Chapter 2

On Some Nonlinear Integral Inequalities in Two Variables

2.1 Introduction

For more than a century, the study of various types of integral and integro-differential inequalities has been focus of great attention by many researchers. Many mathematicians have investigated a number of new integral and integro-differential inequalities in two independent variables which can be used in the analysis of various problems in the theory of partial differential and integral equations. One of the most useful tools in the development of the qualitative theory of partial differential and integral equations is integral inequalities involving functions of many independent variables, which provide explicit bounds on the unknown functions.

During the last few years, many such new inequalities have been discovered, which are motivated by certain applications, see [1, 13, 21, 30, 46, 57, 58] and references are therein. The importance of such inequalities lies in its successful utilization to the situation for which the other available inequalities do not apply directly. These

\(^{\text{A paper based on the text of this Chapter has been submitted for publication.}}\)
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inequalities have been frequently used to obtain global existence, uniqueness, stability, boundedness and other properties of the solutions for wide class of nonlinear partial differential equations. In view of the important role played by these inequalities, it is desirable to find some new inequalities which would be equally important in certain new applications. Also in the study of qualitative analysis of some class of partial differential equations and integro-differential equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a desired bound.

In this chapter, we present some two dimensional nonlinear integral and integro-differential inequalities, which can be used as handy tools in the study of various problems in the theory of partial differential equations, integral equations and integro-differential equations.

Before giving our main results, we introduce the following results which are useful in the procedures of our proof.

**Lemma 2.1.1** (Fangcui Jiang and Fanwei Meng [24]). *Assume that* $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$, *then*

$$a^q \leq \frac{a^{q-p}}{p} k^{q-p} + \frac{p - q}{p} k^q, \text{ for any } k > 0.$$

(2.1.1)

**Theorem 2.1.2** (Wendroff’s Inequality [50]). *Let $u(x, y), c(x, y)$ be nonnegative continuous functions defined for* $x, y \in \mathbb{R}_+$ *and $k \geq 0$ be a constant. If*

$$u(x, y) \leq k + \int_0^x \int_0^y c(s, t)u(s, t)dt\,ds, \text{ for } x, y \in \mathbb{R}_+,$$

(2.1.2)

*then*

$$u(x, y) \leq k \exp \left( \int_0^x \int_0^y c(s, t)dt\,ds \right), \text{ for } x, y \in \mathbb{R}_+.$$

(2.1.3)
2.2 Nonlinear integral inequalities in two variables

In this section, we state and prove some new integral inequalities in two variables, which can be used in the analysis of various problems in the theory of nonlinear partial differential and integral equations.

**Theorem 2.2.1.** Assume that \( u(x,y) \), \( f(x,y) \) and \( g(x,y) \) are nonnegative continuous functions defined for \( x, y \in \mathbb{R}_+ \). If \( c \geq 1 \) is a constant and \( u(x,y) \) satisfies the following integral inequality

\[
u^p(x,y) \leq c + \int_0^x \int_0^y f(s,t)u(s,t)dt\,ds + \int_0^x \int_0^y g(s,t)u(s,t)dt\,ds, \tag{2.2.1}\]

for all \( x, y \in \mathbb{R}_+ \), then

\[
u(x,y) \leq \left[ c \exp \left( (m_1 + m_2) \int_0^x \int_0^y [f(s,t) + g(s,t)] \, dt\,ds \right) \right]^{1/p}, \tag{2.2.2}\]

for all \( x, y \in \mathbb{R}_+ \), where \( p \geq 1 \), \( k > 0 \), \( m_1 = \frac{1}{p} k^{1-p} \) and \( m_2 = \frac{p-1}{p} k^{1/p} \).

**Proof.** Let \( z(x,y) \) be equal to the right side of (2.2.1), i.e.,

\[
z(x,y) = c + \int_0^x \int_0^y f(s,t)u(s,t)dt\,ds + \int_0^x \int_0^y g(s,t)u(s,t)dt\,ds. \tag{2.2.3}\]

Then \( u^p(x,y) \leq z(x,y) \), \( u(x,y) \leq z^{1/p}(x,y) \), \( z(x,y) \geq 1 \) and \( z(0,y) = z(x,0) = z(0,0) = c \) for all \( x, y \in \mathbb{R}_+ \).

Differentiating equation (2.2.3) with respective \( x \) and \( y \) respectively, we have

\[
z_{xy}(x,y) = f(x,y)u(x,y) + g(x,y)u(x,y)
\leq f(x,y)z^{1/p}(x,y) + g(x,y)z^{1/p}(x,y)
= [f(x,y) + g(x,y)] z^{1/p}(x,y). \tag{2.2.4}\]

From (2.2.4) and Lemma 1.1.4, we have

\[
z_{xy}(x,y) \leq [f(x,y) + g(x,y)] (m_1 z(x,y) + m_2).\]
i.e.,
\[
\frac{z_{xy}(x,y)}{z(x,y)} \leq (m_1 + m_2) [f(x,y) + g(x,y)].
\]
Since \(z(x,y) \geq 1\), \(z_x(x,y) \geq 0\), \(z_y(x,y) \geq 0\), we have
\[
\frac{z_{xy}(x,y)}{z(x,y)} - \frac{z_x(x,y)z_y(x,y)}{z^2(x,y)} \leq (m_1 + m_2) [f(x,y) + g(x,y)],
\]
or, equivalently,
\[
\frac{\partial}{\partial y} \left( \frac{z_x(x,y)}{z(x,y)} \right) \leq (m_1 + m_2) [f(x,y) + g(x,y)]. \tag{2.2.5}
\]
Keeping \(x\) fix in (2.2.5), setting \(y = t\) and integrate with respect to \(t\) from 0 to \(y\), we have
\[
\frac{z_x(x,y)}{z(x,y)} \leq (m_1 + m_2) \int_0^y [f(x,t) + g(x,t)] \, dt. \tag{2.2.6}
\]
Keeping \(y\) fix in (2.2.6), setting \(x = s\) and integrate with respect to \(s\) from 0 to \(x\), we have
\[
\log z(x,y) - \log z(x,0) \leq (m_1 + m_2) \int_0^x \int_0^y \log(z(s,t)) \, ds \, dt,
\]
\[
\log z(x,y) - \log c \leq (m_1 + m_2) \int_0^x \int_0^y [f(s,t) + g(s,t)] \, dt \, ds,
\]
\[
z(x,y) \leq c \exp \left( (m_1 + m_2) \int_0^x \int_0^y [f(s,t) + g(s,t)] \, dt \, ds \right). \tag{2.2.7}
\]
The required inequality (2.2.2) follows, from \(u^p(x,y) \leq z(x,y)\) and (2.2.7). This proves the Theorem.

\[\square\]

**Corollary 2.2.2.** Assume that \(u(x,y), f(x,y), g(x,y), p, k, m_1, m_2\) are defined as in Theorem 2.2.1. If \(c(x,y) \geq 1\) is a nondecreasing continuous function defined for each \(x, y \in \mathbb{R}_+\) and satisfies the integral inequality
\[
u^p(x,y) \leq c(x,y) + \int_0^x \int_0^y f(s,t)u(s,t) \, dt \, ds + \int_0^x \int_0^y g(s,t)u(s,t) \, dt \, ds, \tag{2.2.8}
\]
for all $x, y \in \mathbb{R}_+$, then
\[ u(x, y) \leq c(x, y) \left[ \exp \left( (m_1 + m_2) \int_0^x \int_0^y [f(s, t) + g(s, t)] dtds \right) \right]^{\frac{1}{p}}, \tag{2.2.9} \]

for all $x, y \in \mathbb{R}_+$.

**Proof.** Since $c(x, y) \geq 1$ is a nondecreasing continuous function for each $x, y \in \mathbb{R}_+$ and $p \geq 1$ from an inequality (2.2.8), we obtain
\[
\frac{u^p(x, y)}{c^p(x, y)} \leq 1 + \frac{1}{c(x, y)} \int_0^x \int_0^y f(s, t) u(s, t) dtds + \frac{1}{c(x, y)} \int_0^x \int_0^y g(s, t) u(s, t) dtds
\leq 1 + \int_0^x \int_0^y f(s, t) \frac{u(s, t)}{c(s, t)} dtds + \int_0^x \int_0^y g(s, t) \frac{u(s, t)}{c(s, t)} dtds. \tag{2.2.10}
\]

An application of Theorem 2.2.1 to an inequality (2.2.10), we get (2.2.9). This completes the proof. \qed

**Theorem 2.2.3.** Assume that $u(x, y), f(x, y), g(x, y), c$ are defined as in Theorem 2.2.1. If $u(x, y)$ satisfies the following integral inequality
\[ u^p(x, y) \leq c + \int_0^x \int_0^y f(s, t) u^p(s, t) dtds + \int_0^x \int_0^y g(s, t) u^q(s, t) dtds, \tag{2.2.11} \]
for all $x, y \in \mathbb{R}_+$, where $p \geq q \geq 0, p \neq 0$ are constants, then
\[ u(x, y) \leq \left[ c \exp \left( \int_0^x \int_0^y [f(s, t) + g(s, t) (n_1 + n_2)] dtds \right) \right]^{\frac{1}{p}}, \tag{2.2.12} \]
for all $x, y \in \mathbb{R}_+$, where $k > 0$, $n_1 = \frac{q}{p} k^{\frac{q-p}{p}}$ and $n_2 = \frac{p-q}{p} k^{\frac{q}{p}}$.

**Proof.** Define a function $z(x, y)$ by
\[ z(x, y) = c + \int_0^x \int_0^y f(s, t) u^p(s, t) dtds + \int_0^x \int_0^y g(s, t) u^q(s, t) dtds, \tag{2.2.13} \]
then $u^p(x, y) \leq z(x, y)$, $u(x, y) \leq \frac{1}{p} z^p(x, y)$, $z(x, y) \geq 1$ and $z(0, y) = z(x, 0) = z(0, 0) = c$, for all $x, y \in \mathbb{R}_+$.

Differentiating equation (2.2.13) with respective $x$ and $y$ respectively, yields
\[ z_{xy}(x, y) = f(x, y) u^p(x, y) + g(x, y) u^q(x, y) \]
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\[ z_{xy}(x, y) \leq f(x, y)z(x, y) + g(x, y)z^q(x, y). \quad (2.2.14) \]

From an inequality (2.2.14) and Lemma (2.1.1), we have

\[ z_{xy}(x, y) \leq f(x, y)z(x, y) + g(x, y) [n_1z(x, y) + n_2], \]
\[ \frac{z_{xy}(x, y)}{z(x, y)} \leq f(x, y) + g(x, y) [n_1 + n_2]. \]

Since \( z(x, y) \geq 1, \ z_x(x, y) \geq 0, \ z_y(x, y) \geq 0 \), we have

\[ \frac{z_{xy}(x, y)}{z(x, y)} - \frac{z_x(x, y)z_y(x, y)}{z^2(x, y)} \leq f(x, y) + g(x, y) [n_1 + n_2], \]
\[ \frac{\partial}{\partial x} \left( \frac{z_x(x, y)}{z(x, y)} \right) \leq f(x, y) + g(x, y) [n_1 + n_2]. \quad (2.2.15) \]

Keeping \( y \) fix in (2.2.15), setting \( x = s \); integrating with respect to \( s \) from \( 0 \) to \( x \), we have

\[ \frac{z_y(x, y)}{z(x, y)} \leq \int_0^x [f(s, y) + g(s, y) (n_1 + n_2)] \, ds. \quad (2.2.16) \]

Keeping \( x \) fix in (2.2.16), setting \( y = t \); integrating with respect to \( t \) from \( 0 \) to \( y \), we have

\[ \log z(x, y) - \log c \leq \int_0^x \int_0^y ([f(s, t) + g(s, t) (n_1 + n_2)] \, dt \, ds \]
\[ z(x, y) \leq c \exp \left[ \int_0^x \int_0^y [f(s, t) + g(s, t) (n_1 + n_2)] \, dt \, ds \right]. \quad (2.2.17) \]

Using \( u^p(x, y) \leq z(x, y) \) in (2.2.17), we get the desired inequality (2.2.12). The proof is complete.

Remark 2.2.4. In the special case when \( p = q = 1 \) and \( g \equiv 0 \) the Theorem 2.2.3 reduces to Theorem 2.1.2 which is one of the most useful inequality in the development of theory of differential and integral equations due to Wendroff.
Corollary 2.2.5. Assume that $u(x, y), f(x, y), g(x, y), p, q, n_1, n_2$ are defined as in Theorem 2.2.3. If $c(x, y) \geq 1$ is a nondecreasing continuous function defined for each $x, y \in \mathbb{R}_+$ and $u(x, y)$ satisfies the following integral inequality

$$u^p(x, y) \leq c(x, y) + \int_0^x \int_0^y f(s, t)u^p(s, t)dsdt + \int_0^x \int_0^y g(s, t)u^q(s, t)dsdt,$$  \hspace{1cm} (2.2.18)

for all $x, y \in \mathbb{R}_+$, then

$$u(x, y) \leq c(x, y) \left[ \exp \left( \int_0^x \int_0^y [f(s, t) + g(s, t)(n_1 + n_2)] dsdt \right) \right]^{\frac{1}{p}},$$ \hspace{1cm} (2.2.19)

for all $x, y \in \mathbb{R}_+$.

Proof. Since $c(x, y) \geq 1$ is a nondecreasing continuous function for all $x, y \in \mathbb{R}_+$ with $0 \leq s \leq x, 0 \leq t \leq y$, we have

$$\frac{u^p(x, y)}{c^p(x, y)} \leq 1 + \frac{1}{c^p(x, y)} \int_0^x \int_0^y f(s, t)u^p(s, t)dsdt + \frac{1}{c^p(x, y)} \int_0^x \int_0^y g(s, t)u^q(s, t)dsdt$$

$$\leq 1 + \int_0^x \int_0^y f(s, t) \frac{u^p(s, t)}{c^p(s, t)} dsdt + \int_0^x \int_0^y g(s, t) \frac{u^q(s, t)}{c^q(s, t)} dsdt,$$

$$\left( \frac{u(x, y)}{c(x, y)} \right)^p \leq 1 + \int_0^x \int_0^y f(s, t) \left( \frac{u(s, t)}{c(s, t)} \right)^p dsdt + \int_0^x \int_0^y g(s, t) \left( \frac{u(s, t)}{c(s, t)} \right)^q dsdt.$$ \hspace{1cm} (2.2.20)

Now an application of Theorem 2.2.3 to an inequality (2.2.20), we have

$$\frac{u^p(x, y)}{c^p(x, y)} \leq \exp \left[ \int_0^x \int_0^y [f(s, t) + g(s, t)(n_1 + n_2)] dsdt \right].$$

i.e.,

$$u(x, y) \leq c(x, y) \left[ \exp \left( \int_0^x \int_0^y [f(s, t) + g(s, t)(n_1 + n_2)] dsdt \right) \right]^{\frac{1}{p}}.$$

The proof is complete. \hfill \square
2.3 Nonlinear integro-differential inequalities in two variables

This section deals with integro-differential inequalities in two variable which can be used in the study of qualitative properties of the solutions of certain partial differential equations.

**Theorem 2.3.1.** Assume that \( u(x, y), u_{xy}(x, y), f(x, y), g(x, y) \) are nonnegative continuous functions defined for \( x, y \in \mathbb{R}_+ \) and \( u(x, 0) = u(0, y) = 0 \). If \( c \geq 1 \) is a constant and

\[
 u_{xy}(x, y) \leq c + \int_0^x \int_0^y f(s, t)u_{st}(s, t)dt\,ds + \int_0^x \int_0^y g(s, t)u(s, t)dt\,ds, \tag{2.3.1}
\]

for all \( x, y \in \mathbb{R}_+ \), then

\[
 u(x, y) \leq \int_0^x \int_0^y \left[ c\exp \left( (m_1 + m_2) \int_0^\tau \int_0^\sigma [f(\sigma, \tau) + \sigma \tau g(\sigma, \tau)]d\tau d\sigma \right) \right]^{\frac{1}{p}} dt\,ds, \tag{2.3.2}
\]

for all \( x, y \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are as same defined in Theorem 2.2.1.

**Proof.** Define a function \( z(x, y) \) by right hand side of (2.3.1), we have

\[
 u_{xy}(x, y) \leq c + \int_0^x \int_0^y f(s, t)u_{st}(s, t)dt\,ds + \int_0^x \int_0^y g(s, t)u(s, t)dt\,ds, \tag{2.3.3}
\]

and hence

\[
 u(x, y) \leq \int_0^x \int_0^y \left[ c\exp \left( (m_1 + m_2) \int_0^\tau \int_0^\sigma [f(\sigma, \tau) + \sigma \tau g(\sigma, \tau)]d\tau d\sigma \right) \right]^{\frac{1}{p}} dt\,ds. \tag{2.3.4}
\]

Also, we have

\[
 z(0, y) = z(x, 0) = z(0, 0) = c, \quad z(x, y) \geq 1. \tag{2.3.5}
\]

On differentiating \( z(x, y) \) with respective \( x \) and \( y \) respectively, yields

\[
 z_{xy}(x, y) = f(x, y)u_{xy}(x, y) + g(x, y)u(x, y). \tag{2.3.6}
\]
Substituting (2.3.3) in (2.3.6), we obtain
\[ z_{xy}(x, y) \leq f(x, y) z_{x}^{1}(x, y) + g(x, y)u(x, y). \] (2.3.7)

From Lemma 1.1.4 and an inequality (2.3.7), yields
\[ z_{xy}(x, y) \leq f(x, y) (m_{1}z(x, y) + m_{2}) + g(x, y)u(x, y). \] (2.3.8)

Since \( z(x, y) \) is a nondecreasing and \( z(x, y) \geq 1 \), from inequality (2.3.4) and Lemma 1.1.4 we observe that
\[ u(x, y) \leq xy (m_{1} + m_{2}) z(x, y). \] (2.3.9)

Substituting (2.3.9) in (2.3.8), we have
\[ z_{xy}(x, y) \leq f(x, y) (m_{1}z(x, y) + m_{2}) + (m_{1} + m_{2}) xyg(x, y)z(x, y) \]
\[ \leq (m_{1} + m_{2}) [f(x, y) + xyg(x, y)] z(x, y). \]

i.e.,
\[ \frac{z_{xy}(x, y)}{z(x, y)} \leq (m_{1} + m_{2}) [f(x, y) + xyg(x, y)]. \] (2.3.10)

Fix \( y \) in (2.3.10), set \( x = s \) and integrate with respect to \( s \) from 0 to \( x \), we have
\[ \frac{z_{y}(x, y)}{z(x, y)} \leq (m_{1} + m_{2}) \int_{0}^{x} [f(s, y) + syg(s, y)] ds. \] (2.3.11)

Similarly, fix \( x \) in (2.3.11), set \( y = t \) and integrate with respect to \( t \) from 0 to \( y \), we have
\[ z(x, y) \leq c \exp \left( (m_{1} + m_{2}) \int_{0}^{x} \int_{0}^{y} [f(s, t) + stg(s, t)] dtds \right). \] (2.3.12)

By substituting (2.3.3) in (2.3.12) and then integrating, we obtain the desired inequality (2.3.2). This completes the proof.
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**Theorem 2.3.2.** Assume that \( u(x, y), u_{xy}(x, y), f(x, y), g(x, y), c \) are defined as in Theorem 2.3.1. If

\[
u_{xy}^p(x, y) \leq c + \int_0^x \int_0^y f(s, t)u_{st}^q(s, t)dtds + \int_0^x \int_0^y g(s, t)u_{st}(s, t)dtds,
\]

for all \( x, y \in \mathbb{R}_+ \), where \( p \geq q \geq 1 \), then

\[
u(x, y) \leq \int_0^x \int_0^y \left[ c \exp \left( \int_0^s \int_0^t \left[ (n_1 + n_2) f(\sigma, \tau) + (m_1 + m_2) g(\sigma, \tau) \right] d\tau d\sigma \right) \right]^{\frac{1}{p}} dtds
\]

for all \( x, y \in \mathbb{R}_+ \), where \( p, q, n_1, n_2 \) are as same defined in Theorem 2.2.3 and \( m_1, m_2 \) are as same defined in Theorem 2.2.1.

**Proof.** By defining \( z(x, y) \) by right hand side of (2.3.13) and using similar argument as used in above Theorem, we obtain

\[
u_{xy}^p(x, y) \leq z(x, y),
\]

\[
u_{xy}(x, y) \leq (m_1 + m_2)z(x, y)
\]

and

\[
z_{xy}(x, y) = f(x, y)u_{xy}^q(x, y) + g(x, y)u_{xy}(x, y).
\]

Substituting (2.3.15) and (2.3.16) in (2.3.17), we have

\[
z_{xy}(x, y) \leq f(x, y)z_{xy}^q(x, y) + (m_1 + m_2)g(x, y)z(x, y).
\]

From Lemma 2.1.1 and (2.3.18), we observe that

\[
z_{xy}(x, y) \leq [(n_1 + n_2)f(x, y) + (m_1 + m_2)g(x, y)]z(x, y),
\]

equivalently,

\[
\frac{z_{xy}(x, y)}{z(x, y)} \leq [(n_1 + n_2)f(x, y) + (m_1 + m_2)g(x, y)].
\]
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Fix \( x \) in (2.3.19), set \( y = t \) and integrate with respect \( t \) from 0 to \( y \), to obtain the estimate

\[
\frac{z_x(x, y)}{z(x, y)} \leq \int_0^y [(n_1 + n_2)f(x, t) + (m_1 + m_2)g(x, t)] \, dt. \tag{2.3.20}
\]

Similarly, fix \( y \) in (2.3.20), set \( x = s \) and integrate with respect \( s \) from 0 to \( x \), to obtain the estimate

\[
z(x, y) \leq c \exp \left( \int_0^x \int_0^y [(n_1 + n_2)f(s, t) + (m_1 + m_2)g(s, t)] \, dt \, ds \right). \tag{2.3.21}
\]

From (2.3.15) and (2.3.21), we obtain the desired bound for \( u(x, y) \) given in (2.3.14). The proof is complete. \( \square \)

**Theorem 2.3.3.** Assume that \( u(x, y), u_{xy}(x, y), f(x, y), g(x, y), c \) are defined as Theorem 2.3.1. If

\[
u_{xy}^p(x, y) \leq c + \int_0^x \int_0^y f(s, t)u_{st}^p(s, t) \, dt \, ds + \int_0^x \int_0^y g(s, t)u^q(s, t) \, dt \, ds, \tag{2.3.22}
\]

for all \( x, y \in \mathbb{R}_+ \), then

\[
u(x, y) \leq \int_0^x \int_0^y \left[ c \exp \left( \int_0^x \int_0^t [f(\sigma, \tau) + g(\sigma, \tau) [n_1 + n_2](\sigma \tau)^q] \, d\tau \, d\sigma \right) \right]^{\frac{1}{p}} \, dt \, ds, \tag{2.3.23}
\]

for all \( x, y \in \mathbb{R}_+ \), where \( p, q, n_1, n_2 \) are as same defined in Theorem 2.2.3.

**Proof.** Define right hand side of (2.3.22) by \( z(x, y) \), we observe that

\[
u_{xy}^p(x, y) \leq z(x, y), \tag{2.3.24}
\]

\[
u(x, y) \leq \int_0^x \int_0^y z^\frac{1}{p}(s_1,t_1) \, ds_1 \, dt_1, \tag{2.3.25}
\]

\[
z(0, y) = z(x, 0) = z(0, 0) = c, \ z(x, y) \geq 1 \tag{2.3.26}
\]
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and

\[ z_{xy}(x, y) = f(x, y)u_{xy}(x, y) + g(x, y)u^q(x, y). \]  \tag{2.3.27} \]

Substituting (2.3.24) and (2.3.25) in (2.3.27), we obtain

\[
\begin{align*}
z_{xy}(x, y) & \leq f(x, y)z(x, y) + g(x, y) \left( \int_0^x \int_0^y \frac{1}{p}(s_1, t_1)ds_1dt_1 \right)^q \\
& \leq f(x, y)z(x, y) + g(x, y) \left( \frac{1}{p}(x, y) \int_0^x \int_0^y ds_1dt_1 \right)^q \\
& = f(x, y)z(x, y) + g(x, y) \left( \frac{1}{p}(x, y)xy \right)^q \\
& = f(x, y)z(x, y) + g(x, y)z \frac{q}{p}(x, y) (xy)^q. \quad \tag{2.3.28}
\end{align*}
\]

By applying Lemma 2.1.1 to (2.3.28), we obtain

\[
\begin{align*}
z_{xy}(x, y) & \leq [f(x, y) + g(x, y) [n_1 + n_2] (xy)^q] z(x, y), \\
\text{or, } \quad \frac{z_{xy}(x, y)}{z(x, y)} & \leq f(x, y) + g(x, y) [n_1 + n_2] (xy)^q. \quad \tag{2.3.29}
\end{align*}
\]

By integrating (2.3.29), we obtain the estimate

\[
\frac{z_x(x, y)}{z(x, y)} \leq \int_0^y (f(x, t) + g(x, t) [n_1 + n_2] (xt)^q) dt. \quad \tag{2.3.30}
\]

By integrating (2.3.30), we have

\[
z(x, y) \leq c \exp \left[ \int_0^x \int_0^y (f(s, t) + g(s, t) [n_1 + n_2] (st)^q) dtds \right]. \quad \tag{2.3.31}
\]

Consequently, we get the desired inequality given in (2.3.23). The proof is complete.

\[\square\]

Remark 2.3.4. In the special case when \( p = q = 1 \) the inequality (2.3.22) reduces to one of the most useful inequality in the development of theory of differential and integral equations obtained by Pachaptte [53].
2.4 Applications

The study of various types of differential and integral equations have lead to the investigation of number of inequalities contained in earlier sections. In this section, we present applications of the inequalities established in earlier sections to study the properties of a solution of certain partial differential equation.

2.4.1 Nonlinear partial differential equation

Consider the following nonlinear partial differentiable equation

\[
pu^{p-1}(x,y)u_{xy}(x,y) + p(p-1)u^{p-2}(x,y)u_x(x,y)u_y(x,y) = h(x,y,u) + r(x,y,u),
\]

\[
u(x,0) = \sigma(x), u(0,y) = \tau(y), u(0,0) = \alpha,
\]

(2.4.1)

where \(r, h : \mathbb{R}^2_+ \times \mathbb{R} \rightarrow \mathbb{R}, \sigma, \tau : \mathbb{R}_+ \rightarrow \mathbb{R}\) are continuous functions and \(\alpha\) is constant.

We assume that solution \(u(x,y)\) of (2.4.1)-(2.4.2) exist. It is easy to observe that the partial differential equation (2.4.1) with initial condition (2.4.2) can be reduced to the following integral equation

\[
u^p(x,y) = \sigma(x) + \tau(y) - \alpha + \int_0^x \int_0^y h(s,t,u)dt\,ds + \int_0^x \int_0^y r(s,t,u)dt\,ds,
\]

(2.4.3)

for all \(x,y \in \mathbb{R}_+\).

Example 2.4.1. Suppose that the functions in equation (2.4.3) satisfy the conditions

\[
|h(x,y,u)| \leq f(x,y)|u(x,y)|,
\]

(2.4.4)

\[
|r(x,y,u)| \leq g(x,y)|u(x,y)|,
\]

(2.4.5)

\[
|\sigma(x) + \tau(y) - \alpha| \leq c,
\]

(2.4.6)
where $u, f, g, c, p$ are as given in Theorem 2.2.1. Substituting (2.4.4)-(2.4.6) in (2.4.3), we have

$$|u(x,y)|^p \leq c + \int_0^x \int_0^y f(s,t)|u(s,t)|dt ds + \int_0^x \int_0^y g(s,t)|u(s,t)|dt ds. \quad (2.4.7)$$

An application of the Theorem 2.2.1 to an inequality (2.4.7), yields

$$|u(x,y)| \leq \left[ c \exp \left( (m_1 + m_2) \int_0^x \int_0^y [f(s,t) + g(s,t)] dt ds \right) \right]^\frac{1}{p},$$

where $m_1, m_2$ are as defined in Theorem 2.2.1. Thus a solution $u(x,y)$ of (2.4.1)-(2.4.2) is bounded.

Example 2.4.2. Suppose that the following hypothesis holds on functions involved in (2.4.3)

$$|h(x,y,u)| \leq f(x,y)|u(x,y)|^p, \quad (2.4.8)$$
$$|r(x,y,u)| \leq g(x,y)|u(x,y)|^q, \quad (2.4.9)$$
$$1 \leq |\sigma(x) + \tau(y) - \alpha| \leq c(x,y), \quad (2.4.10)$$

where $f, g, c, p, q$ are defined as in Corollary 2.2.5. Substituting (2.4.8)-(2.4.10) in (2.4.3), we obtain

$$|u(x,y)|^p \leq c(x,y) + \int_0^x \int_0^y f(s,t)|u(s,t)|^p dt ds + \int_0^x \int_0^y g(s,t)|u(s,t)|^q dt ds. \quad (2.4.11)$$

Applying Corollary 2.2.5 to an inequality (2.4.11), we obtain an explicit estimate on $u(x,y)$, i.e.,

$$|u(x,y)| \leq c(x,y) \left( \exp \left( \int_0^x \int_0^y [f(s,t) + g(s,t)(n_1 + n_2)] dt ds \right) \right)^\frac{1}{p},$$

where $n_1, n_2$ are as same defined in Theorem 2.2.3.