Chapter 1

On Some Nonlinear Integral Inequalities in One Variable

1.1 Introduction

Being important tools in the study of differential, integral and integro-differential equations, various generalizations of Gronwall’s inequality and their applications have attracted great interests of many mathematicians. Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of nonlinear differential and integral equations. In recent years nonlinear integral inequalities have received considerable attention because of the important applications to a variety of

\footnote{The following papers based on the text of this Chapter has been published:}

problems in diverse fields of nonlinear differential and integral equations.

In the study of the qualitative behavior of solutions of differential and integral equations, integral inequalities established by Gronwall [28], Bellman [10] and Pachpatte [42, 44, 50] play a crucial role. The generalizations and variants of these inequalities are further studied by many mathematicians, see [2, 12, 22, 35, 61, 64] and references are therein.

In this chapter, we extend and improve integral inequalities established by Pachpatte to obtain generalizations and variants of these inequalities. These inequalities find significant applications in the study of various classes of differential and integral equations.

We state some important basic integral inequalities that will be useful in our further discussion.

**Theorem 1.1.1** (Gronwall-Bellman’s Inequality [10]). Let \( u(t), f(t) \) be a nonnegative continuous functions defined on \( J = [\alpha, \alpha + h] \) and \( c \) be a nonegative constant. If

\[
  u(t) \leq c + \int_{\alpha}^{t} f(s)u(s)ds, \quad t \in J,
\]

then

\[
  u(t) \leq c \exp \left( \int_{\alpha}^{t} f(s)ds \right), \quad t \in J.
\]

**Theorem 1.1.2** (Pachpatte’s Inequality [50]). If \( u(t), f(t), g(t) \) are nonnegative continuous functions on \( \mathbb{R}_+ = [0, \infty) \), \( u_0 \) is a nonnegative constant and

\[
  u(t) \leq u_0 + \int_{0}^{t} f(s) \left( u(s) + \int_{0}^{s} g(\sigma)u(\sigma) \right) ds, \quad t \in \mathbb{R}_+,
\]

then

\[
  u(t) \leq u_0 \left[ 1 + \int_{0}^{t} f(s) \exp \left( \int_{0}^{s} [f(\sigma) + g(\sigma)] d\sigma \right) \right] ds, \quad t \in \mathbb{R}_+.
\]


Theorem 1.1.3 (Abdeldaim and Yakout [2]). Suppose that \( x(t), f(t), h(t) \) are nonnegative real valued continuous functions defined on \( \mathbb{R}_+ \) and satisfy the inequality
\[
x^p(t) \leq x_0 + \int_0^t f(s)x^p(s)ds + \int_0^t h(s)x^q(s)ds, \quad \forall \ t \in \mathbb{R}_+, \tag{1.1.5}
\]
where \( x_0 > 0 \) and \( p > q \geq 0 \) are constants. Then
\[
x(t) \leq \exp \left( \frac{1}{p} \int_0^t f(s)ds \right) \left[ x_0^{p_1} + p_1 \int_0^t h(s) \exp \left( -p_1 \int_0^s f(\lambda)d\lambda \right) ds \right]^{\frac{1}{p-q}}, \tag{1.1.6}
\]
for all \( t \in \mathbb{R}_+ \), where \( p_1 = \left\lfloor \frac{p-2}{p} \right\rfloor \).

Lemma 1.1.4 (Zhao [65]). Assume that \( a \geq 0, p \geq 1 \), then
\[
a^{\frac{1}{p}} \leq \frac{1}{p} k^{\frac{1-p}{p}} a + \frac{p-1}{p} \frac{1}{k^p}, \tag{1.1.7}
\]
for any \( k > 0 \).

1.2 Nonlinear integral inequalities in one variable

In this section, we state and prove nonlinear integral inequalities to obtain an explicit bound on solutions of certain nonlinear differential and integral equations.

Theorem 1.2.1. Let \( u(t), f_1(t), f_2(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ = [0, \infty) \) and \( u_0 \) be a nonnegative constant. If
\[
u(t) \leq u_0 + \int_0^t [f_1(s)u(s) + g(s)] ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u^{\frac{1}{p}}(s_1)ds_1] \right) ds, \tag{1.2.1}
\]
for \( t \in \mathbb{R}_+ \), then
\[
u(t) \leq u_0 \exp \left( \int_0^t [f_1(s) + m_1f_2(s)] ds \right)
+ \int_0^t [g(s) + m_2f_2(s)] \exp \left( \int_s^t [f_1(\sigma) + m_1f_2(\sigma)] d\sigma \right) ds, \tag{1.2.2}
\]
for \( t \in \mathbb{R}_+ \), where \( p \geq 1, \ k > 0, \ m_1 = \frac{1}{p}k^{\frac{1-p}{p}} \) and \( m_2 = \frac{p-1}{p}k^{\frac{1}{p}} \).
Proof. Define a function \( z(t) \) by

\[
z(t) = u_0 + \int_0^t [f_1(s)u(s) + g(s)] ds + \int_0^t f_1(s) \left( \int_0^s f_2(s_1)u^{1/2}(s_1) ds_1 \right) ds,
\]

then \( u(t) \leq z(t) \), \( z(0) = u_0 \) and

\[
z'(t) = f_1(t)u(t) + g(t) + f_1(t) \left( \int_0^t f_2(s_1)u^{1/2}(s_1) ds_1 \right)
\]

\[
\leq g(t) + f_1(t) \left( z(t) + \int_0^t f_2(s_1)z^{1/2}(s_1) ds_1 \right). \tag{1.2.3}
\]

From Lemma 1.1.4 and an inequality (1.2.3), we have

\[
z'(t) \leq g(t) + f_1(t) \left( z(t) + \int_0^t f_2(s_1)(m_1z(s_1) + m_2) ds_1 \right).
\]

Let \( v(t) \) be

\[
v(t) = z(t) + \int_0^t f_2(s_1)(m_1z(s_1) + m_2) ds_1,
\]

then \( u(t) \leq z(t) \leq v(t) \), \( v(0) = z(0) = u_0 \) and

\[
v'(t) = z'(t) + f_2(t)(m_1z(t) + m_2)
\]

\[
\leq z'(t) + f_2(t)(m_1v(t) + m_2)
\]

\[
\leq g(t) + f_1(t)v(t) + f_2(t)(m_1v(t) + m_2)
\]

\[
= [f_1(t) + m_1f_2(t)]v(t) + g(t) + m_2f_2(t).
\]

i.e.,

\[
\left[ \frac{v(t)}{\exp \left( \int_0^t [f_1(s) + m_1f_2(s)] ds \right)} \right]' \leq [g(t) + m_2f_2(t)] \exp \left( - \int_0^t [f_1(s) + m_1f_2(s)] ds \right) ds. \tag{1.2.4}
\]

Setting \( t = s \) in (1.2.4) and integrating with respect to \( s \) from 0 to \( t \), we obtain the estimate

\[
v(t) \leq u_0 \exp \left( \int_0^t [f_1(s) + m_1f_2(s)] ds \right)
\]
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\[ + \int_0^t [g(s) + m_2 f_2(s)] \exp \left( \int_s^t [f_1(\sigma) + m_1 f_2(\sigma)] d\sigma \right) ds. \quad (1.2.5) \]

Using the fact that \( u(t) \leq z(t) \leq v(t) \) and above inequality (1.2.5), we obtain the desired bound for \( u(t) \) given in (1.2.2).

**Remark 1.2.2.** If we take \( p = 1 \) and \( g = 0 \), then the Theorem 1.2.1 reduces to inequality established by Pachpatte in Theorem 1.1.2.

**Theorem 1.2.3.** Let \( u(t), f_1(t), f_2(t), g_1(t), g_2(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \) and \( u_0 \) be a nonnegative constant. If

\[ u^p(t) \leq u_0 + \int_0^t [f_1(s)u^p(s) + g_1(s)] ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u(s_1) + g_2(s_1)] ds_1 \right) ds, \]

for \( t \in \mathbb{R}_+ \), then

\[ u^p(t) \leq u_0 \exp \left( \int_0^t [f_1(s) + m_1 f_2(s)] ds \right) + \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)] \exp \left( \int_s^t [f_1(\sigma) + m_1 f_2(\sigma)] d\sigma \right) ds, \quad (1.2.7) \]

for \( t \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are defined as same in Theorem 1.2.1.

**Proof.** Let right hand side of (1.2.6) be \( z(t) \), then \( u^p(t) \leq z(t) \), \( z(0) = u_0 \) and

\[ z'(t) \leq g_1(t) + f_1(t) \left( z(t) + \int_0^t \left[ f_2(s_1)\frac{z}{p}(s_1) + g_2(s_1) \right] ds_1 \right). \quad (1.2.8) \]

Applying Lemma 1.1.4 to (1.2.8), we obtain

\[ z'(t) \leq g_1(t) + f_1(t) \left( z(t) + \int_0^t [f_2(s_1)(m_1z(s_1) + m_2) + g_2(s_1)] ds_1 \right). \]

Define a function \( v(t) \) by

\[ v(t) = z(t) + \int_0^t [f_2(s_1)(m_1z(s_1) + m_2) + g_2(s_1)] ds_1, \]
then \( u^p(t) \leq z(t) \leq v(t) \), \( v(0) = z(0) = u_0 \) and

\[
v'(t) = z'(t) + f_2(t)(m_1 z(t) + m_2) + g_2(t) \\
\leq z'(t) + f_2(t)(m_1 v(t) + m_2) + g_2(t) \\
\leq g_1(t) + f_1(t)v(t) + f_2(t)(m_1 v(t) + m_2) + g_2(t),
\]

or, equivalently,

\[
\begin{bmatrix}
v(t) \\
\exp\left(\int_0^t [f_1(s) + m_1 f_2(s)] \, ds\right)
\end{bmatrix}' \leq [g_1(t) + g_2(t) + m_2 f_2(t)] \exp\left(-\int_0^t [f_1(s) + m_1 f_2(s)] \, ds\right).
\]

Integrating (1.2.9) from 0 to \( t \), we have

\[
v(t) \leq u_0 \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)] \, ds\right) \\
+ \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)] \exp\left(\int_s^t [f_1(\sigma) + m_1 f_2(\sigma)] \, d\sigma\right) \, ds.
\]

As \( u^p(t) \leq v(t) \) and above inequality (1.2.10), we obtain the desired inequality (1.2.7).

This completes the proof.

\[\Box\]

**Corollary 1.2.4.** Let \( u(t), f_1(t), f_2(t), g(t), n(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \) and \( 1 \leq n(t) \) be a nondecreasing. If

\[
w^p(t) \leq n(t) + \int_0^t [f_1(s)u^p(s) + g_1(s)] \, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1) + g_2(s_1)] \, ds_1\right) \, ds,
\]

for \( t \in \mathbb{R}_+ \), then

\[
w^p(t) \leq n^p(t) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)] \, ds\right) \\
+ n^p(t) \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)] \exp\left(\int_s^t [f_1(\sigma) + m_1 f_2(\sigma)] \, d\sigma\right) \, ds,
\]

for \( t \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are defined as same in Theorem 1.2.1.
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Proof. Since \( n(t) \) is nondecreasing, \( n(t) \geq 1 \) and \( p \geq 1 \). From (1.2.11), we have

\[
\frac{u^p(t)}{n^p(t)} \leq 1 + \int_0^t \left[ f_1(s) \frac{u^p(s)}{n^p(s)} + g_1(s) \right] ds + \int_0^t f_1(s) \left( \int_0^s \frac{f_2(s)}{n(s)} u(s) + g_2(s)ds \right) ds.
\]  

(1.2.13)

Now applying Theorem 1.2.3 to (1.2.13), we obtain the desired inequality (1.2.12).

This completes the proof. \( \square \)

Theorem 1.2.5. If \( u(t), f_1(t), f_2(t), f_3(t) \) are nonnegative continuous functions on \( \mathbb{R}_+ \), \( u_0 \) is a nonnegative constant and

\[
u^p(t) \leq u_0 + \int_0^t f_1(s) u^p(s)ds + \int_0^t f_1(s) \left( \int_0^s f_2(s) u^p(s)ds \right) ds + \int_0^t f_1(s) \left( \int_0^s f_3(s) u(s)ds \right) ds,
\]

for \( t \in \mathbb{R}_+ \), then

\[
u^p(t) \leq u_0 \exp \left( \int_0^t [f_1(s) + f_2(s) + m_1 f_3(s)] ds \right)
\]

\[
+ m_2 \int_0^t f_3(s) \exp \left( \int_0^t [f_1(\sigma) + f_2(\sigma) + m_1 f_3(\sigma)] d\sigma \right) ds,
\]

for \( t \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are defined as same in Theorem 1.2.1.

Proof. Define a function \( z(t) \) by right hand side of (1.2.14), we have

\[
z(t) = u_0 + \int_0^t f_1(s) \left[ u^p(s) + \int_0^s f_2(s) u^p(s)ds + \int_0^s f_3(s) u(s)ds \right] ds.
\]

\( u^p(t) \leq z(t), \ z(0) = u_0 \) and

\[
z'(t) = f_1(t) \left[ u^p(t) + \int_0^t f_2(s) u^p(s)ds + \int_0^t f_3(s) u(s)ds \right] ds
\]

\[
\leq f_1(t) \left[ z(t) + \int_0^t f_2(s) \left( z(s) + \int_0^s f_3(s)z^p(s)ds \right) ds \right].
\]

Define a function \( v(t) \) by

\[
v(t) = z(t) + \int_0^t f_2(s) \left( z(s) + \int_0^s f_3(s)z^p(s)ds \right) ds.
\]
then \( w^p(t) \leq z(t) \leq v(t), \ v(0) = z(0) = u_0, \) and

\[
v'(t) = z'(t) + f_2(t) \left( z(t) + \int_0^t f_3(s_2) z^{1\over 2}(s_2) ds_2 \right)
\leq z'(t) + f_2(t) \left( v(t) + \int_0^t f_3(s_2) v^{1\over 2}(s_2) ds_2 \right). \tag{1.2.16}
\]

Applying Lemma 1.1.4 to (1.2.16), to obtain

\[
v'(t) \leq z'(t) + f_2(t) \left( v(t) + \int_0^t f_3(s_2) (m_1 v(s_2) + m_2) ds_2 \right).
\]

Define a function \( w(t) \) by

\[
w(t) = v(t) + \int_0^t f_3(s_2) (m_1 v(s_2) + m_2) ds_2,
\]

then \( w^p(t) \leq z(t) \leq v(t), \ w(0) = v(0) = z(0) = u_0, \) and

\[
w'(t) = v'(t) + f_3(t) (m_1 v(t) + m_2)
\leq z'(t) + f_2(t) w(t) + f_3(t) (m_1 v(t) + m_2)
\leq f_1(t) v(t) + f_2(t) w(t) + f_3(t) (m_1 v(t) + m_2)
\leq f_1(t) w(t) + f_2(t) w(t) + f_3(t) (m_1 w(t) + m_2)
= m_2 f_3(t) + [f_1(t) + f_2(t) + m_1 f_3(t)] w(t),
\]

and hence, we have

\[
\left[ w(t) \over \exp \left( \int_0^t [f_1(s) + f_2(s) + m_1 f_3(s)] ds \right) \right]' 
\leq m_2 f_3(t) \exp \left( - \int_0^t [f_1(s) + f_2(s) + m_1 f_3(s)] ds \right). \tag{1.2.17}
\]

By setting \( t = s \) in (1.2.17) and integrating from 0 to \( t, \) we get

\[
{w(t) \over \exp \left( \int_0^t [f_1(s) + f_2(s) + m_1 f_3(s)] ds \right) } - w(0)
\]
Proof. Since \( n(t) \) is a nondecreasing, \( n(t) \geq 1 \) and \( p \geq 1 \) from (1.2.19), we obtain

\[
\frac{u^p(t)}{n^p(t)} \leq 1 + \int_0^t f_1(s_1) \frac{u^p(s_1)}{n^p(s_1)} ds_1 + \int_0^t f_1(s_1) \left( \int_0^{s_1} f_2(s_2) \frac{u^p(s_2)}{n^p(s_2)} ds_2 \right) ds_1
\]

\[
+ \int_0^t f_1(s_1) \left( \int_0^{s_2} f_2(s_2) \left( \int_0^{s_3} f_3(s_3) \frac{u(s_3)}{n(s_3)} ds_3 \right) ds_2 \right) ds_1.
\]  

(1.2.21)

Now an application of Theorem 1.2.5 to (1.2.21), we get (1.2.20). This completes the proof.  

\[ \square \]

Corollary 1.2.6. Let \( u(t), f_1(t), f_2(t), f_3(t) \) be as same defined in Theorem 1.2.5. Let \( n(t) \) be a nondecreasing and \( n(t) \geq 1 \), for \( t \in \mathbb{R}_+ \). If

\[
w^p(t) \leq n(t) + \int_0^t f_1(s_1)u^p(s_1) ds_1 + \int_0^t f_1(s_1) \left( \int_0^{s_1} f_2(s_2)u(s_2) ds_2 \right) ds_1
\]

\[
+ \int_0^t f_1(s_1) \left( \int_0^{s_2} f_2(s_2) \left( \int_0^{s_3} f_3(s_3) u(s_3) ds_3 \right) ds_2 \right) ds_1,
\]  

(1.2.19)

for \( t \in \mathbb{R}_+ \), then

\[
w^p(t) \leq n^p(t) \exp \left( \int_0^t \left[ f_1(s) + f_2(s) + m_1 f_3(s) \right] ds \right)
\]

\[
+ m_2 n^p(t) \int_0^t f_3(s) \exp \left( \int_0^t \left[ f_1(\sigma) + f_2(\sigma) + m_1 f_3(\sigma) \right] d\sigma \right) ds,
\]  

(1.2.20)

for \( t \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are defined as same in Theorem 1.2.1.

Proof. Since \( n(t) \) is a nondecreasing, \( n(t) \geq 1 \) and \( p \geq 1 \) from (1.2.19), we obtain

\[
\frac{u^p(t)}{n^p(t)} \leq 1 + \int_0^t f_1(s_1) \frac{u^p(s_1)}{n^p(s_1)} ds_1 + \int_0^t f_1(s_1) \left( \int_0^{s_1} f_2(s_2) \frac{u^p(s_2)}{n^p(s_2)} ds_2 \right) ds_1
\]

\[
+ \int_0^t f_1(s_1) \left( \int_0^{s_2} f_2(s_2) \left( \int_0^{s_3} f_3(s_3) \frac{u(s_3)}{n(s_3)} ds_3 \right) ds_2 \right) ds_1.
\]  

(1.2.21)

Now an application of Theorem 1.2.5 to (1.2.21), we get (1.2.20). This completes the proof.  

\[ \square \]
Remark 1.2.7. We note that the inequality established in the Theorem 1.2.5 is a generalization and variants of Gronwall-Bellman’s and Pachpatte’s inequality. In the special case when $f_2 = f_3 = 0$ and $p = 1$, the inequality in the Theorem 1.2.5 reduces to the Gronwall-Bellman’s inequality [10]. If $f_3 = 0$ and $p = 1$ it reduces to Pachpatte’s inequality [50].

1.3 Nonlinear integro-differential inequalities in one variable

In this section, we establish integro-differential inequalities which can be useful to study nonlinear integro-differential equations.

Theorem 1.3.1. Let $u(t), u'(t), f_1(t), f_2(t), g(t)$ be nonnegative continuous functions on $\mathbb{R}_+, u(0) = 0$ and $u_0$ be a nonnegative constant. If

\[(u'(t))^p \leq u_0 + \int_0^t [f_1(s)u'(s) + g(s)] \, ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u(s_1) \, ds_1] \, ds \right) \, ds, \tag{1.3.1}\]

for $t \in \mathbb{R}_+$, then

\[u(t) \leq \int_0^t \left[ \left( u_0 + \int_0^\sigma [g(s) + m_2f_1(s) + \frac{m_2}{m_1}s f_2(s)] \, ds \right) \exp \left( \int_0^\sigma [m_1f_1(s) + sf_2(s)] \, ds \right) \right]^\frac{1}{p} \, d\sigma, \tag{1.3.2}\]

for $t \in \mathbb{R}_+$, where $p, m_1, m_2$ are defined as same in Theorem 1.2.1.

Proof. Denoting $z(t)$ by

\[z(t) = u_0 + \int_0^t [f_1(s)u'(s) + g(s)] \, ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u(s_1) \, ds_1] \, ds \right) \, ds, \tag{1.3.3}\]

we get $z(0) = u_0$, $z(t)$ is a nondecreasing on $\mathbb{R}_+$ and

\[u'(t) \leq \frac{1}{z^p}(t). \tag{1.3.4}\]
By applying Lemma 1.1.4 to (1.3.4) and integrating, we obtain

\[ u(t) \leq m_1 z(t) t + tm_2. \]  

(1.3.5)

Differentiating equation (1.3.3), using (1.3.5) and an application of Lemma 1.1.4, we get

\[ z'(t) = f_1(t) u'(t) + g(t) + f_1(t) \left( \int_0^t f_2(s_1) u(s_1) ds_1 \right) \]
\[ \leq g(t) + m_2 f_1(t) + f_1(t) \left( m_1 z(t) + \int_0^t f_2(s_1) [m_1 s_1 z(s_1) + s_1 m_2] ds_1 \right). \]  

(1.3.6)

Let us define \( v(t) \) by

\[ v(t) = m_1 z(t) + \int_0^t f_2(s_1) (m_1 s_1 z(s_1) + s_1 m_2) ds_1, \]

then \((u'(t))^p \leq z(t) \leq \frac{v(t)}{m_1}, \ v(0) = m_1 z(0) = m_1 u_0 \) and

\[ v'(t) = m_1 z'(t) + f_2(t) (m_1 t z(t) + tm_2) \]
\[ \leq m_1 z'(t) + t f_2(t) (v(t) + m_2) \]
\[ \leq m_1 g(t) + m_1 m_2 f_1(t) + m_1 f_1(t) v(t) + t f_2(t) v(t) + m_2 t f_2(t). \]

i.e,

\[ v'(t) - (m_1 f_1(t) + t f_2(t)) v(t) \leq m_1 g(t) + m_1 m_2 f_1(t) + m_2 t f_2(t), \]

or, equivalently,

\[ \left[ \frac{v(t)}{\exp(\int_0^t [m_1 f_1(s) + s f_2(s)] ds)} \right]' \leq m_1 g(t) + m_1 m_2 f_1(t) + m_2 t f_2(t). \]  

(1.3.7)

Integrating (1.3.7) from 0 to \( t \), yields

\[ \frac{v(t)}{\exp(\int_0^t [m_1 f_1(s) + s f_2(s)] ds)} \leq m_1 u_0 + \int_0^t [m_1 g(s) + m_1 m_2 f_1(s) + m_2 s f_2(s)] ds, \]
1.3 Nonlinear integro-differential inequalities in one variable

\[ v(t) \leq \left( m_1 u_0 + \int_0^t [m_1 g(s) + m_1 m_2 f_1(s) + m_2 s f_2(s)] ds \right) \exp \left( \int_0^t [m_1 f_1(s) + s f_2(s)] ds \right). \]  

(1.3.8)

The required inequality (1.3.2) follows from the inequality \((u'(t))^p \leq z(t) \leq \frac{v(t)}{m_1}\) and an inequality (1.3.8). This proves the theorem. \qed

**Theorem 1.3.2.** Let \(u(t), u'(t), f_1(t), f_2(t)\) be nonnegative continuous functions on \(\mathbb{R}_+, u(0) = 0\) and \(u_0\) be a nonnegative constant. If

\[ (u'(t))^p \leq u_0 + \int_0^t f_1(s)(u'(s))^p ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u(s_1)ds_1] \right) ds, \]  

(1.3.9)

for \(t \in \mathbb{R}_+\), then

\[ u(t) \leq \int_0^t \left[ u_0 \exp \left( \int_0^s [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) \right. \]  

\[ \left. + m_2 \int_0^t \tau f_2(\tau) \exp \left( \int_\tau^s [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) d\tau \right]^{\frac{1}{p}} ds, \]  

(1.3.10)

for \(t \in \mathbb{R}_+\), where \(p, m_1, m_2\) are as same defined in Theorem 1.2.1.

**Proof.** Define a function \(z(t)\) by

\[ z(t) = u_0 + \int_0^t f_1(s)(u'(s))^p ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1)u(s_1)ds_1] \right) ds, \]  

(1.3.11)

then \(z(0) = u_0\), \(z(t)\) is a nondecreasing on \(\mathbb{R}_+\) and

\[ u'(t) \leq z^\frac{1}{p}(t). \]  

(1.3.12)

By applying Lemma 1.1.4 to (1.3.12) and integrating, we obtain

\[ u(t) \leq m_1 z(t) t + tm_2. \]  

(1.3.13)

Differentiating equation (1.3.11), using (1.3.13) and an application of Lemma 1.1.4, we get

\[ z'(t) = f_1(t)(u'(t))^p + f_1(t) \left( \int_0^t [f_2(s_1)u(s_1)ds_1] \right). \]
Let us define $v(t)$ by

$$v(t) = z(t) + \int_0^t f_2(s_1) (m_1 z(s_1) + s_1 m_2) ds_1,$$

then $(u'(t))^p \leq z(t) \leq v(t)$, $v(0) = z(0) = u_0$ and

$$v'(t) = z'(t) + f_2(t) (m_1 z(t) + t m_2)$$

$$\leq z'(t) + f_2(t) (m_1 v(t) + t m_2)$$

$$\leq f_1(t) v(t) + m_1 f_2(t) v(t) + m_2 t f_2(t),$$

$$v'(t) - (f_1(t) + m_1 t f_2(t)) v(t) \leq m_2 t f_2(t).$$

Equivalently

$$\left[ \frac{v(t)}{\exp \left( \int_0^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) \right]' \leq m_2 t f_2(t) \exp \left( - \int_0^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right).$$

(1.3.15)

Setting $t = s$ in (1.3.15) and integrating with respect to $s$ from 0 to $t$, we get

$$\frac{v(t)}{\exp \left( \int_0^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right)} \leq u_0 + \int_0^t m_2 s f_2(s) \exp \left( - \int_0^s [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) ds,$$

$$v(t) \leq u_0 \exp \left( \int_0^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) + \int_0^t m_2 s f_2(s) \exp \left( \int_s^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) ds.$$

(1.3.16)

Thus, the desired inequality (1.3.10) follows from $(u'(t))^p \leq z(t) \leq v(t)$ and (1.3.16).

This completes the proof.

**Theorem 1.3.3.** Let $u(t), u'(t), f_1(t), f_2(t), g(t)$ be nonnegative continuous functions on $\mathbb{R}_+, u(0) = 0$ and $u_0$ be a nonnegative constant. If

$$u'(t) \leq u_0 + \int_0^t [f_1(s) u'(s) + g(s)] ds + \int_0^t f_1(s) \left( \int_0^s [f_2(s_1) u^{1/2}(s_1)] ds_1 \right) ds,$$  

(1.3.17)
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For \( t \in \mathbb{R}_+ \), then

\[
\begin{align*}
    u'(t) &\leq u_0 \exp \left( \int_0^t [f_1(s) + m_1 s f_2(s)] \, ds \right) \\
    &\quad + \int_0^t (g(s) + m_2 f_2(s)) \exp \left( \int_s^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] \, d\sigma \right) \, ds 
\end{align*}
\]

(1.3.18)

for \( t \in \mathbb{R}_+ \), where \( p, m_1, m_2 \) are defined as same in Theorem 1.2.1.

Proof. Denoting \( z(t) \)

\[
z(t) = u_0 + \int_0^t \left[ f_1(s) u'(s) + g(s) \right] \, ds + \int_0^t f_1(s) \left( \int_s^t \left[ f_2(s_1) u^{\frac{1}{p}}(s_1) \right] \, ds_1 \right) \, ds 
\]

(1.3.19)

we obtain \( z(0) = u_0 \), \( z(t) \) is a nondecreasing on \( \mathbb{R}_+ \),

\[
u'(t) \leq z(t) \quad \text{and} \quad u(t) \leq tz(t). \quad (1.3.20)
\]

Differentiating equation (1.3.19) and using (1.3.20), we have

\[
z'(t) = f_1(t) u'(t) + g(t) + f_1(t) \left( \int_0^t f_2(s_1) u^{\frac{1}{p}}(s_1) \, ds_1 \right) \\
    \leq g(t) + f_1(t) \left( z(t) + \int_0^t \left[ f_2(s_1) s_1 z(s_1) \right]^{\frac{1}{p}} \, ds_1 \right). 
\]

(1.3.21)

Now defining \( v(t) \) by

\[
v(t) = z(t) + \int_0^t f_2(s_1) s_1 z(s_1) \, ds_1 \quad \text{and} \quad \text{applying Lemma 1.1.4,}
\]

we obtain

\[
v'(t) - \left[ f_1(t) + m_1 t f_2(t) \right] v(t) \leq g(t) + m_2 f_2(t). 
\]

Equivalently,

\[
    \left[ \frac{v(t)}{\exp(\int_0^t [f_1(s) + m_1 s f_2(s)] \, ds)} \right]' \leq \left[ g(t) + m_2 f_2(t) \right] \exp(- \int_0^t [f_1(s) + m_1 s f_2(s)] \, ds). 
\]

(1.3.22)

Solving (1.3.22), we have

\[
    \frac{v(t)}{\exp(\int_0^t [f_1(s) + m_1 s f_2(s)] \, ds)} \leq u_0 
\]
i.e.,

\[ v(t) \leq u_0 \exp \left( \int_0^t [f_1(s) + m_1 s f_2(s)] ds \right) + \int_0^t [g(s) + m_2 f_2(s)] \exp \left( \int_s^t [f_1(\sigma) + m_1 \sigma f_2(\sigma)] d\sigma \right) ds. \]

Since \( u'(t) \leq v(t) \) and \( 1.3.23 \), we get the desired inequality \( 1.3.18 \). This completes the proof.

### 1.4 More nonlinear integral inequalities in one variable

In 1981, Gripenberg G. [27] studied the qualitative behaviour of solutions of equation

\[ x(t) = k \left( p(t) - \int_0^t A(t-s)x(s) ds \right) \left( f(t) + \int_0^t a(t-s)x(s) ds \right). \]

This equation arises in the study of the spread of an infectious disease that does not induce permanent immunity. For detail meaning of functions arising in \( 1.4.1 \) see [27]. Also in [27] given details of existence and uniqueness of solution \( 1.4.1 \). In 1995, Pachpatte [47] studied boundedness behaviour of solutions of equation \( 1.4.1 \) by establishing the following integral inequality.

**Theorem 1.4.1** (Pachpatte [47]). Let \( u(t), f(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \) and \( c_1, c_2 \) be nonnegative constants. If

\[ u(t) \leq \left( c_1 + \int_0^t f(s) u(s) ds \right) \left( c_2 + \int_0^t g(s) u(s) ds \right), \]

and \( c_1 c_2 \int_0^t R(s) Q(s) ds < 1 \), for all \( t \in \mathbb{R}_+ \), then

\[ u(t) \leq \frac{c_1 c_2 Q(t)}{1 - c_1 c_2 \int_0^t R(s) Q(s) ds}, \quad \text{for all } t \in \mathbb{R}_+, \]
where
\[ R(t) = g(t) \int_0^t f(s)ds + f(t) \int_0^t g(s)ds \quad \text{and} \quad Q(t) = \exp \left( \int_0^t [c_1 g(s) + c_2 f(s)] ds \right). \] (1.4.4)

In this section, we state and prove generalizations and variants of inequalities reported in [47], which are not only useful to study differential and integral equations but also very effective to study certain epidemic models.

**Theorem 1.4.2.** Let \( u(t), f(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( c_1, c_2, p \) be nonnegative constants such that \( c_1 c_2 \geq 1 \) and \( p \geq 1 \). If
\[
\begin{align*}
u^p(t) &\leq \left( c_1 + \int_0^t f(s)u(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right) \quad (1.4.5) \\
\text{and} \quad \left[ \frac{(c_1 c_2)^{\frac{1}{p}}}{p} \int_0^t R(s)Q_1(s)ds \right] &< 1, \quad \text{for all} \ t \in \mathbb{R}_+, \text{then} \\
u(t) &\leq \left[ \frac{(c_1 c_2)^{\frac{1}{p}} Q_1(t)}{1 - \frac{(c_1 c_2)^{\frac{1}{p}}}{p} \int_0^t R(s)Q_1(s)ds} \right], \quad \text{for all} \ t \in \mathbb{R}_+, \quad (1.4.6)
\end{align*}
\]

where
\[ R(t) = g(t) \int_0^t f(s)ds + f(t) \int_0^t g(s)ds \quad \text{and} \quad Q_1(t) = \exp \left( \frac{1}{p} \int_0^t [c_1 g(s) + c_2 f(s)] ds \right). \] (1.4.7)

**Proof.** Define a function \( z(t) \) by
\[
\begin{align*}
z^p(t) &= \left( c_1 + \int_0^t f(s)u(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right), \quad \text{for all} \ t \in \mathbb{R}_+, \quad (1.4.8) \\
\text{then} \quad u^p(t) &\leq z^p(t), \quad 1 \leq z(t) \quad \text{and} \quad z^p(0) = c_1 c_2. \quad \text{Differentiating} \ (1.4.8) \ \text{and using the fact that} \ u(t) \leq z(t) \quad \text{and} \ z(t) \ \text{is a nondecreasing for} \ t \in \mathbb{R}_+, \ \text{we obtain}
\end{align*}
\]
\[
\begin{align*}
px^{p-1}(t)z'(t) &= c_2 f(t)u(t) + c_1 g(t)u(t) + f(t)u(t) \int_0^t g(s)u(s)ds + g(t)u(t) \int_0^t f(s)u(s)ds
\end{align*}
\]
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\[ \leq c_2 f(t)z(t) + c_1 g(t)z(t) + f(t)z(t) \int_0^t g(s)z(s)ds + g(t)z(t) \int_0^t f(s)z(s)ds \]

\[ \leq (c_2 f(t) + c_1 g(t)) z^p(t) + \left( f(t) \int_0^t g(s)ds + g(t) \int_0^t f(s)ds \right) z^{p+1}(t). \]

i.e.,

\[ z'(t) \leq \frac{1}{p} [c_2 f(t) + c_1 g(t)] z(t) + \frac{1}{p} R(t) z^2(t), \quad \text{(1.4.9)} \]

where \( R(t) \) is defined in equation (1.4.7). The inequality (1.4.9) implies the following estimation for \( z(t) \)

\[ z(t) \leq \left[ \left( \frac{(c_1 c_2)^{1/p} Q_1(t)}{1 - (c_1 c_2)^{1/p} \int_0^t R(s)Q_1(s)ds} \right) \right], \quad \text{for all } t \in \mathbb{R}_+, \]

where \( Q_1(t) \) is given in equation (1.4.7). Thus, the desired inequality (1.4.6) follows from \( u(t) \leq z(t) \). The proof is complete.

\[ \square \]

**Theorem 1.4.3.** Let \( u(t), f(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+, c_1, c_2, p, q \) be nonnegative constants such that \( c_1 c_2 \geq 1 \) and \( p > q \geq 1 \) with \( p - q \geq 1 \). If

\[ u^p(t) \leq \left( c_1 + \int_0^t f(s)u^q(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right), \quad \text{for all } t \in \mathbb{R}_+, \quad \text{(1.4.10)} \]

then

\[ u(t) \leq \left[ \left( \frac{(c_1 c_2)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q_2(s)ds}{Q_2(t)} \right) \right]^{\frac{1}{p-q}}, \quad \text{for all } t \in \mathbb{R}_+, \quad \text{(1.4.11)} \]

where \( R(t) \) is as same defined in (1.4.7) and

\[ Q_2(t) = \exp \left( -\frac{p-q}{p} \int_0^t R(s)ds \right). \quad \text{(1.4.12)} \]

**Proof.** Let us define a function \( z(t) \) by

\[ z^p(t) = \left( c_1 + \int_0^t f(s)u^q(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right), \quad \text{for all } t \in \mathbb{R}_+, \quad \text{(1.4.13)} \]
then \( u^p(t) \leq z^p(t) \) and \( z^p(0) = c_1 c_2 \). Differentiating (1.4.13) and using the fact that \( u(t) \leq z(t) \) and \( z(t) \) is a nondecreasing for \( t \in \mathbb{R}_+ \), we obtain

\[
pz^{p-1}(t)z'(t) = c_2 f(t)u^q(t) + c_1 g(t)u(t) + f(t)u^q(t) \int_0^t g(s)u(s)ds + g(t)u(t) \int_0^t f(s)u^q(s)ds \\
\leq c_2 f(t)z^q(t) + c_1 g(t)z(t) + f(t)z^q(t) \int_0^t g(s)z(s)ds + g(t)z(t) \int_0^t f(s)z^q(s)ds \\
\leq [c_2 f(t) + c_1 g(t)]z^q(t) + \left( f(t) \int_0^t g(s)ds + g(t) \int_0^t f(s)ds \right) z^{q+1}(t) \\
\leq [c_2 f(t) + c_1 g(t)]z^q(t) + R(t)z^{q+1}(t),
\]

where \( R(t) \) is defined by (1.4.7). Since \( p - q \geq 1 \), we have

\[
pz^{p-q-1}(t)z'(t) \leq [c_2 f(t) + c_1 g(t)] + R(t)z(t) \\
\leq [c_2 f(t) + c_1 g(t)] + R(t)z^{p-q}(t). \tag{1.4.14}
\]

Let \( z^{p-q}(t) = v(t) \). Then, we have \( v(0) = z^{p-q}(0) = (c_1 c_2)^{\frac{p-q}{p}} \),

\[
(p - q)z^{p-q-1}z'(t) = v'(t) \quad \text{and} \quad z^{p-q-1}z'(t) = \frac{v'(t)}{p - q}.
\]

Now from (1.4.14), we obtain

\[
\left( \frac{p}{p - q} \right) v'(t) \leq [c_2 f(t) + c_1 g(t)] + R(t)v(t), \\
v'(t) \leq \left( \frac{p - q}{p} \right) R(t)v(t) + \left( \frac{p - q}{p} \right) [c_2 f(t) + c_1 g(t)]. \tag{1.4.15}
\]

The inequality (1.4.15) implies the estimate

\[
v(t) \leq \left( c_1 c_2 \right)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q_2(s) ds \\
\frac{Q_2(t)}{Q_2(t)}, \quad \text{for all} \ t \in \mathbb{R}_+,
\]

where \( Q_2(t) \) is given by (1.4.12). Since \( z^{p-q}(t) = v(t) \), we have

\[
z^{p-q}(t) \leq \left( c_1 c_2 \right)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q_2(s) ds \\
\frac{Q_2(t)}{Q_2(t)}, \quad \text{for all} \ t \in \mathbb{R}_+.
\]
This implies
\[ z(t) \leq \left[ (c_1c_2)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q_2(s) ds \right]^{\frac{1}{p-q}} \] for all \( t \in \mathbb{R}_+ \).

As \( u^p(t) = z^p(t) \), we get the desired bound for \( u(t) \) given in (1.4.11). This completes the proof. \( \square \)

**Theorem 1.4.4.** Let \( u(t), f(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \) and \( c_1, c_2 \) be nonnegative constants such that \( c_1 c_2 \geq 1 \). If
\[ u^p(t) \leq \left( c_1 + \int_0^t f(s) u^p(s) ds \right) \left( c_2 + \int_0^t g(s) u^q(s) ds \right), \]
(1.4.16)

\( p \geq q > 0 \) and \( \left[ \frac{q}{p} (c_1 c_2)^{\frac{q}{p}} \int_0^t R(s) Q_3(s) ds \right] < 1 \), for all \( t \in \mathbb{R}_+ \), then
\[ u(t) \leq \left[ \frac{c_1 c_2 \frac{q}{p} Q_3(t)}{1 - \frac{q}{p} (c_1 c_2)^{\frac{q}{p}} \int_0^t R(s) Q_3(s) ds} \right]^{\frac{1}{q}}, \]
for all \( t \in \mathbb{R}_+ \),
(1.4.17)

where, \( R(t) \) is as same defined in (1.4.7) and
\[ Q_3(t) = \exp \left( \frac{q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] ds \right). \]
(1.4.18)

**Proof.** Let us define a function \( z(t) \) by
\[ z^p(t) = \left( c_1 + \int_0^t f(s) u^p(s) ds \right) \left( c_2 + \int_0^t g(s) u^q(s) ds \right), \]
for all \( t \in \mathbb{R}_+ \),
(1.4.19)

then \( u^p(t) \leq z^p(t) \) and \( z^p(0) = c_1 c_2 \). Differentiating (1.4.19) and using the fact that \( u(t) \leq z(t) \) and \( z(t) \) is a monotone nondecreasing for \( t \in \mathbb{R}_+ \), we obtain
\[
pz^{p-1}(t) z'(t) = c_2 f(t) u^p(t) + c_1 g(t) u^q(t) + f(t) u^p(t) \int_0^t g(s) u^q(s) ds
\]
\[
+ g(t) u^q(t) \int_0^t f(s) u^p(s) ds
\]
\[
\leq (c_2 f(t) + c_1 g(t)) z^p(t) + \left( f(t) \int_0^t g(s) ds + g(t) \int_0^t f(s) ds \right) z^{p+q}(t)
\]
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\( \leq (c_2 f(t) + c_1 g(t)) z^p(t) + \left( f(t) \int_0^t g(s) ds + g(t) \int_0^t f(s) ds \right) z^{p+q}(t), \)

or,

\[ p \frac{z'(t)}{z(t)} \leq (c_2 f(t) + c_1 g(t)) + R(t) z^q(t), \tag{1.4.20} \]

where \( R(t) \) is defined in equation (1.4.7). By letting \( z^q(t) = v(t) \), differentiating with respect to \( t \) and using the fact that \( v(t) \) is a nondecreasing for \( t \in \mathbb{R}_+ \), we have

\[ qz^{q-1}z'(t) = v'(t), \]
\[ \frac{z'(t)}{z(t)} = \frac{v'(t)}{z^q(t)}, \]
\[ \frac{z'(t)}{z(t)} = \frac{v'(t)}{qv(t)}. \tag{1.4.21} \]

From inequalities (1.4.20), (1.4.21) and using \( p \geq q > 0 \), we obtain

\[ v'(t) \leq \frac{q}{p} [c_2 f(t) + c_1 g(t)] v(t) + \frac{q}{p} R(t)v^2(t). \tag{1.4.22} \]

The inequality (1.4.22) implies the estimation for \( v(t) \) such that

\[ v(t) \leq \frac{(c_1 c_2)^\frac{q}{p} Q_3(t)}{1 - \frac{q}{p} (c_1 c_2)^\frac{q}{p} \int_0^t R(s) Q_3(s) ds}, \quad \text{for all } t \in \mathbb{R}_+, \]

where \( Q_3(t) \) is as defined in equation (1.4.18). Using \( z^q(t) = v(t) \) in above inequality, we get

\[ z^q(t) \leq \frac{(c_1 c_2)^\frac{q}{p} Q_3(t)}{1 - \frac{q}{p} (c_1 c_2)^\frac{q}{p} \int_0^t R(s) Q_3(s) ds}, \quad \text{for all } t \in \mathbb{R}_+. \]

Since \( u(t) \leq z(t) \), we get the desired inequality given in (1.4.17). This completes the proof.

\[ \square \]

**Remark 1.4.5.** If \( p = q = 1 \), then Theorem 1.4.4 reduces to Theorem 1.4.1 which is established by Pachpatte in [47].
Theorem 1.4.6. Let \( u(t), f(t), g(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( p \geq q > 0 \), \( c_2 \geq 1 \) and \( c_1 \) be nonnegative constants. If

\[
    u^p(t) \leq \left( c_1 + \int_0^t f(s)u(s)ds \right)^p \left( c_2 + \int_0^t g(s)u(s)ds \right)^q, \quad \text{for all } t \in \mathbb{R}_+, \quad (1.4.23)
\]

then

\[
    u(t) \leq \left[ \frac{c_1 c_2 Q(t)}{1 - c_1 c_2 \int_0^t R(s)Q(s)ds} \right] \quad \text{for all } t \in \mathbb{R}_+, \quad (1.4.24)
\]

where \( R(t) \) and \( Q(t) \) are same as defined in (1.4.4).

Proof. Since \( p \geq q \) and \( c_2 \geq 1 \), we have

\[
    u^p(t) \leq \left( c_1 + \int_0^t f(s)u(s)ds \right)^p \left( c_2 + \int_0^t g(s)u(s)ds \right)^q \leq \left[ \left( c_1 + \int_0^t f(s)u(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right) \right]^p.
\]

Define a function \( z(t) \) by

\[
    z(t) = \left( c_1 + \int_0^t f(s)u(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right), \quad (1.4.25)
\]

then \( u^p(t) \leq z^p(t) \) and \( z(0) = c_1 c_2 \). Using the relation \( u(t) \leq z(t) \) and monotonic character of \( z(t) \), we have

\[
    z(t) \leq \left( c_1 + \int_0^t f(s)z(s)ds \right) \left( c_2 + \int_0^t g(s)z(s)ds \right), \quad \text{for all } t \in \mathbb{R}_+, \quad (1.4.26)
\]

Using same argument as used in Theorem 1.4.2, we obtain

\[
    z(t) \leq \left[ \frac{c_1 c_2 Q(t)}{1 - c_1 c_2 \int_0^t R(s)Q(s)ds} \right], \quad \text{for all } t \in \mathbb{R}_+.
\]

Consequently, we get the desired bound for \( u(t) \) given in (1.4.24). This completes the proof.
1.5 Applications

One of the main motivations for the study of different types inequalities given in the previous sections is to apply them as tools in the study of various classes of nonlinear integral and differential equations. In the following section, we give examples to illustrate the application of results established in previous sections.

Example 1.5.1. Consider the following nonlinear integral equation.

\[
    u^p(t) = \left( h_1(t) + \int_0^t B(t-s)u^p(s)ds \right) \left( h_2(t) + \int_0^t C(t-s)u^q(s)ds \right),
\]

(1.5.1)

for all \( t \in \mathbb{R}_+ \), where \( p \geq q > 0 \), \( u(t) \) is a nonnegative continuous function defined on \( \mathbb{R}_+ \) and \( h_1, h_2, B, C \) are continuous functions defined on \( \mathbb{R}_+ \). Here we assume that a solution \( u(t) \) of (1.5.1) under discussion exists on \( \mathbb{R}_+ \). As an application of the Theorem 1.4.4, we present the following example related to a solution of equation (1.5.1). We list the following hypotheses on the functions involved in (1.5.1):

\[
    |h_1(t)| \leq c_1, |h_2(t)| \leq c_2, |B(t-s)| \leq M_1 f_1(s) \quad \text{and} \quad |C(t-s)| \leq N_1 g_1(s),
\]

(1.5.2)

\[
    |h_1(t)| \leq c_1 e^{-\alpha t}, |h_2(t)| \leq c_2 e^{-\alpha t}, |B(t)| \leq M_1 f_1(s)e^{-\alpha(t-2s)} \quad \text{and} \quad |C(t)| \leq N_1 g_1(s)e^{-\alpha(t-2s)},
\]

(1.5.3)

\[
    E_1(t) = \left[ \frac{(c_1 c_2)^{\frac{q}{p}} Q_3(t)}{1 - \frac{q}{p} (c_1 c_2)^{\frac{q}{p}} \int_0^t R(s)Q_3(s)ds} \right]^{\frac{1}{q}} < \infty,
\]

(1.5.4)

for all \( 0 \leq s \leq t, s, t \in \mathbb{R}_+ \), where \( c_1, c_2, M_1, N_1, \alpha \) are nonnegative constants and \( f_1, g_1 \) are nonnegative continuous functions defined on \( \mathbb{R}_+ \) and \( R(t), Q_3(t) \) are same as defined in Theorem 1.4.4.
1.5 Applications

First we discuss the boundedness of solution of nonlinear integral equation (1.5.1).

Suppose that the hypotheses (1.5.2) and (1.5.4) are satisfied and let \( u(t) \) be a solution of (1.5.1). Then from (1.5.1) and (1.5.2), we observe that

\[
|u(t)|^p \leq \left( c_1 + \int_0^t M_1 f_1(s) |u(s)|^p ds \right) \left( c_2 + \int_0^t N_1 g_1(s) |u(s)|^q ds \right),
\]

for all \( t \in \mathbb{R}_+ \). Applying the integral inequality given in Theorem 1.4.4 to (1.5.5) yields

\[
|u(t)| \leq E_1(t), \text{ for all } t \in \mathbb{R}_+.
\]

Thus, a solution \( u(t) \) of (1.5.1) existing on \( \mathbb{R}_+ \) is bounded.

Now, we discuss the behaviour of a solution of nonlinear integral equation (1.5.1) as \( t \to \infty \). Assume that the hypotheses (1.5.3) and (1.5.4) are satisfied, and let \( u(t) \) be a solution of (1.5.1). From (1.5.1) and (1.5.3), we observe that

\[
|u(t)|^p \leq e^{-2\alpha t} \left( c_1 + \int_0^t M_1 f_1(s) |u(s)|^p e^{2\alpha s} ds \right) \left( c_2 + \int_0^t N_1 g_1(s) |u(s)|^q e^{2\alpha s} ds \right).
\]

Multiplying on both sides of (1.5.7) by \( e^{2\alpha t} \), we get

\[
|u(t)|^p e^{2\alpha t} \leq \left( c_1 + \int_0^t M_1 f_1(s) |u(s)|^p e^{2\alpha s} ds \right) \left( c_2 + \int_0^t N_1 g_1(s) |u(s)|^q e^{2\alpha s} ds \right).
\]

Replace \( v(t) \) by \( |u(t)| e^{2\alpha t} \) for \( r > 0 \) in (1.5.8) and applying Theorem 1.4.4 we obtain

\[
v(t) \leq E_1(t) \]

and hence

\[
u(t) \leq E_1(t) e^{-2\alpha t}, \text{ for all } t \in \mathbb{R}_+.
\]

This shows that all the solutions of (1.5.1) approaches zero as \( t \to \infty \).
Example 1.5.2. We calculate the explicit bound on the solution of the following nonlinear integral equation

\[ u^2(t) = \left( 1 + \int_0^t u^2(s) \, ds \right) \left( 1 + \int_0^t u(s) \, ds \right), \tag{1.5.9} \]

where \( u \) is a nonnegative continues function and we assume that a solution \( u(t) \) of (1.5.9) exists on \( \mathbb{R}_+ \).

Applying Theorem 1.4.4 to the equation (1.5.9), we have

\[ u(t) \leq \frac{Q_3(t)}{1 - \frac{1}{2} \int_0^t R(s)Q_3(s) \, ds}, \tag{1.5.10} \]

provided

\[ \left[ \frac{1}{2} \int_0^t R(s)Q_3(s) \, ds \right] < 1, \tag{1.5.11} \]

where \( R(t) \) and \( Q_3(t) \) are defined as in Theorem 1.4.4 and their values are

\[ R(t) = \int_0^t 1 \, ds + \int_0^t 1 \, ds = t + t = 3t \tag{1.5.12} \]

and

\[ Q_3(t) = \exp \left( \frac{1}{2} \int_0^t [1 + 1] \, ds \right) = e^t. \tag{1.5.13} \]

Using (1.5.12) and (1.5.13) in (1.5.11), we get

\[ \left[ \frac{1}{2} \int_0^t R(s)Q_3(s) \, ds \right] = \frac{1}{2} \int_0^t 2se^s \, ds = te^t - e^t + 1. \]

Clearly (1.5.11) holds for \( 0 \leq t < 1 \). Hence the right hand side of (1.5.10) gives the bound on a solution of (1.5.9) in terms of the known quantities as follows:

\[ u(t) \leq \frac{e^t}{1 - \int_0^t se^s \, ds} \leq \frac{1}{1 - t}, \quad \text{for} \quad 0 \leq t < 1. \]
1.5 Applications

Example 1.5.3. Consider the following nonlinear integral equation

\[ u^2(t) = \left(1 + \int_0^t u(s) ds\right) \left(1 + \int_0^t u(s) ds\right), \quad (1.5.14) \]

where \( u \) is a nonnegative continuous functions and assume that every solution \( u(t) \) of (1.5.14) exists on \( \mathbb{R}_+ \). We estimate the bound on a solution of nonlinear integral equation (1.5.14).

Applying Theorem 1.4.3 to equation (1.5.14), we get

\[ u(t) \leq \left[1 + 2 \int_0^t Q_2(s) ds\right] \quad \text{for all } t \in \mathbb{R}_+, \quad (1.5.15) \]

where \( R(t) \) and \( Q_2(t) \) are same as defined in Theorem 1.4.3. In particular

\[ R(t) = \int_0^t ds + \int_0^t ds = 2t \tag{1.5.16} \]

and

\[ Q_2(t) = \exp\left(-\frac{1}{2} \int_0^t 2s ds\right) = e^{-\frac{t^2}{2}}. \tag{1.5.17} \]

Using equations (1.5.16) and (1.5.17) in (1.5.15), we have

\[ u(t) \leq \left[\frac{1 + 2 \int_0^t e^{-\frac{s^2}{2}} ds}{e^{-\frac{t^2}{2}}}\right], \quad \text{for all } t \in \mathbb{R}_+. \]

Example 1.5.4. Here we show that a solution of the following nonlinear integral equation is bounded.

\[ u(t) = 2 + \int_0^t \left[e^s u(s) + \frac{1}{1 + s}\right] ds + \int_0^t e^s \left(\int_0^s \frac{1}{1 + s_1} u^2(s_1) ds_1\right) ds \tag{1.5.18} \]

where \( u(t) \) is defined as in Theorem 1.2.1 and we assume that every solution \( u(t) \) of (1.5.18) exists on \( \mathbb{R}_+ \).

Applying Theorem 1.2.1 to the equation (1.5.18), yields the desired bound for \( u(t) \)

\[
\begin{align*}
\quad u(t) & \leq 2 \exp\left(\int_0^t \left[e^s + \frac{1}{2} k^{-\frac{1}{2}} \frac{1}{1 + s^2}\right] ds\right) \\
& \quad + \int_0^t \left[\frac{1}{1 + s} + \frac{1}{2} k^{-\frac{1}{2}} \frac{1}{1 + s^2}\right] \exp\left(\int_s^t \left[e^\sigma + \frac{1}{2} k^{-\frac{1}{2}} \frac{1}{1 + \sigma^2}\right] d\sigma\right) ds.
\end{align*}
\]
Example 1.5.5. Consider the following equation

\[ u^3(t) = 4 + \int_0^t \left[ e^s u^3(s) + \frac{1}{1+s} \right] ds + \int_0^t e^s \left( \int_0^s \left[ \frac{1}{1+s_1} u(s_1) + e^{-2s_1} \right] ds_1 \right) ds, \tag{1.5.19} \]

where \( u(t) \) is defined as in Theorem 1.2.3 and we assume that every solution \( u(t) \) of (1.5.19) exists on \( \mathbb{R}_+ \).

Applying Theorem 1.2.3 to the equation (1.5.19), we obtain the required bound for \( u(t) \)

\[
u^3(t) \leq 4 \exp \left( \int_0^t \left[ e^s + \frac{1}{3} k^{-\frac{2}{3}} \frac{1}{1+s^2} \right] ds \right) + \int_0^t \left[ \frac{1}{1+s} + e^{-2s} + \frac{2}{3} k \frac{1}{1+s^2} \right] \exp \left( \int_s^t \left[ e^\sigma + \frac{1}{3} k^{-\frac{2}{3}} \frac{1}{1+\sigma^2} \right] d\sigma \right) ds.
\]