CHAPTER-V

ON GENERALIZED $\varphi$-RECURRENT, CONCIRCULAR $\varphi$-RECURRENT AND PROJECTIVE $\varphi$-RECURRENT TRANS SASAKIAN MANIFOLDS

The purpose of the paper is to study some of the properties of generalized $\varphi$-recurrent and generalized Concircular $\varphi$-recurrent Trans Sasakian manifolds and generalize some of the results of [1] and some of those of [2] and [13].

5.1. Introduction. A Riemannian manifold $(M^n, g)$ is called generalized recurrent [3] if its curvature tensor $R$ satisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z],$$

where $A$ and $B$ are associated 1-forms and $B$ is non zero.

The notion of a local Symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extents a weaker version of a local symmetry [4] introduced the notion of a local $\varphi$-symmetry on a Sasakian Manifold. Generalizing the $\varphi$-symmetry, the authors [5] introduced the notion of $\varphi$-recurrent Sasakian Manifolds in 2003. This notion has been studied by many authors for

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different types of Riemannian manifolds. In 2008, generalizing the notion of $\phi$-recurrency the authors [1] have introduced the notion of generalized $\phi$-recurrency to Kenmotsu manifolds. In 2009 D.A.Patil, D.G Prakasha and C.S.Bagewadi extended the study of generalized $\phi$-recurrency to Sasakian manifolds. The authors [13] studied 3-dimensional trans-Sasakian manifolds and showed that these are locally symmetric.

Motivated by the above studies, in the present paper, we extend the study of generalized $\phi$-recurrence of Sasakian and generalized Concircular $\phi$-recurrence of Kenmotsu manifolds to generalized $\phi$-recurrence in Trans Sasakian manifolds, also we studied some curvature properties of 3-dimensional generalized recurrent trans Sasakian manifolds. The paper is organized as follows:

In section 2, we give the preliminaries that is, basic standard formulas of Trans Sasakian manifolds. In Section 3 we introduced the notion of generalized $\phi$-recurrent Trans Sasakian manifolds and obtained the relations between Ricci curvature tensor, the associated 1-forms $A$ and $B$ also obtained the separate theorem for the relationship between the 1-form $A$ and $B$. At the end of section 3, we generalize some of the results of [1] and some of those of [2].

Section 4 is devoted to generalize Concircular $\phi$-recurrent Trans Sasakian manifolds and in fact we have obtained the relationship between the 1-forms for $A$ and $B$. Theorem 4.1 includes some of the results of [2] as special cases.

In Section 5 properties of generalized Einstein projective $\phi$-recurrent manifold are studied. Finally Section 6 is devoted to 3-
dimensional Trans Sasakian manifolds in fact we have studied curvature properties of 3-dimensional generalized recurrent trans-Sasakian manifold and generalize some of the results of [13]

5.2. Trans-Sasakian Manifolds.

This section is devoted to preliminary results on almost contact metric manifolds [6], [7] and trans–Sasakian manifolds by Chaki M.C (please see [8])

A (2n+1) - dimensional smooth manifold M is said to be an almost contact metric manifold if it admits a (1, 1) tensor field $\phi$, an associated vector field $\xi$, a 1-form $\eta$ and a Riemannian metric ‘g’ which satisfy

(5.2.1) $\phi(\xi) = 0$, $\eta(\phi(X)) = 0$, $\phi^2(X) = -X + \eta(X)\xi$,

(5.2.2) $g(\phi X, Y) = -g(X, \phi Y)$, $\eta(X) = g(X, \xi)$, $\eta(\xi) = 1$,

(5.2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, $g(\phi X, X) = 0$,

for all vector fields $X, Y \in \chi(M)$ M, where $\chi(M)$ is the set of all $C^\infty$ vector fields.

An almost contact metric manifold is said to be Trans –Sasakian manifold [9] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ of the Hermitian manifolds, where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(Z, f \frac{d}{dt}) = (\phi Z - f \xi, \eta(Z) \frac{d}{dt}),$$

for any vector field $Z$ on $M$ and a smooth function $f$ on $M \times \mathbb{R}$, and G is the product matrix on $M \times \mathbb{R}$, this may be stated by the condition [7]

(5.2.4) $(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$

where $\alpha$ and $\beta$ are smooth functions on $M$ and $X$ and $Y$ are smooth vector fields on $M$. The manifold with the above trans-Sasakian
structure is called a trans-Sasakian manifold of type \((\alpha, \beta)\). From (2.4) it is easy to see that

\[
\nabla_X \xi = -\alpha \varphi X + \beta \{X - \eta(X)\xi\},
\]

\[
(\nabla_X \eta)(Y) = - \alpha g(\varphi X, Y) + \beta \{g(X, Y) - \eta(X)\eta(Y)\}
\]

For the Trans – Sasakian manifold \(M^{2n+1}(\varphi, \xi, \eta, g)\) the following results hold [10], [11]

\[
R(X, Y)\xi = (\alpha^2 - \beta^2) [\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y + 2\alpha\beta [\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X,
\]

\[
S(X, \xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)](\eta(X)) - ((\varphi X)\alpha) - (2n-1)(X\beta),
\]

\[
R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X),
\]

\[
S(\xi, \xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)],
\]

\[
(\xi\alpha) + 2\alpha\beta = 0
\]

\[
Q\xi = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\xi + \varphi grad\alpha - (2n-1)(grad\beta),
\]

where \(R\) is the curvature of type (1,3) of the manifold and \(Q\) is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor \(S\) and \(g(QX, Y) = S(X, Y)\) for all vector fields \(X\) and \(Y\) on \(M\).
\[(5.2.13)\]
\[
R(\xi, Y)X = (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(X)Y] \\
+ 2\alpha \beta [g(\phi X, Y)\xi + \eta(X)\phi Y] \\
+ (X\alpha)\phi Y + g\{(\phi X), Y\}(\text{grad}\alpha) + (X\beta)(Y - \eta(Y\xi)) \\
- g(\phi X, \phi Y)(\text{grad}\beta).
\]

Since \(R(\xi, Y) = -R(Y, \xi)\), we can give the expression for \(R(Y, \xi)X\), and see that simply by changing the sign in each term of (5.2.13).

The above results will be used in the next sections to come.

5.3. Generalized \(\varphi\)-Recurrent Trans Sasakian Manifolds

**Definition 5.3.1** Trans-Sasakian manifold (\(M, g\)) is called generalized \(\varphi\)-recurrent if its curvature tensor \(R\) satisfies the condition

\[(5.3.1)\]
\[
\varphi^2((\nabla_w R)(X, Y)Z) = A(W)R(X, Y)Z \\
+ B(W)[g(Y, Z)X - g(X, Z)Y],
\]

where \(A\) and \(B\) are associated 1-forms, and \(A \neq 0\).

If \(A = 0\) and \(B = 0\), then a generalized \(\varphi\)-recurrent trans–Sasakian manifold reduces to locally \(\varphi\)-symmetric that is from (5.3.1) we have

\[(5.3.1)a\]
\[
\varphi^2((\nabla_w R)(X, Y)Z) = 0,
\]

By the definition of \(\varphi^2\), we have from (5.3.1),
from which it follows that

\[ \xi \nabla \eta + \nabla - \sum_{i=1}^{2n+1} \eta((\nabla_w R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y, Z) + B(W)2ng(Y, Z) \]

Simplifying and using \( Z = \xi \) in the above equation also we note that

\[ \sum_{i=1}^{2n+1} \eta((\nabla_w R)(e_i, Y)Z)\eta(e_i) = \sum_{i=1}^{2n+1} g((\nabla_w R)(e_i, Y)Z, \xi)g(e_i, \xi) \]

\[ = A(W)S(Y, \xi) + B(W)2ng(Y, \xi) \]

Further simplification gives

(5.3.5)
**Lemma 5.3.1.** In a trans-Sasakian manifold,

\[ \sum_{i=1}^{2n+1} g((\nabla_w R)(e_i, Y)\xi, \xi) g(e_i, \xi) = 0 \]

**Proof:** For a Trans-Sasakian manifold \( M^n \), we shall prove that

\[ g((\nabla_w R)(e_i, Y)\xi, \xi) = 0, \]

So that (5.3.6) follows. We Know that,

\[ g((\nabla_w R)(e_i, Y)\xi, \xi) = g(\nabla_w R(e_i, Y)\xi, \xi) - g(R(\nabla_w e_i, Y)\xi, \xi) - g(R(e_i, \nabla_w Y)\xi, \xi) - g(R(e_i, Y)\nabla_w \xi, \xi), \]

At \( p \in \chi(M) \). Since

\[ \nabla_w e_i = X^j \Gamma_{ji}^h e_h, \]

where \( \Gamma_{ji}^h \) are Riemann-Christoffel symbols and \( \{e_i\} \) are orthonormal basis,

the metric tensor \( g_{ji} = \delta_{ij} \) where \( \delta_{ij} \) is Kronecker delta and therefore

\[ \Gamma_{ji}^h = 0 \]

so that \( \nabla_w e_i = 0 \). Also from (2.8), we have

\[ g(R(e_i, \nabla_w Y)\xi, \xi) = g(\eta(e_i)\nabla_w Y - \eta(\nabla_w Y)e_i, \xi) \]

\[ = \eta(e_i)g(\nabla_w Y, \xi) - \eta(\nabla_w Y)g(e_i, \xi) \]

\[ = \eta(e_i)\eta(\nabla_w Y) - \eta(\nabla_w Y)\eta(e_i) \]

\[ = 0, \]

hence substituting (5.3.8) in (5.3.7), we get

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Using the skew symmetry of $R$, (5.3.9) becomes

$$
g((\nabla_w R)(e_i, Y)\xi, \xi) = g(\nabla_w (R(e_i, Y)\xi), \xi) - g(R(e_i, Y)\nabla_w \xi, \xi) \tag{5.3.10}
$$

For a Trans-Sasakian manifold $M^n$ we have,

$$
g(\nabla_w R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_w \xi) = 0, \tag{5.3.11}
$$

where $\{e_i\}, i = 1, 2, ... 2m + 1$ are the orthonormal basis of the tangent space at any point of the manifold, substituting (3.11) in (3.10) we get

$$
g((\nabla_w R)(e_i, Y)\xi, \xi) = 0 \tag{5.3.12}
$$

from which the lemma 5.3.1 follows. This completes the proof of Lemma 3.1.

Now the equation (5.3.5) reduces to

$$
(\nabla_w S)(Y, \xi) = -A(W)S(Y, \xi) - B(W)2ng(Y, \xi) - A(W)\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}[\eta(YY) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta)] - B(W)\eta(Y, \eta), \tag{5.3.13}
$$

For trans-Sasakian manifold, the L.H.S of (5.3.13) and using preliminary results as listed in (5.2.1)-(5.2.13), we have

$$
g((\nabla_w R)(e_i, Y)\xi, \xi) = 0 \tag{5.3.14}
$$
Further simplifying (5.3.14), we have

\[(\nabla_w S)(Y, \xi) = \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi) \]
\[= \nabla_w \{2n(\alpha^2 - \beta^2) - (\xi\beta)\}[\eta (Y) - ((\phi Y)\alpha) - (2n - 1)(Y\beta)] \]
\[= \{2n(\alpha^2 - \beta^2) - (\xi\beta)\}[\eta (\nabla_w Y) - ((\phi \nabla_w Y)\alpha) - (2n - 1)(\nabla_w Y\beta)] \]
\[= S(Y, -\alpha\phi W + \beta(W - \eta(W)\xi)) \]
\[= W[2n(\alpha^2 - \beta^2) - (\xi\beta)]\{\eta (Y) + [2n(\alpha^2 - \beta^2) - (\xi\beta)]\{\nabla_w \eta (Y) + \eta (\nabla_w Y)\} \]
\[-(\nabla_w \phi)Y\alpha - \phi(\nabla_w Y)\alpha - \phi YW(\alpha) - (2n - 1)(\nabla_w Y)\beta - (2n - 1)YW(\beta) \]
\[-\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}[\eta (\nabla_w Y)] + ((\phi \nabla_w Y)\alpha) + (2n - 1)(\nabla_w Y\beta) \]
\[+ \alpha S(Y, \phi W) - \beta S(Y, W) + \beta W + (W)(\nabla_w Y, \xi) \]

Substituting for \((\nabla_w \eta)(Y)\) and \((\nabla_w \phi)(Y)\) from (5.2.6) and (5.2.4) in (5.3.15), we get
(5.3.16)

\[(\nabla_{\varphi} S)(Y, \xi) = W[(2n(\alpha^2 - \beta^2) - (\xi \beta))(\eta (Y)) \]
\[+ [2n(\alpha^2 - \beta^2) - (\xi \beta)] \{- \alpha g(W, Y) + \beta g(W, \varphi Y) \}
\[- \alpha \{g(W, Y)\xi - \eta(Y)W\} + \beta \{g(W, Y)\xi - \eta(Y)\varphi W\}]\alpha
\[- \varphi YW(\alpha) - (2n - 1)YW(\beta) + \alpha S(Y, \varphi W) - \beta S(Y, W) \]
\[+ \beta \eta(W)\{2n(\alpha^2 - \beta^2) - (\xi \beta)\}(\eta (Y)) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta) \].

Now substituting this in (5.3.13) we get

(5.3.17)

\[W[(2n(\alpha^2 - \beta^2) - (\xi \beta))(\eta (Y)) \]
\[+ [2n(\alpha^2 - \beta^2) - (\xi \beta)] \{- \alpha g(W, Y) + \beta g(W, \varphi Y) \}
\[- \alpha \{g(W, Y)\xi - \eta(Y)W\} + \beta \{g(W, Y)\xi - \eta(Y)\varphi W\}]\alpha
\[- \varphi YW(\alpha) - (2n - 1)YW(\beta) + \alpha S(Y, \varphi W) - \beta S(Y, W) \]
\[+ \beta \eta(W)\{2n(\alpha^2 - \beta^2) - (\xi \beta)\}(\eta (Y)) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta) \],
\[= - A(W)\{2n(\alpha^2 - \beta^2) - (\xi \beta)\}(\eta (Y)) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta) \} - B(W)2n \eta(Y), \]

Multiplying (5.3.17) throughout by -1 and separating
\[\alpha S(Y, \varphi W) - \beta S(Y, W) \text{ from (5.3.17), we get} \]

\[\alpha S(Y, \varphi W) - \beta S(Y, W) = \{- W[(2n(\alpha^2 - \beta^2) - (\xi \beta))(\eta (Y)) \]
\[- [2n(\alpha^2 - \beta^2) - (\xi \beta)] \{- \alpha g(W, Y) + \beta g(W, \varphi Y) \}
\[+ \alpha \{g(W, Y)\xi - \eta(Y)W\} + \beta \{g(W, Y)\xi - \eta(Y)\varphi W\}]\alpha
\[+ \varphi YW(\alpha) + (2n - 1)YW(\beta) \]
\[- \beta \eta(W)\{2n(\alpha^2 - \beta^2) - (\xi \beta)\}(\eta (Y)) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta) \],
\[- A(W)\{2n(\alpha^2 - \beta^2) - (\xi \beta)\}(\eta (Y)) - ((\varphi Y)\alpha) - (2n - 1)(Y\beta) \} - B(W)2n \eta(Y), \]

This leads to the following result:

**Theorem 5.3.1.** In a Generalized $\varphi$-Recurrent Trans- Sasakian Manifold (M, g) the above relation (3.18) holds.
Corollary 5.3.1. [2] A generalized $\varphi$-recurrent Sasakian manifold $(M, g)$ is an Einstein manifold.

Proof. Substituting $\alpha = 1$, and $\beta = 0$ in (5.3.18), then

\begin{equation}
S(Y, \varphi W) = 2n\{A(W) + B(W)\}\eta(Y)
\end{equation}

Taking $Y = \xi$ in the above equation, using the fact that $S(\xi, \varphi(W)) = 0$, we get $A+B=0$ or $A=-B$. That is $A$ and $B$ are in the opposite direction. Equation (3.18) a now becomes

\begin{equation}
S(Y, \varphi W) = 2n\{A(W) + B(W)\}\eta(Y)
\end{equation}

which proves the corollary 5.3.1.

We can also have the following corollary,

Corollary 5.3.2. [2] In a generalized $\varphi$-recurrent Sasakian manifold $(M, g)$ 1-forms $A$ and $B$ are in the opposite directions.

Corollary 5.3.3. [1] A generalized $\varphi$-recurrent Kenmotsu manifold $(M, g)$ is an Einstein manifold.

Proof. Substituting $\alpha = 0$, and $\beta = 1$ in (5.3.18), then

\begin{equation}
-S(Y, W) = 2n\{A(W) - B(W)\}\eta(Y) + 2n\eta(Y) \eta(W)
\end{equation}

Taking $Y = \xi$ in the above equation, using the fact that $S(\xi, W) = 0$, we get $A-B=0$ or $A=B$. That is $A$ and $B$ are in the same direction. Equation (5.3.8) b now becomes

\begin{equation}
S(Y, W) = 2n\{A(W) - B(W)\}\eta(Y)
\end{equation}

which gives

\begin{equation}
S(Y, W) = 2n\{g(Y, W)
\end{equation}

so that manifold is Einstein, which proves the Corollary 5.3.3.

We can also have the following corollary,
Corollary 5.3.4. [1] In a generalized $\varphi$-recurrent Kenmotsu manifold $(M, g)$, the 1-forms $A$ and $B$ are in the same directions.

Now to find the relation between the associated 1-forms $A$ and $B$ for generalized $\varphi$-recurrent trans- Sasakian manifold, consider (5.3.18), substituting $Y = \xi$ in (5.3.18) we find

\[
\alpha S(\xi, \varphi W) - \beta S(\xi, W) = - W[ (2n(\alpha^2 - \beta^2) - (\xi\beta))(\eta(\xi)) \\
- [2n(\alpha^2 - \beta^2) - (\xi\beta)]{-\alpha g(\varphi W, \xi) + \beta g(\varphi W, \varphi \xi)} \\
+ [\alpha \{g(W, \xi)\eta(\xi) W\} + \beta \{g(\varphi W, \xi)\eta(\xi)\varphi W\}]\alpha \\
+ \varphi \xi W(\alpha) + (2n-1)\xi W(\beta) \\
- \beta \eta W(\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}) (\eta(\xi) - ((\varphi \xi)\alpha) - (2n-1)(\xi\beta)), \\
- A(W)\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}[\eta(\xi) - ((\varphi \xi)\alpha) - (2n-1)(\xi\beta)] \\
- B(W)2n \eta(\xi),
\]

Now using preliminary results of Section 2 in (5.3.19) we find

\[
\alpha S(\xi, \varphi W) - \beta S(\xi, W) = - W[ (2n(\alpha^2 - \beta^2) - (\xi\beta)] \\
+ [\alpha \{\eta(W)\xi - W\} + \beta \{\varphi W\}]\alpha \\
+ (2n-1)\xi W(\beta) - \beta \eta W(2n \{\alpha^2 - \beta^2 - (\xi\beta)\} \\
- A(W)2n \{\alpha^2 - \beta^2 - (\xi\beta)\} - B(W)2n
\]

Further simplifying (3.20), we get
This leads to the following result,

**Theorem 5.3.2.** In a Generalized $\varphi$-Recurrent Trans- Sasakian Manifold $(M, g)$ for the associated 1-forms $A$ and $B$, the relation (5.3.21) holds.

**Remark:** Corollaries 3.2 and 3.4 also follow from (5.3.21)

**Corollary 5.3.5.** In a generalized $\varphi$-Recurrent $\alpha$-Sasakian manifold $(M, g)$, following relation

$$2n\{\alpha^2 A(W) + B(W)\} = \alpha((\varphi^2 W)\alpha) - 2nW(\alpha^2) + \alpha\{\eta(W)(\xi\alpha) - W\alpha\}$$

holds.

**Remark.** Corollary 5.3.2 also follows from (5.3.21)

**Corollary 5.3.6.** In a generalized $\varphi$-Recurrent $\beta$-Kenmotsu manifold $(M, g)$, the following relation

$$\xi\beta - \beta\alpha$$

holds.
- $2n((\beta^2 + (\xi\beta))A(W) + B(W)2n = (2n-1)(W\beta) + W[2n(\beta^2 + (\xi\beta))]
+ (2n-1)\xi W(\beta)$

holds.

**Remark.** Corollary 5.3.4 also follows from (5.3.22).

### 5.4. On Generalized Concircular $\varphi$ -Recurrent Trans Sasakian Manifolds

**Definition 5.4.1.** A trans-Sasakian manifold $(M, g)$ of dimension $2n+1$ is called a Generalized Concircular $\varphi$-Recurrent if its Concircular curvature tensor $\overline{C}$ (Yano K. Kon. M 1984), satisfies the following condition for arbitrary vector fields $X, Y, Z$ and $W$

$$\varphi^2(\nabla_w \overline{C}(X, Y)Z) = A(W)\overline{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]$$

where $A(W)$ and $B(W)$ are the associated $1$-forms and $r$ is the scalar curvature of the manifold $(M, g)$ and $\overline{C}$ is the Concircular curvature tensor of type $(1,3)$ is given by

$$\overline{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y]$$

Let us consider a generalized Concircular $\varphi$-recurrent trans Sasakian manifold; then by (5.4.1), (5.4.2) reduces to

$$-\nabla_w \overline{C}(X, Y)Z + \eta(\nabla_w \overline{C}(X, Y)Z)\xi
= A(W)\overline{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]$$

from which it is easy to see that
(5.4.4)

\[-g(\nabla_w \overline{C}(X, Y)Z, U) + \eta(\nabla_w \overline{C}(X, Y)Z)g(\xi, U)\]

\[= A(W)g\{\overline{C}(X, Y)Z, U\} + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\]

Substituting for \(\overline{C}\) from (5.4.1) in (5.4.4) we get

(5.4.5)

\[-g(\nabla_w \{R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]\}, U)\]

\[+ \eta(\nabla_w \{R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]\})g(\xi, U)\]

\[= A(W)g\{R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]\}, U\} + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\]

Simplifying (5.4.5), we get

(5.4.6)

\[-g(\nabla_w \{R(X, Y)Z, U\} + \frac{W(r)}{2n(2n + 1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\]

\[+ [g(\nabla_w \{R(X, Y)Z, \xi\} - \frac{W(r)}{2n(2n + 1)}\{g(Y, Z)g(X, \xi) - g(X, Z)g(Y, \xi)\}]\eta(U)\]

\[= A(W)[g(R(X, Y)Z, U) - \frac{r}{2n(2n + 1)}\{g(Y, Z)g(X, U)\]

\[- g(X, Z)g(Y, U)\}]

\[+ B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\]

Let \(\{e_i\}, i = 1, 2, \ldots, 2n + 1\), be an orthonormal basis of the tangent space at any point of the manifold. Now putting \(Y = Z = e_i\) in (5.4.6) and taking the summation over \(i, 1 \leq i \leq 2n + 1\) we get
wherein we have used the fact that repeated indices imply the summation and

\[ g(\epsilon, \epsilon) = 2n + 1, g(X, \epsilon) g(\epsilon, U) = g(X, U), \]
\[ g(X, \epsilon) g(\epsilon, \xi) = g(X, \xi) = \eta(X) \]

After the simplification of (5.4.7), we get

(5.4.8)

\[
- g(\nabla w (R(X, \epsilon) e_i, U) + \frac{W(r)}{2n(2n + 1)} [g(\epsilon, \epsilon) g(X, U) - g(X, \epsilon) g(\epsilon, U)]
\]
\[
+ [g(\nabla w (R(X, \epsilon) e_i, \xi) - \frac{W(r)}{2n(2n + 1)} [g(\epsilon, \epsilon) g(X, \xi) - g(X, \epsilon) g(\epsilon, \xi)] \eta(U)
\]
\[
= A(W) [g(R(X, \epsilon) e_i, U) - \frac{r}{2n(2n + 1)} g(\epsilon, \epsilon) g(X, U)]
\]
\[
+ B(W) [g(\epsilon, \epsilon) g(X, U) - g(X, \epsilon) g(\epsilon, U)]
\]

Further simplifying (5.4.8), we get
Now replacing \( U \) by \( \xi \) and using (5.2.1) in (5.4.9) we get,

\[
-(\nabla_w S)(X, U) + \frac{W(r)}{(2n+1)}[g(X, U)] + [\nabla_w S](X, \xi) - \frac{W(r)}{(2n+1)} \{\eta(X)\} \eta(U)
\]

\[
= A(W)[S(X, U) - \frac{r}{(2n+1)}g(X, U)] + B(W)2ng(X, U)
\]

Now replacing \( U \) by \( \xi \) and using (5.2.1) in (5.4.9) we get,

\[
-(\nabla_w S)(X, \xi) + \frac{W(r)}{(2n+1)}[g(X, \xi)] + [\nabla_w S](X, \xi) - \frac{W(r)}{(2n+1)} \{\eta(X)\} \eta(\xi)
\]

\[
= A(W)[S(X, \xi) - \frac{r}{(2n+1)}g(X, \xi)] + B(W)2ng(X, \xi)
\]

Canceling the same positive and negative terms from L.H.S of and simplifying, we get

\[
2n \eta(X) B(W) = A(W) \left[ \frac{r}{(2n+1)} \eta(X) - \{2n(\alpha^2 - \beta^2) - (\xi \beta)\}\{\eta(X)\} \right]
\]

\[
- ((\varphi X)\alpha) - (2n-1)(X\beta)
\]

where \( r \) is the scalar curvature of \( M \).

Take \( X = \xi \), (5.4.11) reduces to,

\[
2n B(W) = A(W) \left[ \frac{r}{2n+1} - 2n((\alpha^2 - \beta^2 - \xi \beta)) \right]
\]

This leads to the following result.

**Theorem 5.4.1.** In a Generalized Concircular \( \varphi \)-Recurrent Trans-Sasakian Manifold
(M, g), for the associated 1-forms A and B, the relation (4.12) holds.

**Corollary 5. 4.1** [2]. In a Generalized Concircular $\varphi$-Recurrent Sasakian Manifold $(M, g)$ the associated 1-forms A and B are related by,

\[(5.4.12)\quad B = \frac{r}{2n(2n+1)} - 1 \ A\]

**Proof.** Substituting $\alpha = 1$ and $\beta = 0$ in (5.4.12) the corollary 5.4.1 follows.

**Corollary 5.4.2** [1]. In a Generalized Concircular $\varphi$-recurrent Kenmotsu Manifold $(M, g)$ the associated 1-forms A and B are related by,

\[(5.4.13)\quad B = \frac{r}{2n(2n+1)} + 1 \ A\]

**Proof.** Substituting $\alpha = 0$ and $\beta = 1$ in (5.4.12), (5.4.14) of the corollary 5.4.2 follows

\[\xi\text{-Sectional Curvature}\]

The $\xi$-sectional curvature $K(\xi, X)$ of a trans-Sasakian manifold for a unit vector field $X$ orthogonal to $\xi$ is given by $K(\xi, X) = g(R(\xi, X)\xi, X)$ hence (5.2.9), yields

\[(5.4.14)\quad K(\xi, X) = -\{\alpha^2 - (\xi \beta) - \beta^2\}\]

If $\{\alpha^2 - (\xi \beta) - \beta^2\} = 0$, then the manifold is of vanishing $\xi$-sectional curvature, for instance for a Cosymplectic manifold $M$, $\xi$-sectional curvature vanishes.

**Corollary 5.4.3.** A Generalized Concircular $\varphi$-Recurrent Trans-Sasakian Manifold
(M, g) is such that $r = -2n(2n + 1)K(\xi, X)$, then M is Concircular $\varphi$-Recurrent, where $K(\xi, X)$ is the non vanishing $\xi$-Sectional curvature of M.

**Proof.** If the condition of the corollary is satisfied, then from (5.4.12) and (5.4.14) we have

$$2n B(W) = A(W)[\frac{r}{2n + 1} + 2nK(\xi, X)]$$

From (5.4.15) $B = 0$, Hence proof follows from (5.4.1).

5.5. **On Generalized Einstein Projective $\varphi$-Recurrent Trans Sasakian Manifolds.**

In this section, we study some of the properties the generalized Einstein $\varphi$-Recurrent Trans Sasakian Manifolds by introducing the definition of generalized projective $\varphi$-recurrent trans-Sasakian manifolds.

**Definition 5.5.1.** The projective curvature tensor $P$ of type $(1, 3)$, [12] on a Riemannian manifold $M$ of dimension $n$ is defined by

$$P(X, Y) Z = R(X, Y) Z - (1/n - 1) [S(Y, Z) X - S(X, Z) Y]$$

For any vector fields $X$, $Y$ and $Z$ and $S$ is the Ricci curvature tensor on $M$.

By the above definition, for a trans-Sasakian manifold $M$ of dimension $2n+1$, the projective curvature Tensor $P$ of type $(1, 3)$ is given by

$$P(X, Y) Z = R(X, Y) Z - (1/2n) [S(Y, Z) X - S(X, Z) Y]$$

If the $M$ is Einstein, then

$$R(X, Y) = \frac{r}{2n + 1} g(X, Y)$$
Now substituting (5.5.2) in (5.5.1), we get

(5.5.3) \[ P(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y] \]

From definition 5.5.1, we can have

**Definition 5.5.2.** A trans-Sasakian manifold \((M, g)\) is called Generalized projective \(\varphi\)-recurrent if its projective curvature tensor, satisfies the condition,

(5.5.4) \[ \varphi^{\gamma}(\nabla_{\nu}P(X, Y)Z) = A(W)P(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y] \]

where \(A(W)\) and \(B(W)\) are the associated 1-forms ,B is non zero.

Now comparing, (5.4.2) and (5.5.3), we have

Now comparing, (5.4.2) and (5.5.3), we have

\[ \overline{C}(X, Y)Z = P(X, Y)Z \]

Hence we can state

**Theorem 5.5.1.** In a Generalized Einstein projective \(\varphi\)-Recurrent Trans-Sasakian Manifold \((M, g)\), for the associated 1-forms \(A\) and \(B\), the relation (5.4.12) holds.

**Corollary 5.5.1 [2].** In a Generalized Einstein Projective \(\varphi\)-Recurrent Sasakian Manifold \((M, g)\), the associated 1-forms \(A\) and \(B\) are related by

\[ B = \left[ \frac{r}{2n(2n+1)} - 1 \right] A \]

**Corollary 5.5.2 [1].** In a Generalized Einstein projective \(\varphi\)-recurrent Kenmotsu Manifold \((M, g)\), the associated 1-forms \(A\) and \(B\) are related by,

\[ B = \left[ \frac{r}{2n(2n+1)} + 1 \right] A \]
5.6. Curvature property of generalized $\varphi$ recurrent Trans Sasakian Manifolds

In this section we consider a 3-dimensional trans-Sasakian manifold $M$ and study its curvature properties.

For 3-dimensional generalized $\varphi$-recurrent trans-Sasakian manifold, its curvature tensor field of type $(1, 3)$ [13] is given by

\[(5.6.1)\]

\[
R(X,Y)Z = \left(\frac{r}{2} + 2\xi \beta - 2(\alpha^2 - \beta^2)(g(Y,Z)X - g(X,Z)Y) - g(Y,Z)\left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right)\eta(X)\xi - \eta(X)(\varphi \text{grad} \alpha - \text{grad} \beta) + (X\beta + (\varphi X)\alpha)\xi\right)
+
\left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right)\eta(Y)\xi - \eta(Y)(\varphi \text{grad} \alpha - \text{grad} \beta) + (Y\beta + (\varphi Y)\alpha)\xi
\]

Where $X, Y, Z \in \chi(M)$

Differentiating (6.1) (please see for details [13], we get,
Suppose that $\alpha$ and $\beta$ are constants and $X$, $Y$, $Z$, $W$ are orthogonal to $\xi$, then (5.6.2), reduces to

$$(\nabla_w R)(X, Y)Z = \left[\frac{dr(W)}{2} + 2(\nabla_w (\xi\beta) - 4(\alpha(W) - d\beta(W))[g(Y.Z)X - g(X, Z)Y]\right] - g(Y, Z)\left[\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha(W) - d\beta(W))\right] \eta(X)\xi$$

$$+ \left\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right\} \{\nabla_w \eta)(X)\xi + \eta(X)(\nabla_w \xi)\right\}$$

$$- (\nabla_w \eta)(Y)(\phi(\text{grad}\alpha) - \text{grad}\beta) - \eta(Y)(\nabla_w (\phi(\text{grad}\alpha) - \text{grad}\beta))$$

$$+ (\nabla_w (\xi\beta + (\phi\alpha))\xi + (\xi\beta + (\phi\alpha))\nabla_w \xi)\right\}$$

$$+ g(X, Z)\left[\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha(W) - d\beta(W))\right] \eta(Y)\xi$$

$$+ \left\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right\} \{\nabla_w \eta)(Y)\xi + \eta(Y)(\nabla_w \xi)\right\}$$

$$- (\nabla_w \eta)(Y)(\phi(\text{grad}\alpha) - \text{grad}\beta) - \eta(Y)(\nabla_w (\phi(\text{grad}\alpha) - \text{grad}\beta))$$

$$+ (\nabla_w (\xi\beta + (\phi\alpha))\xi + (\xi\beta + (\phi\alpha))\nabla_w \xi)\right\}$$

$$- [(\nabla_w (Z\beta + (\phi\alpha))\eta(Y) + (Z\beta + (\phi\alpha))\nabla_w \eta)Y$$

$$+ ((\nabla_w (Y\beta + (\phi\alpha))\eta(Z) + (Y\beta + (\phi\alpha))\nabla_w \eta)Z$$

$$+ (\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha(W) - d\beta(W))\right] \eta(Y)\eta(Z)$$

$$+ \left\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right\} \{\nabla_w \eta)(Y)\eta(Z) + \eta(Y)(\nabla_w \eta)Z\} X$$

$$+ [(\nabla_w (Z\beta + (\phi\alpha))\eta(X) + (Z\beta + (\phi\alpha))\nabla_w \eta)X$$

$$+ ((\nabla_w (X\beta + (\phi\alpha))\eta(Z) + (X\beta + (\phi\alpha))\nabla_w \eta)Z$$

$$+ (\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha(W) - d\beta(W))\right] \eta(X)\eta(Z)$$

$$+ \left\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right\} \{\nabla_w \eta)(X)\eta(Z) + \eta(X)(\nabla_w \eta)Z\} Y$$
Applying $\varphi^2$ on sides of (5.6.3), then using $\varphi \xi = 0$ and (5.6.3), we get

(5.6.4) $\varphi^2 (\nabla_w R)(X, Y)Z = \{-\frac{dr(W)}{2}\}[g(Y, Z)X - g(X, Z)Y]$

By definition of Generalized $\varphi$ -Recurrent Trans Sasakian Manifolds

$\varphi^2 ((\nabla_w R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]$,

i.e., by (3.1), we get

(5.6.4a)

$A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]$

(5.6.5) $A(W)R(X, Y)Z = \{B(W) - \frac{dr(W)}{2}\}[g(Y, Z)X - g(X, Z)Y]$

Putting $W = \{e_i\}$, where $\{e_i\}, i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$ we obtain

(5.6.6)

$R(X, Y)Z = \lambda (g(Y, Z)X - g(X, Z)Y)$

Where

$\lambda = \frac{B(e_i)}{A(e_i)} - \frac{dr(e_i)}{2A(e_i)}$
is a scalar and A is assumed to be a non zero 1-form. Then by Schurs
Theorem $\lambda$ will be constant on the manifold. Therefore $(M^3, g)$ is of
constant curvature $\lambda$, thus we can state

**Theorem 5.6.1.** A 3-dimensional connected generalized $\phi$ recurrent
Trans Sasakian Manifold $(M^3, g)$ is of constant curvature provided
$\alpha \beta$ are constants.

**Corollary 5.6.1[2].** A 3-dimensional generalized $\phi$ recurrent
Sasakian Manifold $(M^3, g)$ is of constant curvature

**Proof.** Corollary follows by taking $\alpha = 1, \beta = 0$ in Theorem 5.6.1

If $B = 0$ from (5.3.1) we have

$$\varphi^2 ((V_w R)(X, Y)Z) = A(W)R(X, Y)Z,$$

that is $M$ is a 3-dim. $\phi$ recurrent trans-Sasakian manifold then from
(5.6.6), we have

$$R(X, Y)Z = -\frac{dr(e_i)}{2A(e_i)}(g(Y,Z)X - g(X,Z)Y).$$

Thus we can state.

**Corollary 5.6.2.** A 3-dimensional $\phi$ recurrent Trans Sasakian
Manifold $(M^3, g)$ is of constant curvature **provided** $\alpha \beta$ are constants.

**Corollary 5.6.3[[13]]** A 3-dimensional connected trans-Sasakian
manifold of type $(\alpha, \beta)$ is locally $\phi$-symmetric if and only if the scalar
curvature is constant provide $\alpha$ and $\beta$ are constants

**Proof:** If $A = 0$ and $B = 0$ then (5.3.1)a holds so that from (5.6.4), the
proof follows.
References:


