CHAPTER- VI

ON WEAKLY $\varphi$-SYMMERIC TRANS- SASAKIAN MANIFOLDS

The purpose of the paper is to introduce a new concept such as Weakly $\varphi$- Symmetric Trans- Sasakian Manifold, and study its properties. A series of corollaries from the main theorems are also obtained and a concrete example for the existence of such manifolds is provided.

6.1. Introduction.

In Mathematics, a weakly symmetric space is a notion introduced by the Norwegian mathematician Atle Selberg in the 1950 as a generalization of a symmetric space, due to Elie Cartan. Geometrically the spaces are defined as complete Riemannian manifolds such that any two points can be exchanged by an isometry, the symmetric case being when the isometry is required to have period two. The classification of weakly symmetric spaces relies on that of periodic automorphism of complex bi semi simple Lie algebras. They provide examples of Gelfand pairs, although the corresponding theory of spherical functions in harmonic analysis, known for symmetric spaces, has not yet been developed.

The notion of Weakly Symmetric manifolds are introduced by Tamassy and Binh, 1989. Study after study shows that this notion of weakly symmetric manifold took a different turn in 1999 (please see De and Bandyopadhyay, 1999) and thus we have the final definition as stated below.
Definition 6.1.1. (De and Bandyopadhyay, 1999): A non-flat Riemannian manifold \((M^n, g)\) \((n>2)\) is called Weakly Symmetric if its curvature tensor \(R\) of type \((0, 4)\) satisfies the condition:

\[
+ B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\
+ D(V)R(Y, Z, X, U),
\]

for all the vector fields \(X, Y, Z, U, V\) in \(\chi(M^n)\). where \(A, B, D\) are the associated 1-forms (not simultaneously zero) and \(\nabla\) denotes the operator of the covariant differentiation with respect to Riemannian metric ‘\(g\)’. The \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\).

In 1993, Tamassy and Binh again introduced one more notion of Weakly Ricci Symmetric Manifolds as follows.

In 1985, Oubina introduced first time the notion of trans-Sasakian manifolds, which contains both the classes of Sasakian and Cosymplectic Structures closely related to the conformal Kahler manifolds. Trans-Sasakian manifolds of the type \((0, \alpha), (\alpha, 0)\) and \((0, \beta)\) are the co-Symplectic ,\(\alpha\)-Sasakian and \(\beta\)-Kenmotsu manifolds respectively. In particular if \(\alpha=1, \beta=0\) and \(\alpha=0, \beta=1\); then a trans – Sasakian manifold reduces to Sasakian and Kenmotsu manifolds respectively. Thus trans-Sasakian structures provide a large class of generalized quasi Sasakian structures. In 2002, Kim et.al (Kim et.al, 2002) studied the generalized Ricci-recurrent trans-Sasakian manifolds. In 2003 De and Tripathi (De and Tripathi, 2003) studied Ricci semi –Symmetric trans Sasakian manifolds. In 2008 ,Trans-Sasakian manifolds are also studied by Shaikh et.al ,2008 (Shaikh et.al ,2008), In 2009 ,A.A Shaikh and S.K.Hui (see A.A Shaikh and
S.K.Hui, 2009) have continued the study of Trans-Sasakian manifolds and obtained interesting results based on non-vanishing $\xi$-sectional curvature. We are inspired by this paper and we are forced to introduce two new definitions and obtained interesting theorems parallel to those of A.A Shaikh and S.K.Hui, 2009.

In 2010, we have introduced in one of the papers, the notion of Weak $\varphi$-Symmetries of Kenmotsu manifolds (see Pujar S.S and Naik S.S, 2010). The purpose of the present paper is to introduce a new concept on Weakly $\varphi$-Symmetric trans-Sasakian manifold, and study some properties of the same. We also study a series of corollaries in sections 3 as special cases of the main theorems. In section 2, preliminary results of Trans Sasakian manifolds are given. In section 3 and 4, new definition for weakly $\varphi$-Symmetric trans-Sasakian manifolds and example for the existence of weakly $\varphi$-Symmetric trans-Sasakian Manifold are introduced and properties of these manifolds are also studied. In particular some properties, of Weakly $\varphi$-Symmetric trans-Sasakian Manifolds of types $(\alpha, 0)$, $(0, \beta)$, $(1, 0)$ and $(0, 1)$ are also studied.

6.2. Trans-Sasakian Manifolds.

This section is devoted to preliminary results on almost contact metric manifolds (D.E.Blair1976); (Blair, D.E and Oubina, J.A, 1990).and trans–Sasakian manifolds by Chaki M.C (please see Chaki M.C, 1994)

A $(2n+1)$ - dimensional smooth manifold $M$ is said to be an almost contact metric manifold if it admits a $(1, 1)$ tensor field $\varphi$, an associated vector field $\xi$, a 1-form $\eta$ and a Riemannian metric ‘$g$’ which satisfy
(6.2.1) \[ \varphi(\xi) = 0, \eta(\varphi(X)) = 0, \quad \varphi^2(X) = -X + \eta(X)\xi, \]

(6.2.2) \[ g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \]

(6.2.3) \[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, X) = 0 \]

for all vector fields \( X, Y \) on \( M \).

An almost contact metric manifold is said to be trans–Sasakian manifold (see Oubina, 1985) if \((M \times \mathbb{R}, J, G)\) belongs to the class \( W_4 \) of the Hermitian manifolds, where \( J \) is the almost complex structure on \( M \times \mathbb{R} \) defined by

\[ J(Z, f \frac{d}{dt}) = (\varphi Z - f \xi, \eta(Z) \frac{d}{dt}), \]

for any vector field \( Z \) on \( M \) and a smooth function \( f \) on \( M \), and \( G \) is the product matrix on \( M \times \mathbb{R} \), this may be stated by the condition (Blair, D.E and Oubina.,1990)

\[ (\nabla_X \varphi)(Y) = \alpha \{g(X,Y)\xi - \eta(Y)X\} + \beta \{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}, \quad (6.2.4) \]

where \( \alpha \) and \( \beta \) smooth functions on \( M \) and \( X \) and \( Y \) are smooth vector fields on \( M \). The manifold with the above trans-Sasakian structure is called a trans-Sasakian manifold of type \((\alpha, \beta)\). From (2.4) it is easy to see that

(6.2.5) \[ \nabla_X \xi = -\alpha \varphi X + \beta \{X - \eta(X)\xi\}, \]

(6.2.6) \[ \nabla_X \eta = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y) \]

For the trans–Sasakian manifold \( M^{2n+1}(\varphi, \xi, \eta, g) \) the following results hold (see De.U.C and Tripathi M.M.2003, and A.A.shaikh and S.K,Hui, 2009).

(6.2.7)

\[ R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y \]

\[ + 2\alpha \beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X, \]

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\begin{align}
\eta ( R ( X, Y ) Z) &= (\alpha^2 - \beta^2) [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] \\
&\quad - 2 \alpha \beta [g(\varphi X, Z) \eta(Y) - g(\varphi Y, Z) \eta(X)] \\
&\quad - (Y\alpha)g(\varphi X, Z) - (X\beta)\{g(Y, Z) - \eta(Y) \eta(Z)\} \\
&\quad + (X\alpha)g(\varphi Y, Z) - (Y\beta)\{g(X, Z) - \eta(X) \eta(Z)\}
\end{align}

\begin{align}
S(X, \xi) &= [2n(\alpha^2 - \beta^2) - (\xi\beta)](\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta), \\
R(\xi, X)\xi &= [(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - X, \\
S(\xi, \xi) &= [2n(\alpha^2 - \beta^2) - (\xi\beta)], \\
(\xi\alpha) + 2\alpha\beta &= 0 \\
Q\xi &= [2n(\alpha^2 - \beta^2) - (\xi\beta)]\xi + \varphi \text{grad} \alpha - (2n - 1)(\text{grad}\beta)],
\end{align}

where R is the curvature of type (1,3) of the manifold and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S and g(QX,Y) = S(X,Y) for all vector fields X and Y on M

\begin{align}
R(\xi, Y) X &= (\alpha^2 - \beta^2) [g(X, Y) \xi - \eta(X)Y] \\
&\quad + 2 \alpha \beta [g(\varphi X, Y) \xi + \eta(X)\varphi Y] \\
&\quad + (X\alpha)\varphi Y + g\{\varphi X, Y\} (\text{grad}\alpha) + (X\beta)(Y - \eta(Y)\xi) \\
&\quad - g(\varphi X, \varphi Y)(\text{grad}\beta).
\end{align}

Since R(\xi, Y) = -R(Y, \xi)X, we can give the expression for R(Y, \xi)X, and see that simply by changing the sign in each term of (6.2.14).

\section*{6.3. Weakly $\varphi$-Symmetric Trans-Sasakian Manifolds}

The notion of a weakly symmetric manifold was introduced by L.Tamassy and T.Q Binh. Such a manifold have been studied by T.Q.Binh,M,Prvanovic and U.C De and S.Bandyopadhyay.
A non-flat Riemannian manifold \((M^n, g)\) \((n>2)\) is called weakly symmetric if its curvature tensor \(R\) satisfies the condition

\[
\]

\[
+ D(W)R(Y, Z)X + g(R(Y, Z)W, X)\rho,
\]

where \(\nabla\) denotes the Levi-Civita connection on \((M^n, g)\) and \(A, B, C, D\) and \(\rho\) are 1-forms and a vector field respectively which are non zero simultaneously. Such a manifold has been denoted by \((WS)_n\). In 1999 De and Bandyopadhyay proved the existence of a weakly symmetric manifold by an example. It was proved in 1995 by M .Prvnovic that 1-forms and the vector field must be related as follows

\[
B(X) = C(X) = D(X), g(X, \rho) = D(X), \forall X.
\]

That is the weakly symmetric manifold is characterized by the condition

\[
(6.3.2)
\]

\[
(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + D(W)R(Y, Z)X + g(R(Y, Z)W, X)\rho,
\]

\[
g(X, \rho) = D(X), \text{for all } X.
\]

The 1-forms \(A\) and \(D\) are the first and the second associated 1-forms respectively.

In this paper we have introduced the following definition

**Definition 6. 3.1.** A non flat Riemannian manifold \((M^n, g)\) \((n>2)\) is called Weakly \(\varphi\) symmetric if its curvature tensor \(R\) of type \((1, 3)\) satisfies the condition:

\[
(6.3.3)
\]
+ D(Z)R(Y, X)W + D(W)R(Y, Z)X + R(Y, Z, W, X)\rho \)

for all vector fields \( X, Y, Z, W, V \in \mathfrak{X}(M^n) \) and \( A, D \) are associated 1-forms of the manifold and ,

\[
(6.3.4) \quad R(Y, Z, W, V) = g(R(Y, Z) W, V). \quad \text{and} \quad g(\rho, V) = D(V)
\]

By observing definition 3.1, it is quite natural to think that this definition may be extended to trans-Sasakian manifolds. Thus we have the following definition.

**Definition 6.3.2.** A Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) is said to be Weakly \(\varphi\)-Symmetric if its curvature tensor \(R\) satisfies the condition \((6.3.3)\).

The \(\xi\)-sectional curvature \(K(\xi, W)\) of a trans- Sasakian manifold for a unit vector field \(W\) orthogonal to \(\xi\) is given by

\[
K(\xi, W) = g\{R(\xi, W)\xi, W\}
\]

From the relation \((6.2.10)\), we have

\[
R(\xi, W)\xi = [(\alpha^2 - \beta^2 - (\xi\beta))(\eta (W)\xi - W),
\]

\[
K(\xi, W) = g(R(\xi, W)\xi, W) = [(\alpha^2 - \beta^2 - (\xi\beta))][\eta(W)g(\xi, W) - g(W, W)].
\]

\[
= -[\alpha^2 - \beta^2 - (\xi\beta)]g(W, W) - g(W, \xi)g(W, \xi)
\]

which gives expression for \(\xi\)-sectional curvature

\[
K(\xi, W) = -[\alpha^2 - \beta^2 - (\xi\beta)].
\]

If \([\alpha^2 - \beta^2 - (\xi\beta)] = 0\), then the manifold is of vanishing \(\xi\)-sectional curvature.

**Theorem 6.3.1.** In a Weakly \(\varphi\)-Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\)of non vanishing \(\xi\)-sectional curvature the relation
(6.3.5) \[ A(\xi) + 2D(\xi) = \frac{2\beta(\xi\beta) + \xi(\xi\beta) - 2\alpha(\xi\alpha)}{\alpha^2 - \xi\beta - \beta^2} \]

holds.

Proof. Operating by $g$ on both sides of (6.3.3) and substituting for $\varphi^2$ in (6.3.3),

(6.3.6) \[
g\{\phi^2(V_\chi R)(Y, Z)W, U\} = A(X)g(R(Y, Z)W, U) + D(Y)g(R(X, Z)W, U) + D(Z)g(R(Y, X)W, U) + D(W)g(R(Y, Z)X, U)
+ D(V)g(R(Y, Z)W, X)
\]

By virtue of (6.2.1), relation (6.3.6) can be expressed as

(6.3.7) \[
-g((V_\chi R)(Y, Z)W, U)) + \eta [(V_\chi R)(Y, Z)W]g(U, \xi)
= A(X)g\{R(Y, Z)W, U\} + D(Y)g\{R(X, Z)W, U\} + D(Z)g\{R(Y, X)W, U\} + D(W)g\{R(Y, Z)X, U\}
+ D(V)g\{R(Y, Z)W, X\}.
\]

If $\xi$ is orthogonal to $U$ then $\eta(U) = 0 = g(U, \xi)$

Setting $Y = U = e_i$, for $i = 1, 2, 3, \ldots, 2n+1$, where $\{e_i\}$ are orthonormal basis of the tangent space at each point $P$ of the manifold and assuming that the vector field $V$ is orthogonal to $\xi$ and using the properties of $R$ in fifth term on the right hand side of (6.3.7) and taking the sum from $i=1$ to $i=2n+1$ it is easy to see that the relation (6.3.7) reduces to

(6.3.8) \[
\]

where $S$ is the Ricci curvature tensor of the type $(0, 2)$

Putting $X = Z = W = \xi$ in (3.8), we find
\[-(\nabla_\xi S)(\xi, \xi) = A(\xi) S(\xi, \xi) + D[R(\xi, \xi)\xi] + D(\xi) S(\xi, \xi) + D(\xi) S(\xi, \xi) + D[R(\xi, \xi)\xi].\] (6.3.9)

Now substituting for \(S(\xi, \xi), R(\xi, \xi)\xi\), from (6.2.11), and (6.2.10) respectively in (6.3.9), and after simplification the result (6.3.5) of the Theorem 3.1 follows, as by the hypothesis of the Theorem the manifold \(M^{2n+1}\) is of non vanishing \(\xi\)-sectional curvature.

This completes the proof of the theorem.

**Remark 6.3.1.** For weakly \(\varphi\)-symmetric trans-Sasakian manifold of type \((1, 0)\) and \((0, 1)\) (i.e. weakly \(\varphi\)-Symmetric Sasakian and Kenmotsu manifolds), it is easy to see from (3.5) of Theorem 3.1 that

\[A(\xi) + 2D(\xi) = 0\] (6.3.10)

**Theorem 6.3.2.** In a Weakly \(\varphi\)-Symmetric Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \(\xi\)-sectional curvature the associated 1-form \(D\) is given by

\[D(W) = \frac{(2\beta(\xi\beta)) + \xi(\xi\beta) - 4n\alpha(\xi\alpha)}{(2n-1)\alpha^2 - (\xi\beta) - \beta^2} \eta(W) + \frac{(2n-1)W(\beta - \xi\beta) + \phi W(\alpha - \xi\alpha)}{(2n-1)\alpha^2 - (\xi\beta) - \beta^2} \eta(W) + \frac{2(n-1)(\alpha^2 - \beta^2)\eta(W)D(\xi)}{(2n-1)\alpha^2 - (\xi\beta) - \beta^2} + \frac{-2\beta(\xi\beta) + \xi(\xi\beta) - 2\alpha(\xi\alpha)}{(2n-1)(\alpha^2 - (\xi\beta) - \beta^2)^2} \{2n(\alpha^2 - \beta^2) - \xi\beta\} \eta(W)

for any vector field \(W\).
Proof. Taking $X = Z = \xi$ in (6.3.8), we get

$$-(\nabla_{\xi}S)(\xi, W) = A(\xi) S(\xi, W) + D\{R(\xi, \xi) W\} + D(\xi)S(\xi, W)$$

$$+ D(W) S(\xi, \xi) + D\{R(\xi, W)\xi\}$$

and

$$-(\nabla_{\xi}S)(\xi, W) = (A(\xi) + D(\xi)) S(\xi, W)$$

$$+ D(W) S(\xi, \xi) + D\{R(\xi, W)\xi\}$$

and

$(6.3.11)$

$$-(\nabla_{\xi}S)(\xi, W) = A(\xi) S(\xi, W) + D(\xi)S(\xi, W)$$

$$+ D(W) S(\xi, \xi) + D\{R(\xi, W)\xi\}$$

$$= \{A(\xi) + D(\xi)\} S(W, \xi) + D(W)[2n(\alpha^2 - \beta^2 - (\xi\beta))]$$

$$+ D\{(\alpha^2 - \beta^2 - (\xi\beta))(\eta(W)\xi - W)\}$$

$$-(\nabla_{\xi}S)(\xi, U) = \{A(\xi) + D(\xi)\} S(W, \xi)$$

$$+ [D(W)(2n - 1) + \eta(W)D(\xi)][(\alpha^2 - \beta^2 - (\xi\beta)]$$

Also the left hand side of $(6.3.11)$ reduces to,

$$-(\nabla_{\xi}S)(\xi, W) = -(\nabla_{\xi}(S(\xi, W)) - S(\nabla_{\xi}\xi, W) - S(\xi, \nabla_{\xi} W))$$

$$= - \nabla_{\xi}(S(\xi, W)) + S(\nabla_{\xi}\xi, W) + S(\xi, \nabla_{\xi} W)$$

$$= - \nabla_{\xi}[\{2n(\alpha^2 - \beta^2) - (\xi\beta)\} \eta(W) - ((\phi W)\alpha) - (2n - 1)W\beta)]$$

$$+ S(\xi, \beta W - \beta W)$$

$$= - \{2n(2\alpha(\xi\alpha) - 2\beta(\xi\beta)) - \xi(\xi\beta)\}\eta(W)$$

$$+ \{2n(\alpha^2 - \beta^2) - (\xi\beta)\} \{(\nabla_{\xi}\eta) W + \eta(\nabla_{\xi} W)\}$$

$$+ ((\nabla_{\xi}\phi) W)\alpha - \phi(\nabla_{\xi} W)\alpha - (\phi W)\nabla_{\xi}\alpha$$

$$- (2n - 1)(\nabla_{\xi} W)\beta - (2n - 1)W(\nabla_{\xi}\beta)$$

$$+ \{2n(\alpha^2 - \beta^2) - (\xi\beta)\} \eta(\beta W - \beta W)$$

$$- (\phi(\beta W - \beta W)\alpha) - (2n - 1)(\beta W - \beta W)\beta$$
Now using (6.2.4), (6.2.5) and (6.2.6) in the right hand side of above expression and further simplifying, it is easy to see that
\[-(\nabla_\xi S)(\xi,W)\]
\[= -\{2n(2\alpha(\xi\alpha) - 2\beta(\xi\beta)) - \xi(\xi\beta)\} \eta(W)\]
\[+ \beta 2n(\alpha^2 - \beta^2) + 2n\beta((\alpha^2 - \beta^2)) \eta(W)\]
\[+ \xi\beta^2 - \xi\beta^2 \eta(W) + \phi\beta\xi\alpha - \phi\beta\alpha W\]
\[+ \phi W(\xi\alpha) + (2n - 1)\beta\xi\beta - (2n - 1)\beta W\beta\]
\[+ (2n - 1)W(\xi\beta) + 2n(\alpha^2 - \beta^2) - \beta\xi\beta\]
\[+ \beta\alpha\phi W - (2n - 1)\beta\xi\beta + (2n - 1)\beta W\beta]\]
which after canceling same positive and negative terms, we get
\[-(\nabla_\xi S)(\xi, W) = -\{2n(2\alpha(\xi\alpha) - 2\beta(\xi\beta)) - \xi(\xi\beta)\} \eta(W)\]
\[+ \phi W(\xi\alpha) + (2n - 1)W(\xi\beta)\]

\[(6.3.12)\]
Substituting (6.3.12) in (6.3.11) we get
\[(6.3.13)\]
\[-2n(2\alpha(\xi\alpha) + 2\beta(\xi\beta)) + \xi(\xi\beta)\} \eta(W) + \phi W(\xi\alpha) + (2n - 1)W(\xi\beta)\]
\[= \{A(\xi) + D(\xi)\} S(W, \xi)\]
\[+ [D(W)(2n - 1) + \eta(W)D(\xi)]\]
\[\{[\alpha^2 - \beta^2 - (\xi\beta)\}\]
Now by using (6.2.9), and substituting for \(A(\xi) + 2D(\xi)\) from (6.3.5) in (6.3.13), the proof of Theorem 3.2 follows, as by the hypothesis of the Theorem the manifold \(M^{2n+1}\) is of non vanishing \(\xi\)-sectional curvature which completes the proof of the Theorem 3.2.
**Theorem 6.3.3.** In a Weakly $\varphi$-Symmetric Trans-Sasakian Manifold $(M^{2n+1}, g)$ ($n > 1$) of non vanishing $\xi$-sectional curvature for the associated 1-forms $A$ and $D$, the following relation holds.

\[ 2nA(X) + 2D(X) = \frac{2D(\xi)[\{(\alpha^2 - \xi\beta - \beta^2)\eta(X)\}}{\alpha^2 - \xi\beta - \beta^2} \]

\[ - 2n\frac{2\eta(X)\eta(X) - (\xi\beta)\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)}{\alpha^2 - \xi\beta - \beta^2} D(\xi) \]

\[ - 2n\frac{2\eta(X)\eta(X) - (\xi\beta)\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)}{\alpha^2 - \xi\beta - \beta^2} D(\xi) \]

\[ - 2n\frac{2\eta(X)\eta(X) - (\xi\beta)\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)}{\alpha^2 - \xi\beta - \beta^2} D(\xi) \]

\[ - 2n\frac{2\eta(X)\eta(X) - (\xi\beta)\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta)}{\alpha^2 - \xi\beta - \beta^2} D(\xi) \]

\[ 2\beta[(\varphi X)\alpha + (2n - 1)(X\beta) - (\xi\beta)\eta(X)] \]

\[ (\alpha^2 - \xi\beta - \beta^2) \]

holds.

**Proof.** Substituting $Z = W = \xi$ in (6.3.8) we get

(6.3.14)

\[ - (\nabla_X S)(\xi, \xi) = A(X) S(\xi, \xi) + D\{R(X, \xi)\xi\} + D(\xi)S(X, \xi) \]

\[ + D(\xi) S(\xi, X) + D\{R(X, \xi)\xi\}, \]

\[ = A(X)(2n(\alpha^2 - \xi\beta - \beta^2) + [D(\xi) + D(\xi)]S(X, \xi) \]

\[ + D(-\alpha^2 - \xi\beta - \beta^2)(\eta(X)\xi - X) \]

\[ + D(-\alpha^2 - \xi\beta - \beta^2)(\eta(X)\xi - X) \]

\[ \nabla_X S(\xi, \xi) = 2(\alpha^2 - \xi\beta - \beta^2)[\{(\eta(X))(D(\xi) - D(X)] \]

\[ - 2n(\alpha^2 - \xi\beta - \beta^2)A(X) - 2D(\xi)S(X, \xi) \]

Now consider the left hand side of (6.3.14)
\[ \nabla_X S(\xi, \bar{\xi}) = \nabla_X (S(\xi, \bar{\xi})) - S(\nabla_X \xi, \bar{\xi}) - S(\xi, \nabla_X \bar{\xi}) \]
\[ = \nabla_X (S(\xi, \bar{\xi})) - 2S(\nabla_X \xi, \bar{\xi}) \]
\[ = \nabla_X (2n(\alpha^2 - \xi\beta - \beta^2)) - 2S(-\alpha\varphi X + \beta(X - \eta(X))\xi), \]

Further simplifying, we get
\[ \nabla_X S(\xi, \bar{\xi}) = 2n[2\alpha(X\alpha) - 2\beta(X\beta) - X(\xi\beta)] \]
\[ + 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n - 1)(\varphi X)\beta] \]
\[ + 2\beta[(\varphi X)\alpha + (2n - 1)\{(X\beta) - (\xi\beta)\eta(X)\}] \]

Substituting in (6.3.14)
(6.3.15)
\[ 2n[2\alpha(X\alpha) - 2\beta(X\beta) - X(\xi\beta)] \]
\[ + 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n - 1)(\varphi X)\beta] \]
\[ + 2\beta[(\varphi X)\alpha + (2n - 1)\{(X\beta) - (\xi\beta)\eta(X)\}] \]
\[ = 2(\alpha^2 - \xi\beta - \beta^2)[\eta(X)\Delta(\xi) - \Delta(X)] \]
\[ - 2n(\alpha^2 - \xi\beta - \beta^2)A(X) - 2\Delta(\xi)S(X, \bar{\xi}) \]

Further simplifying we get,
(6.3.16)
\[ (\alpha^2 - \xi\beta - \beta^2)(2nA(X) + 2\Delta(X)) \]
\[ = 2\Delta(\xi)[(\alpha^2 - \xi\beta - \beta^2)]\eta(X) - S(X, \bar{\xi})] \]
\[ - 2n[2\alpha(X\alpha) - 2\beta(X\beta) - X(\xi\beta)] \]
\[ - 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n - 1)(\varphi X)\beta] \]
\[ - 2\beta[(\varphi X)\alpha + (2n - 1)\{(X\beta) - (\xi\beta)\eta(X)\}] \]

Simplifying further (6.3.16) we get,
(6.3.17)
2nA(X) + 2D(X)

\[ \frac{2D(\xi)[\{(\alpha^2 - \xi\beta - \beta^2)\} \eta(X)}{\alpha^2 - \xi\beta - \beta^2} \]

\[ - 2 \frac{2n(\alpha^2 - \beta^2) - (\xi\beta))]}{\alpha^2 - \xi\beta - \beta^2} \eta(X) - ((\phi X)\alpha) - (2n - 1)(\xi\beta), D(\xi) \]

\[ - \frac{2n[2\alpha(\xi\alpha) - 2\beta(\xi\beta) - X(\xi\beta)]}{(\alpha^2 - \xi\beta - \beta^2)} \]

\[ - \frac{2\alpha[(\xi\alpha) - \eta(X)(\xi\alpha) - (2n - 1)(\phi X)\beta}{(\alpha^2 - \xi\beta - \beta^2)} \]

\[ - \frac{2\beta[(\phi X)\alpha + (2n - 1)(\xi\beta)\eta(X)]}{(\alpha^2 - \xi\beta - \beta^2)} \]

Now by equation (6.3.17) and the hypothesis of the Theorem, the manifold \( M^{2n+1} \) is of non-vanishing \( \xi \)-sectional curvature which completes the proof of the Theorem 3.5.

In particular if, \( \phi \text{grad}\alpha = \text{grad}\beta \), then,

\[ \xi\beta = g(\xi, \text{grad}\beta) \]

\[ = g(\xi, \phi \text{grad}\alpha) \]

\[ = \eta(\phi \text{grad}\alpha) \]

\[ = 0 \]

Substituting this value of \( \xi\beta \) in (6.3.17) we get,
(6.3.18)

\[ 2nA(X) + 2D(X) = \frac{2D(\xi)((\alpha^2 - \beta^2))\eta(X)}{(\alpha^2 - \beta^2)} - 2 \frac{2n(\alpha^2 - \beta^2)(\eta(X) - ((\varphi X)\alpha) - (2n-1)(X\beta))D(\xi)}{(\alpha^2 - \beta^2)} - \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta)]}{(\alpha^2 - \beta^2)} - \frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)(\varphi X)\beta]}{(\alpha^2 - \beta^2)} - \frac{2\beta[(\varphi X)\alpha + (2n-1)(\xi\beta)]}{(\alpha^2 - \beta^2)} \]

for any vector field X provided that \((\alpha^2 - \beta^2) \neq 0\).

If \(\alpha^2 - \beta^2 = 0\), then in view of \((\xi\alpha) + 2\alpha\beta = 0\) that is both \(\alpha, \beta\) are positive and their sum is zero, it is possible only when \(\alpha = 0 = \beta\) and hence the manifold is cosymplectic. This leads to the following:

**Corollary 6.3.1.** If a weakly \(\varphi\) - symmetric non cosymplectic trans – Sasakian manifold \((M^{2n+1}, g)\) \((n > 1)\) satisfies the condition \(\varphi \text{grad} \alpha = \text{grad} \beta\); and is of non zero \(\xi\) sectional curvature, then the relation (6.3.18) holds.

**Proof.** Substituting \(\varphi \text{grad} \alpha = \text{grad} \beta\) so that \(\xi\beta = 0\), putting \(\xi\beta = 0\) in (6.3.17), (6.3.18) follows, hence the proof of Corollary 3.1

**Corollary 6.3.2.** In a weakly \(\varphi\) - symmetric Sasakian manifold \((M^{2n+1}, g)\) \((n > 1)\) with non zero \(\xi\) sectional curvature, the relation (6.3.19) \(2nA(X) + 2D(X) + (2n-1)D(\xi)\eta(X) = 0\) everywhere.

If \(\beta = 0\) and \(\alpha = 1\), then (6.3.18) yields

\[ 2nA(X) + 2D(X) + (2n - 1)\eta(X) = 0 \]

hence the proof of Corollary 3.2.
**Corollary 6.3.3.** If an $\alpha$-Sasakian manifold is weakly $\varphi$-Symmetric with non zero $\xi$ sectional curvature, then the following relation holds

\[(6.3.20)\]

\[2nA(X) + 2D(X) = 2D(\xi)\eta(X) + \frac{2n(\alpha^2)(\eta(X) + (\varphi X)\alpha)_\alpha D(\xi)}{\alpha^2} - \frac{2n[2\alpha(X\alpha)]}{\alpha^2} - \frac{2\alpha\eta(X) - \eta(X)(\xi\alpha)}{\alpha^2} - \frac{2\alpha\eta(X) + \eta(X)(\xi\alpha)}{\alpha^2} \]

**Proof.** For the weakly $\varphi$-symmetric trans--Sasakian manifold $(M^{2n+1}, g)$ $(n>1)$ of type $(\alpha, 0)$ i.e.; for $\alpha$-Sasakian manifold , $\beta = 0$, and $\alpha \neq 0$ so that (6.3.18) after simplification yields (6.3.20), hence the proof of Corollary 3.3 completes.

**Corollary 6.3.4.** If a $\beta$-Kenmotsu manifold is weakly $\varphi$-Symmetric with non zero $\xi$ sectional curvature, then the following relation holds

\[(6.3.21)\]

\[2nA(X) + 2D(X) = 2D(\xi)\eta(X) + \frac{4n\eta(X) + (2n - 1)(X\beta)D(\xi)}{\beta} + \frac{4n(X\beta)}{\beta} - \frac{2[(2n - 1)(X\beta)]}{\beta} \]

**Proof.** For the weakly $\varphi$-symmetric trans--Sasakian manifold $(M^{2n+1}, g)$ $(n>1)$ of type $(0, \beta)$ i.e.; for $\beta$-Kenmotsu manifold , $\beta = 0$, and $\alpha = 0$ so that (6.3.18) after simplification yields (6.3.21), hence the proof of Corollary 3.4 completes.

**Corollary 6.3.5.** If a Kenmotsu manifold is Weakly $\varphi$-Symmetric with non zero $\xi$ sectional curvature, then the following relation holds

\[(6.3.22)\]

\[2nA(X) + 2D(X) - 2(D(\xi) + 2n)\eta(X) = 0 \]
For the weakly $\varphi$-symmetric trans–Sasakian manifold $(M^{2n+1}, g)$ (n>1) of type $(0,1)$ i.e. for Kenmotsu manifold, $\beta = 1$, and $\alpha = 0$ so that (6.3.18) after simplification yields (6.3.22), hence the proof of Corollary 3.5 completes.

6.4. Examples for the existence of weakly $\varphi$-symmetric Trans–Sasakian manifold $(M^{2n+1}, g)$ (n>1) of type $(\alpha, \beta)$.

6.4.1. Let us consider a 3–dimensional manifold

Let ,

$$e_1 = e^x \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad e_2 = e^y \frac{\partial}{\partial y}, \quad \text{and} \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent vector fields at each point of the manifold $M$. We define an indefinite Riemannian metric $g$ on $M$ as

$$g = \left( e^{-2z} + y^2 \right) (dx)^2 + e^{-2z} (dy)^2 + (dz)^2$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \Gamma(TM)$ and $\varphi$ be the tensor field of the type $(1,1)$ defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0,$$

then applying linearity property of $\varphi$ and $g$, we have

$$\varphi^2 Z = -Z + \eta(Z) e_3, \quad g(\varphi Z, \varphi U) = g(Z, U) - \eta(Z) \eta(U)$$

for any $Z, U \in M$, and also we have

$$\eta(e_1) = 0, \quad \eta(e_2) = 0, \quad \eta(e_3) = 1$$
Hence for \( e_3 = \xi \), \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

Let \( \nabla \) be the Levi Civita connection with respect to \( g \). Further

\[
[e_1, e_2] = (ye^{2z}e_2 - e^{2z}e_3), \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.
\]

By Using Koszul’s formula for the Levi-Civita connection with respect to \( g \) one obtains,

\[
\nabla_{e_1}e_3 = -e_1 + \frac{1}{2} e^{2z}e_2 , \quad \nabla_{e_2}e_3 = -e_2 - \frac{1}{2} e^{2z}e_1 , \quad \nabla_{e_1}e_3 = 0 ,
\]

\[
\nabla_{e_1}e_2 = -\frac{1}{2} e^{2z}e_3 , \quad \nabla_{e_2}e_2 = e_3 + ye^{2z}e_1 , \quad \nabla_{e_1}e_2 = -\frac{1}{2} e^{2z}e_1 ,
\]

\((6.4.1)\)

\[
\nabla_{e_3}e_1 = e_3 , \quad \nabla_{e_1}e_1 = -ye^{2z}e_2 + \frac{1}{2} e^{2z}e_3 , \quad \nabla_{e_1}e_1 = \frac{1}{2} e^{2z}e_2 ,
\]

Now, for \( \xi = e_3 \), the above results satisfy

\[
\nabla_{X}\xi = -\alpha \phi X + \beta (X - \eta(X)\xi) ,
\]

with \( \alpha = -\frac{1}{2} e^{2z} \) and \( \beta = -1 \), consequently \( M(\phi, \xi, \eta, g) \) is a 3-dimensional trans-Sasakian manifold.

Let \( R \) be the curvature tensor of \( g \) of type \((1, 3)\), by using the above relations, one can easily calculate the non vanishing components of the curvature tensor as follows.

\[
R(e_1, e_2)e_1 = ye^{2z}e_3 + \left(\frac{3}{4} e^{4z} + 1\right)e_2 , \quad R(e_1, e_2)e_2 = -\left(\frac{3}{4} e^{4z} + 1\right)e_1 ,
\]

\[
R(e_1, e_2)e_3 = -ye^{2z}e_1 , \quad R(e_1, e_3)e_1 = \left(1 - \frac{e^{4z}}{4}\right)e_3 + ye^{2z}e_2
\]

\[
R(e_1, e_3)e_2 = -ye^{2z}e_1 , \quad R(e_1, e_3)e_3 = \frac{1}{4} e^{4z} - 1)e_1 ,
\]

\[
R(e_2, e_1)e_2 = (1 - \frac{1}{4} e^{4z})e_3 , \quad R(e_2, e_3)e_3 = (\frac{1}{4} e^{4z} - 1)e_2 ,
\]
Let us conceder operating by \( g \) on both sides of (6.3.3) we get
\[
g(\phi^2 (\nabla_x R)(Y, Z)W, U)
\]
\[
= A(X)g(R(Y, Z)W, U) + D(Y)g(R(X, Z)W, U)
\]
\[
+ D(Z)g(R(Y, X)W, U) + D(W)g(R(Y, Z)X, U) + D(V)g(R(Y, Z)W, X)
\]
(6.4.2)  
\[
g(\phi^2 (\nabla_{e_i} R)(Y, Z)W, U)
\]
\[
= A(e_i)g(R(Y, Z)W, U) + D(Y)g(R(e_i, Z)W, U)
\]
\[
+ D(Z)g(R(Y, e_i)W, U) + D(W)g(R(Y, Z)e_i, U) + D(U)g(R(Y, Z)W, e_i)
\]
where \( e_i : i = 1, 2, 3, \) if \( R(Y, Z, W, U) \neq 0 \)
then we can find from (6.4.2) the 1-forms as
(6.4.3) and (6.4.4)
\[
A(e_i) = \frac{g(\phi^2 (\nabla_{e_i} R)(Y, Z)W, U)}{R(Y, Z, W, U)}, i = 1, 2, 3.
\]
and
\[
D(Y)g(R(e_i, Z)W, U) + D(Z)g(R(Y, e_i)W, U)
\]
\[
+ D(W)g(R(Y, Z)e_i, U) + D(U)g(R(Y, Z)W, e_i) = 0
\]
Any vector fields \( Y, Z, W, U \) can be written as  
\[
Y = b_1 e_1 + b_2 e_2 + b_3 e_3, \quad Z = c_1 e_1 + c_2 e_2 + c_3 e_3, \quad W = d_1 e_1 + d_2 e_2 + d_3 e_3
\]
\[
U = f_1 e_1 + f_2 e_2 + f_3 e_3.
\]
Substituting these vector fields in (6.4.4) we find
\[
p_1 D(e_1) + q_1 D(e_2) + r_1 D(e_3) = 0,
\]
\[
p_2 D(e_1) + q_2 D(e_2) + r_2 D(e_3) = 0,
\]
\[
p_3 D(e_1) + q_3 D(e_2) + r_3 D(e_3) = 0
\]
where \( p_i, q_i, \) and \( r_i \) for \( i = 1, 2, 3 \) are the functions of \( x, y, z \).
The set of equations in (6.4.4) are homogeneous in \( D \), the trivial solution always exists, so that \( D(e_i) = 0, \) \( i = 1, 2, 3 \). Thus one can state,
Theorem 6.3.5. A weakly $\varphi$–symmetric trans-Sasakian manifold $(M^3, g)$, can be a $\varphi$–recurrent trans-Sasakian manifold.

If $p_i, q_i$, and $r_i$ for $i = 1, 2, 3$ are such that

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

Then the system (6.4.4) has infinite number of solutions, hence one can state

Theorem 6.3.6. There exists weakly $\varphi$–symmetric trans-Sasakian manifold $(M^3, g)$ which is neither $\varphi$–recurrent nor locally $\varphi$–symmetric.
References:


