^{2x} \left(2 + \frac{1}{2} e^{4x}\right) - \frac{1}{2} b_2a_2e^{2x} \left(\frac{3}{4} e^{4x} - 1\right)\]

\[-b_2a_2e^{4x} + 3b_2a_3y e^{3x} + \frac{1}{2} a_1b_2e^{2x} \left(\frac{1}{4} e^{4x} - 1\right)\]

\[= \lambda_4.\]

Using (7.1.1), we get

(7.4.5) \(g(\phi^2(\nabla_e, Q)(X), Y) = A(e_1)S(X, Y) + B(X)S(e_1, Y) + C(Y)S(X, e_1),\)

(7.4.6) \(g(\phi^2(\nabla_e, Q)(X), Y) = A(e_2)S(X, Y) + B(X)S(e_2, Y) + C(Y)S(X, e_2),\)

(7.4.7) \(g(\phi^2(\nabla_e, Q)(X), Y) = A(e_3)S(X, Y) + B(X)S(e_3, Y) + C(Y)S(X, e_3).\)

Setting,

(7.4.8) \[A(e_1) = \frac{\lambda_2}{\lambda_1}, A(e_2) = \frac{\lambda_3}{\lambda_1}, A(e_3) = \frac{\lambda_4}{\lambda_1};\]

from (7.4.5), (7.4.6) and (7.4.7) it is easy to see that,

(7.4.9) \(B(X)S(e_1, Y) + C(Y)S(X, e_1) = 0\)

(7.4.10) \(B(X)S(e_2, Y) + C(Y)S(X, e_2) = 0\)

(7.4.11) \(B(X)S(e_3, Y) + C(Y)S(X, e_3) = 0\)

From these homogeneous equations in B and C, one obtains non trivial solutions.
\( B(e_i) = -C(e_i) \)

Hence we can take arbitrarily as,

\[ B(e_i) = (a_2 b_3 - a_1 b_1) e^{3z}, \quad C(e_i) = (a_1 b_1 - a_2 b_3) e^{3z} \]

Also from (4.9), (4.10) and (4.11) we find

\[ B(e_2) = 0, \quad C(e_2) = 0, \quad B(e_3) = 0, \quad C(e_3) = 0, \]

From 1-forms given by (7.4.8), (7.4.12) and (7.4.13), the manifold under consideration is weakly \( \phi \)-Ricci symmetric trans-Sasakian manifold. This leads to the following:

**Theorem 7.4.1.** There exists a trans-Sasakian manifold \((M^3, g)\) which is weakly \( \phi \)-Ricci symmetric but neither locally weakly \( \phi \)-Ricci-symmetric nor weakly \( \phi \)-Ricci recurrent.

**Proof:** Follows from (7.2.15) and (7.2.16).

**Theorem 7.4.2.** There exists a trans-Sasakian manifold \((M^3, g)\) which is Weakly \( \phi \)-Ricci recurrent.

**Proof:** From homogeneous equations (7.4.9), (7.4.10) and (7.4.11) in B and C there always exist a trivial solution so that \( B = C = 0 \), hence proof follows from (7.1.1), (7.2.16) and (7.4.8).
References:


