Chapter 5

On Some Simultaneous Nonlinear Integral Inequalities

5.1 Introduction

The differential and integral inequalities occupy a very privileged position in the theory of differential and integral equations. On the basis of various motivations, in the recent years simultaneous integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of nonlinear differential, integral equations and science. Some effective integral inequalities are established by Gronwall [28], Bellman [10] and see in [1 12 44 45 48 50] to study differential and integral equations which are further generalized by mathematicians [22 23 35 37 64]. In 1977, D. E. Greene established some important simultaneous integral inequalities and after that B.G. Pachpatte obtained useful generalization of Greene’s inequalities.

In this chapter, we establish some new simultaneous integral inequalities, which can be used as handy tools to study the properties of solutions of system of certain nonlinear integral and differential equations.
We need the following inequalities for further discussion.

**Theorem 5.1.1** (Bellman’s Inequality [10]). Let \( u(t), n(t), f(t) \) be nonnegative continuous functions defined on \( \mathbb{R}_+ \) and \( n(t) \) be a nondecreasing on \( \mathbb{R}_+ \). If
\[
  u(t) \leq n(t) + \int_0^t f(s)u(s)ds, \quad t \in \mathbb{R}_+, \tag{5.1.1}
\]
then
\[
  u(t) \leq n(t) \exp \left( \int_0^t f(s)ds \right), \quad t \in \mathbb{R}_+. \tag{5.1.2}
\]

**Theorem 5.1.2** (Bihari’s Inequality [12]). Let \( u(t) \) and \( f(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \). Let \( w(u) \) be a continuous nondecreasing functions on \( \mathbb{R}_+ \) and \( w(u) > 0 \) on \((0, \infty)\). If
\[
  u(t) \leq k + \int_0^t f(s)w(u(s))ds, \quad \text{for all } t \in \mathbb{R}_+, \tag{5.1.3}
\]
where \( k \) is a nonnegative constant, then for all \( 0 \leq t \leq t_1 \),
\[
  u(t) \leq G^{-1} \left( G(k) + \int_0^t f(s)ds \right), \tag{5.1.4}
\]
where
\[
  G(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad r > 0, r_0 > 0, \tag{5.1.5}
\]
\( G^{-1} \) is the inverse function of \( G \) and \( t_1 \in \mathbb{R}_+ \) chosen such that \( G(k) + \int_0^t f(s)ds \in \text{Dom}(G^{-1}) \), for all \( t \in \mathbb{R}_+ \) lying in the interval \( 0 \leq t \leq t_1 \).

**Theorem 5.1.3** (Pachpatte [50]). Let \( u(t), f(t), g(t), h(t) \) be nonnegative continuous functions on \( \mathbb{R}_+ \). Let \( w(u) \) be a continuous, nondecreasing, submultiplicative functions on \( \mathbb{R}_+ \) and \( w(t) > 0 \) on \((0, \infty)\). If
\[
  u(t) \leq u_0 + g(t) \int_0^t f(s)u(s)ds + \int_0^t h(s)w(u(s))ds, \quad \text{for all } t \in \mathbb{R}_+, \tag{5.1.6}
\]
where \( u_0 \) is a positive constant, then for all \( 0 \leq t \leq t_1 \),

\[
    u(t) \leq a(t)G^{-1}(G(u_0) + \int_0^t h(s)w(a(s))ds),
\]

(5.1.7)

where

\[
    a(t) = 1 + g(t) \int_0^t f(s) \exp \left( \int_s^t g(\sigma)f(\sigma)d\sigma \right) ds,
\]

(5.1.8)

\[
    G(r) = \int_{r_0}^r \frac{ds}{w(s)}, \quad r > 0, r_0 > 0
\]

(5.1.9)

and \( G^{-1} \) is the inverse function of \( G \) and \( t_1 \in \mathbb{R}_+ \) chosen such that \( G(u(0)) + \int_0^t h(s)w(a(s))ds \in \text{Dom}(G^{-1}) \), for all \( t \in \mathbb{R}_+ \) lying in the interval \( 0 \leq t \leq t_1 \).

The following definitions is useful in further discussion:

**Definition 5.1.4.** A function \( f \) is said to be subadditive if \( f(x) + f(y) \geq f(x + y) \) whenever \( x, y, x + y \in I \) and \( f \) is said to be superadditive if \( f(x) + f(y) \leq f(x + y) \) whenever \( x, y, x + y \in I \).

**Definition 5.1.5.** A function \( f \) is said to be submultiplicative if \( f(xy) \leq f(x)f(y) \) whenever \( x, y, xy \in I \).

### 5.2 Simultaneous integral inequalities in one Variable

In this section, we focus on simultaneous integral inequalities in one variable.

**Theorem 5.2.1.** Let \( u(t), v(t), h_i(t)(i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( w(t) \) be a continuous, nondecreasing, superadditive functions on \( \mathbb{R}_+ \) with \( w(.) > 0 \) on \( (0, \infty) \) and \( k_1, k_2 \) be nonnegative constants. If

\[
    u(t) \leq k_1 + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds
\]

(5.2.1)
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\( v(t) \leq k_2 + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds, \) \hspace{1cm} (5.2.2)

for \( t \in \mathbb{R}_+ \), then for \( 0 \leq t \leq t_1 \),

\( u(t), v(t) \leq G^{-1}\left( G(\bar{m}) + \int_0^t h(s)ds \right) \), \hspace{1cm} (5.2.3)

where

\( h(t) = \max\{h_1(t) + h_3(t), h_2(t) + h_4(t)\}, \quad \bar{m} = k_1 + k_2, \) \hspace{1cm} (5.2.4)

\( G \) is as same defined in Theorem 5.1.2, \( G^{-1} \) is the inverse function of \( G \) and \( t_1 \in \mathbb{R}_+ \) chosen such that \( (G(\bar{m}) + \int_0^t h(s)ds) \in \text{Dom}(G^{-1}) \), for all \( t \in \mathbb{R}_+ \) lying in the interval \( 0 \leq t \leq t_1 \).

Proof. From (5.2.1) and (5.2.2), we have

\[
 u(t) + v(t) \leq k_1 + k_2 + \int_0^t [h_1(s) + h_3(s)]w(u(s))ds + \int_0^t [h_2(s) + h_4(s)]w(v(s))ds \\
 \leq k_1 + k_2 + \int_0^t [w(u(s)) + w(v(s))]h(s)ds \\
 \leq \bar{m} + \int_0^t w(u(s) + v(s))h(s)ds. \] \hspace{1cm} (5.2.5)

Substituting \( z(t) = u(t) + v(t) \) in (5.2.5), we get

\( z(t) \leq \bar{m} + \int_0^t h(s)w(z(s))ds. \) \hspace{1cm} (5.2.6)

Now an application Bihari’s inequality to (5.2.6), we obtain

\( z(t) \leq G^{-1}\left( G(\bar{m}) + \int_0^t h(s)ds \right), \) \hspace{1cm} (5.2.7)

Since \( u(t) + v(t) = z(t) \) and \( u(t), v(t) \) are nonnegative from (5.2.7), we get (5.2.3). This completes the proof. \( \square \)
Corollary 5.2.2. Let $u(t), v(t), h_i(t) (i = 1, 2, 3, 4)$ be nonnegative continuous functions on $\mathbb{R}_+$ and $k_1, k_2$ are nonnegative constants. If

\[ u(t) \leq k_1 + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds, \]

(5.2.8)

\[ v(t) \leq k_2 + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds, \]

(5.2.9)

for all $t \in \mathbb{R}_+$, where $p \geq 1$, then for all $0 \leq t \leq t_1$,

\[ u(t), v(t) \leq G^{-1} \left( G(\bar{m}) + \int_0^t h(s)ds \right)^{\frac{1}{p}}, \]

(5.2.10)

where $G$ is as same defined in Theorem 5.1.2 and $G^{-1}, t_1, h, \bar{m}$ are as same defined in Theorem 5.2.1.

Proof. Inequalities (5.2.8) and (5.2.9), we can write as follows:

\[ u(t) \leq k_1 + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds, \]

(5.2.11)

\[ v(t) \leq k_2 + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds, \]

(5.2.12)

where $w(s) = s^p$. An application of Theorem 5.2.1 to above system, we get (5.2.10).

This completes the proof. \qed

Theorem 5.2.3. Let $u(t), v(t)$ be continuous functions on $\mathbb{R}_+$, $h_i(t) (i = 1, 2, 3, 4), k_1, k_2, w$ be as same defined in Theorem 5.2.1 and $p \geq 1$. If

\[ 1 \leq w^p(t) \leq k_1 + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds \]

(5.2.13)

\[ 1 \leq v^p(t) \leq k_2 + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds, \]

(5.2.14)

for $t \in \mathbb{R}_+$, then for $0 \leq t \leq t_1$,

\[ u(t), v(t) \leq \left[ G^{-1} \left( G(\bar{m}) + \int_0^t h(s)ds \right) \right]^{\frac{1}{p}}, \]

(5.2.15)

where $G$ is as same defined in Theorem 5.1.2 and $G^{-1}, t_1, h, \bar{m}$ are as same defined in Theorem 5.2.1.
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Proof. Adding inequalities (5.2.13) and (5.2.14), we obtain

\[ u^p(t) + v^p(t) \leq \bar{m} + \int_0^t w(u(s) + v(s))h(s)ds. \]  \hspace{1cm} (5.2.16)

Since \( u, v \geq 1 \) and \( p \geq 1 \) from (5.2.16), we have

\[ u^p(t) + v^p(t) \leq \bar{m} + \int_0^t w(u^p(s) + v^p(s))h(s)ds. \]  \hspace{1cm} (5.2.17)

If \( z(t) = u^p(t) + v^p(t) \), then from (5.2.17), we have

\[ z(t) \leq \bar{m} + \int_0^t w(z(s))h(s)ds. \]  \hspace{1cm} (5.2.18)

Now an application of Bihari’s inequality to (5.2.18), we obtain

\[ z(t) \leq G^{-1} \left( G(\bar{m}) + \int_0^t h(s)ds \right). \]  \hspace{1cm} (5.2.19)

The required inequality (5.2.15) follows by using \( u^p(t) + v^p(t) = z(t) \), (5.2.19) and the fact that \( u(t), v(t) \) are nonnegative. This completes the proof. \( \square \)

Corollary 5.2.4. Let \( u(t), v(t), h_i(t)(i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( k_1, k_2 \) be nonnegative constants, \( p \geq 1 \) and \( q \geq 1 \). If

\[ 1 \leq u^p(t) \leq k_1 + \int_0^t h_1(s)u^q(s)ds + \int_0^t h_2(s)v^q(s)ds \]  \hspace{1cm} (5.2.20)

\[ 1 \leq v^p(t) \leq k_2 + \int_0^t h_3(s)u^q(s)ds + \int_0^t h_4(s)v^q(s)ds, \]  \hspace{1cm} (5.2.21)

for all \( t \in \mathbb{R}_+ \), then for \( 0 \leq t \leq t_1 \),

\[ u(t), v(t) \leq \left[ G^{-1} \left( G(\bar{m}) + \int_0^{t_1} h(s)ds \right) \right]^{\frac{1}{p}}, \]  \hspace{1cm} (5.2.22)

where \( G \) is as same defined in Theorem 5.1.2 and \( G^{-1}, t_1, h, \bar{m} \) are as same defined in Theorem 5.2.1.
Proof. From inequalities \ref{5.2.20} and \ref{5.2.21}, we observe that

\begin{align}
  u^p(t) & \leq k_1 + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds \quad \text{(5.2.23)} \\
v^p(t) & \leq k_2 + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds \quad \text{(5.2.24)}
\end{align}

where \( w(s) = s^q \). Applying Theorem \ref{5.2.3} to above system, we get the required inequality \ref{5.2.22}. This completes the proof. \hfill \Box

**Theorem 5.2.5.** Let \( u(t), v(t), f_i(t), h_i(t), (i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( w(t) \) be a continuous, nondecreasing, submultiplicative, superadditive function on \( \mathbb{R}_+ \) with \( w(.) > 0 \) on \( (0, \infty) \) and \( k_1, k_2 \) be nonnegative constants such that \( k_1 + k_2 > 0 \). If

\begin{align}
  u(t) & \leq k_1 + \int_0^t f_1(s)u(s)ds + \int_0^t f_2(s)v(s)ds + \int_0^t h_1(s)w(u(s))ds + \int_0^t h_2(s)w(v(s))ds \\
v(t) & \leq k_2 + \int_0^t f_3(s)u(s)ds + \int_0^t f_4(s)v(s)ds + \int_0^t h_3(s)w(u(s))ds + \int_0^t h_4(s)w(v(s))ds
\end{align}

for \( t \in \mathbb{R}_+ \), then for \( 0 \leq t \leq t_1 \),

\begin{equation}
  u(t), v(t) \leq a(t)G^{-1}\left(G(\bar{m}) + \int_0^t h(s)w(a(s))ds\right), \quad \text{(5.2.27)}
\end{equation}

where

\begin{align}
  f(t) & = \max\left\{f_1(t) + f_3(t), f_2(t) + f_4(t)\right\}, \quad \text{(5.2.28)} \\
a(t) & = 1 + \int_0^t f(s)\exp\left(\int_s^t f(\sigma)d\sigma\right)ds, \quad \text{(5.2.29)}
\end{align}

where \( G \) is as same defined in Theorem \ref{5.1.2}, \( h, \bar{m}, \) are as same defined in Theorem \ref{5.2.2} and \( t_1 \in \mathbb{R}_+ \) chosen so that \( G(\bar{m}) + \int_0^t h(s)w(a)ds \in \text{Dom}(G^{-1}) \) for all \( t \in \mathbb{R}_+ \) lying in the interval \( 0 \leq t \leq t_1 \).

Proof. From \ref{5.2.25} and \ref{5.2.26}, we have

\begin{align}
  u(t) + v(t) & \leq k_1 + k_2 + \int_0^t [f_1(s) + f_3(s)]u(s)ds + \int_0^t [f_2(s) + f_4(s)]v(s)ds
\end{align}
Let $\int^t_0 [h_1(s) + h_3(s)]w(u(s))ds + \int^t_0 [h_2(s) + h_4(s)]w(v(s))ds$

\[\leq k_1 + k_2 + \int^t_0 f(s)[u(s) + v(s)]ds + \int^t_0 h(s)[w(u(s)) + w(v(s))]ds\]

\[\leq k_1 + k_2 + \int^t_0 f(s)[u(s) + v(s)]ds + \int^t_0 h(s)w(u(s) + v(s))ds\]

\[\leq \bar{m} + \int^t_0 f(s)[u(s) + v(s)]ds + \int^t_0 h(s)w(u(s) + v(s))ds. \quad (5.2.30)\]

Substitute $z(t) = u(t) + v(t)$ in (5.2.30), we have

\[z(t) \leq \bar{m} + \int^t_0 f(s)z(s)ds + \int^t_0 h(s)w(z(s))ds. \quad (5.2.31)\]

Now an application of Theorem 5.1.3 to (5.2.31), we obtain

\[z(t) \leq a(t)G^{-1}\left(G(\bar{m}) + \int^t_0 h(s)w(a(s))ds\right). \quad (5.2.32)\]

Since $u(t) + v(t) = z(t)$ and $u(t), v(t)$ are nonnegative from (5.2.32), we get the desired bound for $u$ and $v$ given in (5.2.27). This completes the proof. \qed

**Theorem 5.2.6.** Let $u(t), v(t)$ be continuous on $\mathbb{R}_+$ and $p \geq 1$. If

\[1 \leq u^p(t) \leq k_1 + \int^t_0 f_1(s)u(s)ds + \int^t_0 f_2(s)u(s)ds + \int^t_0 h_1(s)w(u(s))ds + \int^t_0 h_2(s)w(v(s))ds\]

\[1 \leq v^p(t) \leq k_2 + \int^t_0 f_3(s)u(s)ds + \int^t_0 f_4(s)u(s)ds + \int^t_0 h_3(s)w(u(s))ds + \int^t_0 h_4(s)w(v(s))ds, \quad (5.2.34)\]

for $t \in \mathbb{R}_+$, then for $0 \leq t \leq t_1$,

\[u(t), v(t) \leq \left[a(t)G^{-1}\left(G(\bar{m}) + \int^t_0 h(s)w(a(s))ds\right)\right]^\frac{1}{p}, \quad (5.2.35)\]

where $w, f, h, a$ are as same defined Theorem 5.2.5 and $G$ is as same defined in Theorem 5.1.2.

**Proof.** Using same argument as used in Theorem 5.2.3 and Theorem 5.2.5, we obtain an estimate

\[u^p(t) + v^p(t) \leq \bar{m} + \int^t_0 f(s)[u^p(s) + v^p(s)]ds + \int^t_0 h(s)w(u^p(s) + v^p(s))ds. \quad (5.2.36)\]
Substitute $z(t) = u^p(t) + v^p(t)$ in (5.2.30), we have

$$z(t) \leq \bar{m} + \int_0^t f(s)z(s)ds + \int_0^t h(s)w(z(s))ds.$$  (5.2.37)

Now an application of Theorem 5.1.3 to an inequality (5.2.37), we obtain

$$z(t) \leq a(t)G^{-1}\left(G(\bar{m}) + \int_0^t h(s)w(a(s))ds\right).$$  (5.2.38)

As $u^p(t) + v^p(t) = z(t)$ and $u(t), v(t)$ are nonnegative from (5.2.38), we get the desired inequality (5.2.35). This completes the proof. □

**Theorem 5.2.7.** Let $u(t), v(t), u'(t), v'(t), h_i(t) (i = 1, 2, 3, 4)$ be nonnegative continuous functions on $\mathbb{R}_+$ and $u(0) + v(0) = 0$. Let $w(t)$ be a continuous, nondecreasing, superadditive, submultiplicative function on $\mathbb{R}_+$ with $w(.) > 0$ on $(0, \infty)$ and $k_1, k_2$ be nonnegative constants. If

$$u'(t) \leq k_1 + \int_0^t h_1(s)w(u(s) + u'(s))ds + \int_0^t h_2(s)w(v(s) + v'(s))ds$$  (5.2.39)

$$v'(t) \leq k_2 + \int_0^t h_3(s)w(u(s) + u'(s))ds + \int_0^t h_4(s)w(v(s) + v'(s))ds,$$  (5.2.40)

then for $0 \leq t \leq t_1$,

$$u(t) + u'(t), v(t) + v'(t) \leq (1 + t)G^{-1}\left(G(\bar{m}) + \int_0^t h(s)w(1 + s)ds\right),$$  (5.2.41)

where $h, \bar{m}$ are as same defined in Theorem 5.2.2, $G$ is as same defined in Theorem 5.1.2 and $t_1 \in \mathbb{R}_+$ chosen such that

$$G(\bar{m}) + \int_0^t h(s)w(1 + s)ds \in Dom(G^{-1}),$$

for all $t \in \mathbb{R}_+$ lying in the interval $0 \leq t \leq t_1$.

**Proof.** Using the same argument used in the Theorem 5.2.5, we obtain

$$u'(t) + v'(t) \leq \bar{m} + \int_0^t h(s)w(u(s) + v(s) + u'(s) + v'(s))ds.$$  (5.2.42)
Substituting \( z(t) = u(t) + v(t) \) in (5.2.42), we have

\[
z'(t) \leq \bar{m} + \int_0^t h(s)w(z(s) + z'(s))ds. \tag{5.2.43}
\]

Now an application of Theorem 2.9.5 of [50] to an inequality (5.2.43), we obtain

\[
z(t) + z'(t) \leq (1 + t)G^{-1} \left( G(\bar{m}) + \int_0^t h(s)w(1 + s)ds \right). \tag{5.2.44}
\]

As \( u(t) + v(t) = z(t) \) and \( u(t), v(t), u'(t), v'(t) \) are nonnegative from (5.2.44), we get (5.2.41). This completes the proof.

**Theorem 5.2.8.** Let \( u(t), v(t), u'(t), v'(t), h_i(t) (i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( \mathbb{R}_+ \), \( w(t) \) be a continuous, nondecreasing, superadditive function on \( \mathbb{R}_+ \) with \( w(.) > 0 \) on \( (0, \infty) \) and \( k_1, k_2 \) be nonnegative constants. If

\[
u'(t) \leq k_1 + \int_0^t h_1(s)w(u(s) + u'(s))ds + \int_0^t h_2(s)w(v(s) + v'(s))ds \tag{5.2.45}
\]

\[
v'(t) \leq k_2 + \int_0^t h_3(s)w(u(s) + u'(s))ds + \int_0^t h_4(s)w(v(s) + v'(s))ds, \tag{5.2.46}
\]

for \( t \in \mathbb{R}_+ \), then for \( 0 \leq t \leq t_1 \),

\[
u(t), v(t) \leq \bar{m} + \int_0^t h(s)w \left( \Omega^{-1} \left[ \Omega(\bar{m} + u(0)) + \int_0^s h(\sigma)d\sigma \right] \right) ds, \tag{5.2.47}
\]

where \( 1 \leq h, \bar{m} \) are as same defined in Theorem 5.2.2

\[
\Omega(r) = \int_{r_0}^r \frac{ds}{s + w(s)}, \quad r > 0, r_0 > 0 \tag{5.2.48}
\]

and \( t_1 \in \mathbb{R}_+ \) chosen such that

\[
\Omega(\bar{m} + u(0)) + \int_0^t h(\sigma)d\sigma \in Dom(\Omega^{-1}),
\]

for all \( t \in \mathbb{R}_+ \) lying in the interval \( 0 \leq t \leq t_1 \).
Proof. Adding inequalities (5.2.45) and (5.2.46), we obtain
\[ u'(t) + v'(t) \leq \bar{m} + \int_0^t h(s)w(u(s) + v(s) + u'(s) + v'(s))ds. \] (5.2.49)

Substitute \( z(t) = u(t) + v(t) \) in (5.2.49), we have
\[ z'(t) \leq \bar{m} + \int_0^t h(s)w(z(s) + z'(s))ds. \] (5.2.50)

Now an application of Theorem 2.9.3 of [50] to (5.2.50), we obtain
\[ z(t) \leq \Omega^{-1} \left( \Omega(\bar{m} + u(0)) + \int_0^t h(\sigma)d\sigma \right). \] (5.2.51)

As \( u(t) + v(t) = z(t) \) and \( u'(t), v'(t) \) are nonnegative from (5.2.51), we get (5.2.47).
This completes the proof. \[ \square \]

5.3 Simultaneous integral inequalities in two Variables

In this section, we presents some simultaneous integral inequalities in two variables.

The following Theorems are useful in our main result.

Theorem 5.3.1 (Wendroff’s inequality [50]). Let \( u(x, y), n(x, y), f(x, y) \) be a nonnegative continuous functions defined for \( x, y \in \mathbb{R}_+ = [0, \infty) \) and \( n(x, y) \) be a nondecreasing in each variable \( x, y \in \mathbb{R}_+ \), if
\[ u(x, y) \leq n(x, y) + \int_0^x \int_0^y f(s, t)u(s, t)dsdt, \text{ for } x, y \in \mathbb{R}_+, \] (5.3.1)
then
\[ u(x, y) \leq n(x, y) \exp \left( \int_0^x \int_0^y f(s, t)dsdt \right), \text{ } x, y \in \mathbb{R}_+. \] (5.3.2)
Theorem 5.3.2 (Byung-IL Kim [14]). Let \( u(x, y), a(x, y), b(x, y), c(x, y) \) be a nonnegative continuous functions defined for \( x \geq 0, y \geq 0 \), and let \( a(x, y) \) be a nondecreasing in each variable \( x \geq 0, y \geq 0 \). Suppose that
\[
    u(x, y) \leq a(x, y) + \int_0^x b(s, y)u^p(s, y)ds + \int_0^y c(s, t)u^p(s, t)dsdt, \tag{5.3.3}
\]
for \( x, y \in \mathbb{R}_+ \), where \( p \geq 0, p \neq 1 \) be a constant and \( \int_0^x b(s, y)u^p(s, y)ds \) be nondecreasing in \( y \geq 0 \). Then
\[
    u(x, y) \leq \left[ a^p(x, y) + q \left( \int_0^x b(s, y)ds + \int_0^y f(x, y)dt ds \right) \right]^{\frac{1}{q}}, \tag{5.3.4}
\]
for \( x, y \in \mathbb{R}_+ \), where \( q = 1 - p \).

Now we presents our main result in the following Theorems.

Theorem 5.3.3. Let \( u(x, y), v(x, y), h_i(x, y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions for \( x, y \in \mathbb{R}_+ \), \( k_1, k_2 \) be nonnegative constants and if
\[
1 \leq u^p(x, y) \leq k_1 + \int_0^x \int_0^y h_1(s, t)u(s, t)dsdt + \int_0^x \int_0^y h_2(s, t)v(s, t)dsdt \tag{5.3.5}
\]
\[
1 \leq v^p(x, y) \leq k_2 + \int_0^x \int_0^y h_3(s, t)u(s, t)dsdt + \int_0^x \int_0^y h_4(s, t)v(s, t)dsdt, \tag{5.3.6}
\]
for \( x, y \in \mathbb{R}_+ \), then
\[
    u^p(x, y), v^p(x, y) \leq \bar{m} \exp \left( \int_0^x \int_0^y h(s, t)dsdt \right), \tag{5.3.7}
\]
for \( x, y \in \mathbb{R}_+ \), where
\[
h(x, y) = \max \{ h_1(x, y) + h_3(x, y), h_2(x, y) + h_4(x, y) \}, \quad \bar{m} = k_1 + k_2. \tag{5.3.8}
\]
and \( p \geq 1 \).

Proof. Adding inequalities (5.3.5) and (5.3.6), we have
\[
u^p(x, y) + v^p(x, y) \leq k_1 + k_2 + \int_0^x \int_0^y [h_1(s, t) + h_3(s, t)]u^p(s, t)dsdt
\]
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\[ + \int_0^x \int_0^y [h_2(s, t) + h_4(s, t)] v^p(s, t) ds dt \]
\[ \leq k_1 + k_2 + \int_0^x \int_0^y [u^p(s, t) + v^p(s, t)] h(s, t) ds dt \]
\[ \leq \bar{m} + \int_0^x \int_0^y [u^p(s, t) + v^p(s, t)] h(s, t) ds dt. \]  
(5.3.9)

Substituting \( z(x, y) = u^p(x, y) + v^p(x, y) \) in (5.3.9), we have

\[ z(x, y) \leq \bar{m} + \int_0^x \int_0^y z(s, t) h(s, t) ds dt. \]  
(5.3.10)

Now an application of Theorem 5.3.1 to (5.3.10), we get

\[ z(x, y) \leq \bar{m} \exp \left( \int_0^x \int_0^y h(s, t) ds dt \right). \]  
(5.3.11)

As \( u(x, y), v(x, y) \) are nonnegative continuous functions and \( z(x, y) = u^p(x, y) + v^p(x, y) \) from (5.3.11), we get (5.3.7). The proof is complete. \( \square \)

**Theorem 5.3.4.** Let \( u(x, y), v(x, y), h_i(x, y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions for \( x, y \in \mathbb{R}_+ \) \( k_1, k_2 \) be nonnegative constants and

\[ u(x, y) \leq k_1 + \int_0^x \int_0^y h_1(s, t) u^p(s, t) ds dt + \int_0^x \int_0^y h_2(s, t) v^p(s, t) ds \]  
(5.3.12)

\[ v(x, y) \leq k_2 + \int_0^x \int_0^y h_3(s, t) u^p(s, t) ds dt + \int_0^x \int_0^y h_4(s, t) v^p(s, t) ds dt. \]  
(5.3.13)

If \( \bar{m}^{1-p} > (p - 1) \int_0^x \int_0^y h(s, t) ds dt, \) for \( x, y \in \mathbb{R}_+ \), then

\[ u(x, y), v(x, y) \leq \left( \bar{m}^{1-p} - (p - 1) \left( \int_0^x \int_0^y h(s, t) ds dt \right) \right)^{\frac{1}{1-p}}, \]  
(5.3.14)

for \( x, y \in \mathbb{R}_+ \), where \( \bar{m}, h(x, y) \) are as same defined in (5.3.8) and \( p > 1 \).

**Proof.** From (5.3.12) and (5.3.13), we have

\[ u(x, y) + v(x, y) \leq \bar{m} + \int_0^x \int_0^y [u^p(s, t) + v^p(s, t)] h(s, t) ds dt \]
\[ \leq \bar{m} + \int_0^x \int_0^y [u(s, t) + v(s, t)]^p h(s, t) ds dt. \]  
(5.3.15)
Substituting \( z(x, y) = u(x, y) + v(x, y) \) in (5.3.15), we get
\[
z(x, y) \leq \bar{m} + \int_0^x \int_0^y z^p(s, t)h(s, t)dsdt.
\] (5.3.16)

Now from Theorem 5.3.2 and (5.3.16), we have
\[
z(x, y) \leq \left( \bar{m}^{1-p} - (p - 1) \left( \int_0^x \int_0^y h(s, t)dsdt \right) \right)^{\frac{1}{1-p}}.
\] (5.3.17)

As \( u(x, y), v(x, y) \) are nonnegative and \( z(x, y) = u(x, y) + v(x, y) \) from (5.3.17), we get (5.3.14). The proof is complete.

**Theorem 5.3.5.** Let \( u(x, y), v(x, y), h_i(x, y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions for \( x, y \in \mathbb{R}_+ \), \( k_1, k_2 \) be nonnegative constants and
\[
u(x, y) \leq \left( k_2 + \int_0^x \int_0^y h_3(s, t)u(s, t)dsdt + \int_0^x \int_0^y h_4(s, t)v(s, t)dsdt \right)^p.
\] (5.3.19)

If \( \bar{m}^{1-p} > (p - 1) \int_0^x \int_0^y h(s, t)dsdt \) for \( x, y \in \mathbb{R}_+ \), then
\[
u(x, y) \leq \left( \bar{m}^{1-p} - (p - 1) \left( \int_0^x \int_0^y h(s, t)dsdt \right) \right)^{\frac{p}{1-p}}.
\] (5.3.20)

for \( x, y \in \mathbb{R}_+ \), where \( \bar{m} \) and \( h(x, y) \) are as same defined in (5.3.8) and \( p > 1 \).

**Proof.** From (5.3.18) and (5.3.19), we have
\[
u(x, y) \leq \left( k_1 + k_2 + \int_0^x \int_0^y [h_1(s, t) + h_3(s, t)]u(s, t)dsdt \right)^p
\]
\[
+ \int_0^x \int_0^y [h_2(s, t) + h_4(s, t)]v(s, t)dsdt \right)^p
\]
\[
\leq \left( \bar{m} + \int_0^x \int_0^y [u(s, t) + v(s, t)]h(s, t)dsdt \right)^p.
\] (5.3.21)

Substituting \( z(x, y) = u(x, y) + v(x, y) \) in (5.3.21), we have
\[
z(x, y) \leq \left( \bar{m} + \int_0^x \int_0^y z(s, t)h(s, t)dsdt \right)^p.
\] (5.3.22)
Define a function $\bar{z}(x, y)$ by
\[
\bar{z}(x, y) = \bar{m} + \int_0^x \int_0^y z(s, t)h(s, t)\, ds\, dt,
\]
then $z(x, y) \leq \bar{z}^p(x, y)$ and
\[
\bar{z}(x, y) \leq \bar{m} + \int_0^x \int_0^y \bar{z}^p(s, t)h(s, t)\, ds\, dt. \tag{5.3.23}
\]

Applying Theorem 5.3.2 to (5.3.23), we get
\[
z^{\frac{1}{p}}(x, y) \leq \bar{z}(x, y) \leq \left( \bar{m}^{1-p} - (p - 1) \left( \int_0^x \int_0^y h(s, t)\, ds\, dt \right) \right)^{\frac{1}{1-p}}. \tag{5.3.24}
\]

As $u(x, y), v(x, y)$ are nonnegative and $z(x, y) = u(x, y) + v(x, y)$ from (5.3.24), we get (5.3.20). The proof is complete.

**Theorem 5.3.6.** Let $u(x, y), v(x, y), h_i(x, y) (i = 1, 2, 3, 4)$ be nonnegative continuous functions for $x, y \in \mathbb{R}_+$, $k_1, k_2$ be nonnegative constants and
\[
1 \leq u^p(x, y) \leq k_1 + \int_0^x \int_0^y h_1(s, t)u^q(s, t)\, ds\, dt + \int_0^x \int_0^y h_2(s, t)v^q(s, t)\, ds
\]
\[
1 \leq v^p(x, y) \leq k_2 + \int_0^x \int_0^y h_3(s, t)u^q(s, t)\, ds\, dt + \int_0^x \int_0^y h_4(s, t)v^q(s, t)\, ds\, dt. \tag{5.3.25}
\]

If $\bar{m}^{1-q} > (q - 1) \int_0^x \int_0^y h(s, t)\, ds\, dt$ for $x, y \in \mathbb{R}_+$, then
\[
u(x, y), v(x, y) \leq \left( \bar{m}^{1-q} - (q - 1) \left( \int_0^x \int_0^y h(s, t)\, ds\, dt \right) \right)^{\frac{p}{1-q}}. \tag{5.3.26}
\]

for $x, y \in \mathbb{R}_+$, where $\bar{m}$, $h(x, y)$ are as same defined in (5.3.8), $p > 1$ and $q > 1$.

**Proof.** From inequalities (5.3.25) and (5.3.26), we observe that
\[
u^p(x, y) + v^p(x, y) \leq \bar{m} + \int_0^x \int_0^y [u^q(s, t) + v^q(s, t)]h(s, t)\, ds\, dt \leq \bar{m} + \int_0^x \int_0^y [u^p(s, t) + v^p(s, t)]q h(s, t)\, ds\, dt. \tag{5.3.28}
\]
Substituting \( z(x, y) = u^p(x, y) + v^p(x, y) \) in (5.3.28), we get

\[
z(x, y) \leq \bar{m} + \int_0^x \int_0^y z^q(s, t)h(s, t)dsdt.
\] (5.3.29)

Now from Theorem 5.3.2 and (5.3.29), we get

\[
z(x, y) \leq \left( \bar{m} - q - (q - 1) \left( \int_0^x \int_0^y h(s, t)dsdt \right) \right)^{\frac{1}{1-q}}.
\] (5.3.30)

As \( u(x, y), v(x, y) \) are nonnegative and \( z(x, y) = u^p(x, y) + v^p(x, y) \) from (5.3.30), we get (5.3.27). The proof is complete.

Theorem 5.3.7. Let \( u(x, y), v(x, y), h_i(x, y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( \mathbb{R}_+ \). Let \( w' \) be a continuous, nonnegative, superadditive functions on \( \mathbb{R}_+ \) and \( w > 0 \) on \((0, \infty)\). If

\[
\begin{align*}
u(x, y) &\leq a(x) + \int_0^x \int_0^y h_1(s, t)w(u(s, t))dsdt + \int_0^x \int_0^y h_2(s, t)w(v(s, t))dsdt \quad (5.3.31) \\
v(x, y) &\leq b(y) + \int_0^x \int_0^y h_3(s, t)w(u(s, t))dsdt + \int_0^x \int_0^y h_4(s, t)w(v(s, t))dsdt, \quad (5.3.32)
\end{align*}
\]

for \( x, y \in \mathbb{R}_+ \), where \( a(x), b(y) > 0, a'(x) \geq 0, b'(y) \geq 0 \) continuous function for \( x, y \in \mathbb{R}_+ \), then for \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \),

\[
u(x, y), v(x, y) \leq G^{-1} \left[ G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))}ds + \int_0^x \int_0^y h(s, t)dsdt \right],
\] (5.3.33)

where \( h(x, y) \) is as same defined in (5.3.8), \( G \) is as same defined in Theorem 5.1.2 and \( x_1, y_1 \) are chosen such that

\[
G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))}ds + \int_0^x \int_0^y h(s, t)dsdt \in \text{Dom}(G^{-1})
\]

for all \( x, y \in \mathbb{R}_+ \) lying in the subinterval \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \) of \( R_+ \).

Proof. Adding (5.3.31) and (5.3.32) and using fact that \( w \) is a superadditive, we observe that

\[
u(x, y) + v(x, y) \leq a(x) + b(y) + \int_0^x \int_0^y \left[ w(u(s, t)) + w(v(s, t)) \right]h(s, t)dsdt
\]
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\[ \leq a(x) + b(y) + \int_0^x \int_0^y [w(u(s, t) + v(s, t))] h(s, t) ds dt. \quad (5.3.34) \]

Substituting \( z(x,y) = u(x,y) + v(x,y) \) in (5.3.34), we have

\[ z(x, y) \leq a(x) + b(y) + \int_0^x \int_0^y h(s, t) w(z(s, t)) ds dt. \quad (5.3.35) \]

Now an application of Theorem 5.2.1 [50] to (5.3.35), we obtain

\[ z(t) \leq G^{-1} \left[ G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))} ds + \int_0^x \int_0^y h(s, t) ds dt \right]. \quad (5.3.36) \]

As \( u(x, y) + v(x, y) = z(x, y) \) and \( u(x, y), v(x, y) \) are nonnegative from (5.3.36), we get (5.3.33). The proof is complete. \( \square \)

**Theorem 5.3.8.** Let \( u(x, y), v(x, y), h_i(x, y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( x, y \in \mathbb{R}_+ \). Let \( w' \) be a continuous, nonnegative, superadditive functions on \( \mathbb{R}_+ \) and \( w > 0 \) on \( (0, \infty) \). If

\[ 1 \leq u^p(x, y) \leq a(x) + \int_0^x \int_0^y h_1(s, t) w(u(s, t)) ds dt + \int_0^x \int_0^y h_2(s, t) w(v(s, t)) ds dt, \quad (5.3.37) \]

\[ 1 \leq v^p(x, y) \leq b(y) + \int_0^x \int_0^y h_3(s, t) w(u(s, t)) ds dt + \int_0^x \int_0^y h_4(s, t) w(v(s, t)) ds dt, \quad (5.3.38) \]

for \( x, y \in \mathbb{R}_+ \), where \( a(x), b(y) \) are as same defined in Theorem 5.3.7, then for \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \),

\[ u^p(x, y), v^p(x, y) \leq G^{-1} \left[ G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))} ds + \int_0^x \int_0^y h(s, t) ds dt \right], \quad (5.3.39) \]

where \( h(x, y) \) is as same defined in (5.3.8) and \( G \) is as same defined in Theorem 5.1.2.

**Proof.** Adding (5.3.37) and (5.3.38), and using fact that \( w \) is a superadditive and nondecreasing, we obtain

\[ u^p(x, y) + v^p(x, y) \leq a(x) + b(y) + \int_0^x \int_0^y w(u^p(s, t) + v^p(s, t)) h(s, t) ds dt. \quad (5.3.40) \]
Substituting \( z(x,y) = u^p(x,y) + v^p(x,y) \) in (5.3.40), we have

\[
z(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y h(s,t)w(z(s,t))dsdt.
\] (5.3.41)

Now an application of Theorem 5.2.1 [50] to (5.3.41), we obtain

\[
z(x,y) \leq G^{-1}\left[ G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))}ds + \int_0^x \int_0^y h(s,t)dsdt \right].
\] (5.3.42)

As \( u^p(x,y) + v^p(x,y) = z(x,y) \) and \( u(x,y), v(x,y) \) are nonnegative from (5.3.42), we obtain (5.3.39). The proof is complete.

\[\square\]

**Theorem 5.3.9.** Let \( u(x,y), v(x,y), h_i(x,y)(i = 1, 2, 3, 4) \) be nonnegative continuous functions on \( x, y \in \mathbb{R}_+ \). Let \( w' \) be a continuous, nonnegative, superadditive functions on \( \mathbb{R}_+ \) and \( w > 0 \) on \( (0, \infty) \). If

\[
u(x,y) \leq b(x,y) + \int_0^x \int_0^y h_3(s,t)w(u(s,t))dsdt + \int_0^x \int_0^y h_4(s,t)w(v(s,t))dsdt,
\] (5.3.44)

for \( x, y \in \mathbb{R}_+ \), where \( a(x,y), b(x,y) \) are positive continuous function for \( x, y \in \mathbb{R}_+ \) and all partial derivative of \( a, b \) are nonnegative continuous on \( \mathbb{R}_+ \), then for \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \),

\[
u(x,y), v(x,y) \leq G^{-1}\left[ G(m(x,y)) + \int_0^x \int_0^y h(s,t)dsdt \right],
\] (5.3.45)

where \( h(x,y) \) is as same defined in (5.3.8), \( G \) is as same defined in Theorem 5.1.2, \( m(x,y) = a(x,y) + b(x,y) \) and \( x_1, y_1 \in \mathbb{R}_+ \) are chosen so that \( G(m(x,y)) + \int_0^x \int_0^y h(s,t)dsdt \in \text{Dom}(G^{-1}) \) for all \( x, y \in \mathbb{R}_+ \) lying in the subinterval \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \) of \( \mathbb{R}_+ \).

**Proof.** The proof of Theorem 4.2.5 is similar to Theorem 4.2.3 and we omit the proof here. \[\square\]
5.4 Applications

In the following sections, we illustrate the applications of the inequalities obtained in previous sections.

5.4.1 Simultaneous integral equations in one variable

Example 5.4.1. Consider the following general simultaneous integral equation

\[ u(t) = a(t) + \int_0^t F(x, s, u(s), v(s))ds \quad (5.4.1) \]
\[ v(t) = b(t) + \int_0^t G(t, s, u(s), v(s))ds, \quad (5.4.2) \]

for \( t \in \mathbb{R}_+ \), where \( u, v, a, b, F \) are continuous functions for \( t \in \mathbb{R}_+ \). Suppose that the functions \( a, b, F, G \) in equations (5.4.1)-(5.4.2) satisfy the following conditions:

\[ |a(t)| \leq k_1, \quad |b(t)| \leq k_2, \quad (5.4.3) \]
\[ |F(t, s, u, v)| \leq h_1(s)w(|u|) + h_2(s)w(|v|), \quad (5.4.4) \]
\[ G(t, s, u, v) \leq h_3(s)w(|u|) + h_4(s)w(|v|), \quad (5.4.5) \]

where \( h_i(t)(i = 1, 2, 3, 4), k_1, k_2 \) and \( w \) are as same defined in Theorem 5.2.1 and \((u, v)\) be a solution of (5.4.1) and (5.4.2), then we can obtain an explicit bound for \((u, v)\).

From (5.4.1)-(5.4.5), we observe that

\[ |u(t)| \leq k_1 + \int_0^t h_1(s)w(|u(s)|)ds + \int_0^t h_2(s)w(|v(s)|)ds, \quad (5.4.6) \]
\[ |v(t)| \leq k_2 + \int_0^t h_3(s)w(|u(s)|)ds + \int_0^t h_4(s)w(|v(s)|)ds. \quad (5.4.7) \]

An application of Theorem 5.2.1 to above system, we get

\[ |u(t)|, |v(t)| \leq G^{-1} \left( G(\tilde{m}) + \int_0^t h(s)ds \right), \quad (5.4.8) \]

where \( G, \tilde{m}, h \) are as same defined in Theorem 5.2.1.
Example 5.4.2. Using the Corollary 5.2.2, we show that a solution of the following system is bounded.

\[
\begin{align*}
  u(t) &= 2 + \int_0^t \sin s \, u^2(s) \, ds + \int_0^t v^2(s) \, ds \\
  v(t) &= 1 + \int_0^t \cos s \, u^2(s) \, ds + \int_0^t v^2(s) \, ds,
\end{align*}
\]

where \(u(t), v(t)\) are as same defined in Corollary 5.2.2 and we assume that a solution \((u(t), v(t))\) of (5.4.9)-(5.4.10) exists.

Applying Corollary 5.2.2 to (5.4.9)-(5.4.10), we get

\[
\begin{align*}
  u(t), v(t) &< G^{-1}\left( G(\bar{m}) + \int_0^t h(s) \, ds \right) \\
  &= G^{-1}\left( G(3) + \int_0^t 2ds \right) \\
  &= G^{-1}\left( \frac{1}{r_0} - \frac{1}{3} + 2t \right) \\
  &= \frac{3}{1 - 6t}, \quad \text{for} \quad 0 \leq t < \frac{1}{6}.
\end{align*}
\]

Example 5.4.3. We calculate an explicit bound on a solution of the system of nonlinear integral equations

\[
\begin{align*}
  u(t) &= 3 + \int_0^t \sin s \, e^{u(s)} \, ds + \int_0^t e^{v(s)} \, ds \\
  v(t) &= 4 + \int_0^t e^{u(s)} \, ds + \int_0^t \cos s \, e^{v(s)} \, ds,
\end{align*}
\]

where \(u(t), v(t)\) are as same defined in Theorem 5.2.1 and we assume that a solution \((u(t), v(t))\) of (5.4.11)-(5.4.12) exists.

Applying Theorem 5.2.1 to (5.4.11)-(5.4.12), we get

\[
\begin{align*}
  u(t), v(t) &< G^{-1}\left( G(\bar{m}) + \int_0^t h(s) \, ds \right),
\end{align*}
\]

where \(G\) is as same defined in Theorem 5.1.2 and \(h, \bar{m}\), are as same defined in Theorem 5.2.1.
\[ G^{-1} \left( G(t) + \int_0^t 2ds \right) = G^{-1} \left( \frac{1}{r_0} - \frac{1}{r} + 2t \right) = \frac{7}{1-14t}, \quad \text{for} \quad 0 \leq t < \frac{1}{14}. \]

### 5.4.2 Simultaneous integral equation in two variables

**Example 5.4.4.** Consider the following general simultaneous integral equation

\[
u(x, y) = a(x, y) + \int_0^x \int_0^y F(x, y, s, t, u(s, t), v(s, t))ds dt \tag{5.4.14}
\]

\[
v(x, y) = b(x, y) + \int_0^x \int_0^y G(x, y, s, t, u(s, t), v(s, t))ds dt, \tag{5.4.15}
\]

where \(a, b, u, v \in C[\Delta, \mathbb{R}], F, G \in C[\Delta^2 \times \mathbb{R} \times \mathbb{R}], \Delta^2 = \Delta \times \Delta\). Suppose that the functions \(F, G\) in equations \(5.4.14\)-\(5.4.15\) satisfy the following conditions:

\[
|F(x, y, s, t, u, v)| \leq h_1(t, s)w(|u|) + h_2(t, s)w(|v|), \tag{5.4.16}
\]

\[
|G(x, y, s, t, u, v)| \leq h_3(t, s)w(|u|) + h_4(t, s)w(|v|), \tag{5.4.17}
\]

where \(h_i(x, y)(i = 1, 2, 3, 4)\) and \(w\) are as same defined in Theorem \(5.3.9\). If \((u, v)\) is a solution of \(5.4.14\) and \(5.4.15\), then we obtain an explicit bound on \((u, v)\). From \(5.4.14\)-\(5.4.17\), we obtain

\[
|u(x, y)| \leq |a(x, y)| + \int_0^x \int_0^y h_1(s, t)w(|u(s, t)|)ds dt + \int_0^x \int_0^y h_2(t, s)w(|v(s, t)|)ds dt, \tag{5.4.18}
\]

\[
|v(x, y)| \leq |b(x, y)| + \int_0^x \int_0^y h_3(s, t)w(|u(s, t)|)ds dt + \int_0^x \int_0^y h_4(t, s)w(|v(s, t)|)ds dt. \tag{5.4.19}
\]

An application of Theorem \(5.3.9\) to above system, we get

\[
|u(x, y)|, |v(x, y)| \leq G^{-1} \left[ G(m(x, y)) + \int_0^x \int_0^y h(s, t)ds dt \right], \tag{5.4.20}
\]

where \(m(x, y) = |a(x, y)| + |b(x, y)|\) and \(G, h\) are as same defined in Theorem \(5.3.9\).
Example 5.4.5. Consider the following system of nonlinear integral equations

\[
\begin{align*}
    u(x, y) &= \frac{1}{12} + \int_0^x \int_0^y \cos s \; u^2(s, t) \, ds \, dt + \int_0^x \int_0^y v^2(s, t) \, ds \, dt, \\
    v(x, y) &= \frac{1}{12} + \int_0^x \int_0^y u^2(s, t) \, ds \, dt + \int_0^x \int_0^y \sin s \; v^2(s, t) \, ds \, dt,
\end{align*}
\]

(5.4.21) (5.4.22)

where \(u(t)\) and \(v(t)\) are defined as in Theorem 5.3.4 and we assume that a solution \((u(t), v(t))\) of (5.4.21)-(5.4.22) exists on \(R_+\).

Applying Theorem 5.3.4 to (5.4.21)-(5.4.22), we get

\[
    u(x, y), v(x, y) < \left( \bar{m}^{1-p} - (p-1) \left( \int_0^x \int_0^y h(s, t) \, ds \, dt \right) \right)^{\frac{1}{1-p}},
\]

(5.4.23)

provided

\[
    \bar{m}^{1-p} > (p-1) \int_0^x \int_0^y h(s, t) \, ds \, dt
\]

(5.4.24)

holds, where \(p, \bar{m}, h(x, y)\) are same as defined in Theorem 5.3.4 and their values are \(p = 2, \bar{m} = \frac{1}{6}\) and \(h(t) = 2\).

Substituting the values of \(p, \bar{m}, h\) in (5.4.24), we obtain \(3 > xy\). Clearly (5.4.24) holds for \(x, y \in [0, 1]\). Hence the right hand side of (5.4.23) gives the bound on a solution of (5.4.21)-(5.4.22) in terms of the known quantities

\[
    u(x, y) < \frac{1}{6 - 2xy} \quad \text{and} \quad v(x, y) < \frac{1}{6 - 2xy},
\]

for \(x, y \in [0, 1]\).

Example 5.4.6. We calculate the explicit bound on a solution of the following nonlinear integral equation

\[
    1 \leq u(x, y) = v(x, y) = 1 + \int_0^x \int_0^y u^2(s, t) \, ds \, dt + \int_0^x \int_0^y v^2(s, t) \, ds \, dt,
\]

(5.4.25)

where \(u(x, y), v(x, y)\) are as same defined in Theorem 5.3.7 and we assume that a solution \(u(x, y), v(x, y)\) of (5.4.25) exists for \(x, y \in R_+\).
Equation (5.4.25) we can be written in form of the following system

\[
\begin{align*}
    u(x, y) &= 1 + \int_0^x \int_0^y u^2(s, t) \, ds \, dt + \int_0^x \int_0^y v^2(s, t) \, ds \, dt \\
    v(x, y) &= 1 + \int_0^x \int_0^y u^2(s, t) \, ds \, dt + \int_0^x \int_0^y v^2(s, t) \, ds \, dt.
\end{align*}
\]

(5.4.26)

(5.4.27)

Applying Theorem 5.3.7 to (5.4.26)-(5.4.27), we get

\[
\begin{align*}
    u(x, y), v(x, y) &< G^{-1}
    \left[
    G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w(a(s) + b(0))} \, ds + \int_0^x \int_0^y h(s, t) \, ds \, dt
    \right],
\end{align*}
\]

(5.4.28)

where \( G \) is same as defined in Theorem 5.3.7. To find the value of above estimate, we compare (5.4.26)-(5.4.27) with (5.3.31)-(5.3.32), we get

\[
  \begin{align*}
    w(t) &= t^2, \\
    k_1 &= k_2 = h_1 = h_2 = h_3 = h_4 = 1, \text{ and hence } h(t) = 2 \text{ and } M = 2.
  \end{align*}
\]

Now, Let us find the values of \( G \) and \( G^{-1} \)

\[
G(r) = \int_{r_0=1}^r \frac{ds}{s^2} = 1 - \frac{1}{r} \quad \text{and hence } \quad G^{-1}(t) = \frac{1}{1-t}
\]

(5.4.29)

and

\[
G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w[a(s) + b(0)]} \, ds + \int_0^x \int_0^y h(s, t) \, ds \, dt = G(2) + 2xy = \frac{1}{2} + 2xy.
\]

(5.4.30)

Thus

\[
G^{-1} \left( \frac{1}{2} + 2xy \right) = \frac{1}{1 - \frac{1}{2} - 2xy} = \frac{1}{\frac{1}{2} - 2xy} = \frac{2}{1 - 4xy}.
\]

(5.4.31)

From equations (5.4.29)-(5.4.31) it is clear that

\[
G(a(0) + b(y)) + \int_0^x \frac{a'(s)}{w[a(s) + b(0)]} \, ds + \int_0^x \int_0^y h(s, t) \, ds \, dt \in \text{Dom}(G^{-1})
\]

for \( 0 \leq x < \frac{1}{2}, 0 \leq y < \frac{1}{2} \). Hence

\[
1 \leq u(x, y), v(x, y) < \frac{2}{1 - 4xy}, \quad \text{for } x, y \in \left[ 0, \frac{1}{2} \right).
\]
Future scope for research

It is to be noted that several interesting problem in the field of nonlinear differential, integral and integro-differential equations are not yet studied in the literature. These problems may be studied using various types of inequalities and hence there is still an immense scope to find out many more inequalities. In view of research carried out regarding inequalities, it seems that following inequalities are worth to study.

- Nonlinear retarded type integral inequalities for several variables.
- Nonlinear fractional integral inequalities.
- Nonlinear delay integral inequalities.
- Finite difference inequalities of more general types.
- Nonlinear integral inequalities on time-scale.

We hope that these types inequalities will be devolved in near future and will open new fields of applications.