Chapter 10

Nonhomogeneous Heat Conduction Problem and its Thermal Deflection due to Internal Heat Generation in a Thin Hollow Circular Disk

10.1 Introduction

Nowacki [1] determined the steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper surface with zero temperature on the lower surface and with the circular edge thermally insulated. Roy Choudhury [2] discussed the normal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. This satisfies the time-dependent heat conduction equation. Ootao et al. [3] studied the theoretical analysis of a three-dimensional transient thermal stress problem for a nonhomogeneous hollow circular cylinder due to a moving heat source in the axial direction.
from the inner and outer surfaces. Khobragade et al. [4] solved an inverse axially symmetric quasi-static problem of thermoelasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate and suitable heat exchange conditions are met on the upper and lower surfaces of the plate is solved with the help of a generalized integral transform technique. Recently, Deshmukh et al. [5] studied the two dimensional non-homogeneous boundary value problem of heat conduction and studied the thermal deflection of a thin clamped circular plate due to heat generation.

In this chapter, an attempt is made to solve a nonhomogeneous heat conduction problem in a thin hollow circular disk occupying the space $D: a \leq r \leq b, 0 \leq z \leq h$ under unsteady-state temperature field due to internal heat generation within it and discussed the temperature change and thermal deflection. Initially, the disk is kept at an arbitrary temperature $F(r, z)$. For times $t > 0$ heat is generated within the thin hollow circular disk at a rate of $g(r, z, t)$ Btu/hr ft$^3$, while the boundary surfaces at $(r = a), (r = b), (z = 0)$ and $(z = h)$ are kept temperatures $f_1(z, t)$ and $f_2(z, t), f_3(r, t)$ and $f_4(r, t)$ respectively. The governing heat conduction equation has been solved by using finite Hankel transform and the generalized finite Fourier transform. The results are obtained in series form in terms of Bessels
functions. As a special case different metallic disk have been con-
sidered. The results for temperature change and thermal deflection
have been computed numerically and illustrated graphically.

10.2 Formulation of the problem

Consider a thin hollow circular disk of thickness $h$ and radius $r$ oc-
cupying the space $D$: $a \leq r \leq b$, $0 \leq z \leq h$ under an unsteady
temperature field due to internal heat generation within it, as shown
in figure 10.1. Initially, the disk is kept at an arbitrary temperature
$F(r, z)$. For times $t > 0$ heat is generated within the thin hollow
circular disk at a rate of $g(r, z, t)$ Btu/hr ft$^3$, while the boundary
surfaces at $(r = a)$, $(r = b)$, $(z = 0)$ and $(z = h)$ are kept tempera-
tures $f_1(z, t)$ and $f_2(z, t)$, $f_3(r, t)$ and $f_4(r, t)$ respectively.

Under these realistic prescribed conditions, temperature and ther-
mal deflection in a thin hollow circular disk due to internal heat
generation are required to be determined.

The temperature of the hollow circular disk $T(r, z, t)$ at time $t$ satis-
ifies the differential equation as,

$$
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r, z, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t},
$$

(10.2.1)
with the boundary conditions,

\[ T = f_1(z, t), \quad \text{at } r = a, \text{ for } t > 0, \quad (10.2.2) \]

\[ T = f_2(z, t), \quad \text{at } r = b, \text{ for } t > 0, \quad (10.2.3) \]

\[ T = f_3(r, t), \quad \text{at } z = 0, \text{ for } t > 0, \quad (10.2.4) \]

\[ T = f_4(r, t), \quad \text{at } z = h, \text{ for } t > 0, \quad (10.2.5) \]

and initial condition,

\[ T = F(r, z), \quad \text{in } a \leq r \leq b, \ 0 \leq z \leq h, \text{ for } t = 0, \quad (10.2.6) \]

where \( k, \alpha \) are the thermal conductivity and thermal diffusivity of the material of the hollow circular disk.

\[ \begin{align*}
T = f_1(z, t), & \quad \text{at } r = a, \text{ for } t > 0, \\
T = f_2(z, t), & \quad \text{at } r = b, \text{ for } t > 0, \\
T = f_3(r, t), & \quad \text{at } z = 0, \text{ for } t > 0, \\
T = f_4(r, t), & \quad \text{at } z = h, \text{ for } t > 0, \\
T = F(r, z), & \quad \text{in } a \leq r \leq b, \ 0 \leq z \leq h, \text{ for } t = 0
\end{align*} \]

**Figure 10.1:** Geometry of the heat conduction problem.
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Here the circular disk is assumed sufficiently thin. The differential equation satisfied the deflection function \( w(r, t) \) as defined in [6] as,

\[
\nabla^2 \nabla^2 w = -\frac{1}{(1 - \nu)D} \nabla^2 M_T, \quad (10.2.7)
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (10.2.8)
\]

and \( M_T \) is the thermal moment of the disk, \( \nu \) is the Poisson’s ratio of the disk material, \( D \) is the flexural rigidity of the disk denoted by

\[
D = \frac{Eh^3}{12(1 - \nu^2)}; \quad (10.2.9)
\]

The term \( M_T \) is defined as

\[
M_T = a_t E \int_0^h \left( z - \frac{h}{2} \right) T(r, z, t) dz, \quad (10.2.10)
\]

\( a_t \) and \( E \) are the coefficients of the linear thermal expansion and the Young’s modulus, respectively.

Since the inner and outer edges of the hollow circular disk are clamped,

\[
w = 0 \quad \text{at } r = a \quad \text{and } r = b, \quad (10.2.11)
\]

Initially, \( T = w = F(r, z) \), at \( t = 0 \).

Equations (10.2.1) to (10.2.11) constitute the mathematical formulation of the problem under consideration.
10.3 Solution of the heat conduction problem

To obtain the expression for temperature function $T(r, z, t)$; firstly we define the finite Fourier transform and its inverse transform over the variable $z$ in the range $0 \leq z \leq h$ defined in [7] as,

$$
T(r, \eta_p, t) = \int_{z'=0}^{h} K(\eta_p, z').T(r, z', t).dz'
$$

(10.3.1)

$$
T(r, z, t) = \sum_{n=1}^{\infty} K(\eta_p, z).T(r, \eta_p, t)
$$

(10.3.2)

where

$$
K(\eta_p, z) = \sqrt{\frac{2}{h}} \sin(\eta_p z).
$$

and $\eta_1, \eta_2, \ldots$ are the positive roots of the transcendental equation

$$
\sin(\eta_p h) = 0, \quad p = 1, 2, 3, \ldots
$$

i.e.

$$
\eta_p = \frac{p \pi}{h}, \quad p = 1, 2, 3, \ldots
$$

Applying the finite Fourier transform defined in equation (10.3.1) to equation (10.2.1) and using the conditions (10.2.2)-(10.2.6), one obtains

$$
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \eta_p^2 + \frac{g(r, \eta_p, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
$$

(10.3.3)
with
\[ \bar{T} = \mathcal{F}_1(\eta_p, t), \quad \text{at } r = a, \text{ for } t > 0, \]  
\[ \bar{T} = \mathcal{F}_2(\eta_p, t), \quad \text{at } r = b, \text{ for } t > 0, \]  
\[ \bar{T} = \mathcal{F}(r, \eta_p), \quad \text{in } a \leq r \leq b, \text{ for } t = 0, \]

where \( \bar{T} = T(r, \eta_p, t). \)

Secondly, we define finite Hankel transform and its inverse transform over the variable \( r \) in the range \( a \leq r \leq b \) as defined in [7] respectively as,

\[ \bar{T}(\beta_m, \eta_p, t) = \int_{r'=a}^{b} r'.K_0(\beta_m, r').T(r', \eta_p, t).dr'. \]  
(10.3.7)

\[ T(r, \eta_p, t) = \sum_{m=1}^{\infty} K_0(\beta_m, r).\bar{T} (\beta_m, \eta_p, t) \]  
(10.3.8)

where

\[ K_0(\beta_m, r) = \frac{\pi}{\sqrt{2}} \frac{\beta_m J_0(\beta_m b) Y_0(\beta_m b)}{J_0(\beta_m a) Y_0(\beta_m a)} \left[ \frac{J_0(\beta_m r) Y_0(\beta_m r)}{J_0(\beta_m b) Y_0(\beta_m b)} \right] - \frac{J_0(\beta_m r) Y_0(\beta_m r)}{J_0(\beta_m b) Y_0(\beta_m b)} \]

and \( \beta_1, \beta_2, \beta_3, \ldots \) are the positive root of transcendental equation

\[ \frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)} = 0. \]

Applying the finite Hankel transform defined in equation (10.3.7)
to equation (10.3.3) and using the conditions (10.3.4)-(10.3.6), one obtains

\[
\frac{\partial \overline{T}(\beta_m, \eta_p, t)}{\partial t} + \alpha (\beta_m^2 + \eta_p^2) \overline{T}(\beta_m, \eta_p, t) = A(\beta_m, \eta_p, t) \quad (10.3.9)
\]

\[
\overline{T}(\beta_m, \eta_p, t) = \overline{F}(\beta_m, \eta_p), \quad \text{for } t = 0, \quad (10.3.10)
\]

where

\[
A(\beta_m, \eta_p, t) = \frac{\alpha}{k} f(\beta_m, \eta_p, t) + \alpha \left\{ \frac{dK_0(\beta_m, \eta_p)}{dr} f_1(\eta_p, t) \bigg|_{r=a}^{r=b} - \frac{dK_0(\beta_m, r)}{dr} f_2(\eta_p, t) \bigg|_{r=b}^{r=a} + \frac{dK_0(\eta_p, \beta_m)}{d\beta_m} f_3(\beta_m, t) \bigg|_{z=0}^{z=b} \right\} \quad (10.3.11)
\]

Solution of the equation (10.3.9) is obtained as

\[
\overline{T}(\beta_m, \eta_p, t) = e^{-\alpha(\beta_m^2 + \eta_p^2)t} \left[ \overline{F}(\beta_m, \eta_p) + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} A(\beta_m, \eta_p, t').dt' \right] \quad (10.3.12)
\]

Finally taking inverse finite Hankel transform defined in equation (10.3.8) and inverse finite Fourier transform defined in equation (10.3.2), one obtains the expressions of the temperature \(T(r, z, t)\) as

\[
T(r, z, t) = \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} K(\eta_p, z) K_0(\beta_m, r).e^{-\alpha(\beta_m^2 + \eta_p^2)t} \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').F(r', z').dr'.dz' \right\}
\]

\[
+ \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left[ \frac{\alpha}{k} \int_{r'=a}^{b} \int_{z'=0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').g(r', z', t').dr'.dz' \right]
\]
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\[ + \alpha a K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z'). f_1(z', t') \, dz' \]
\[ - \alpha b K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z'). f_2(z', t') \, dz' \]
\[ + \sqrt{\frac{2}{\pi}} \alpha \eta_p \int_{r'=a}^{b} r'. K_0(\beta_m, r'). f_3(r', t') \, dr' \]
\[ + \sqrt{\frac{2}{\pi}} \alpha \eta_p \cos(\eta_p h) \int_{r'=a}^{b} r'. K_0(\beta_m, r'). f_4(r', t') \, dr' \] \, dt' \}

\[ (10.3.13) \]

QUASI-STATIC THERMAL DEFLECTION

Assume the solution of (10.2.7) satisfying conditions (10.2.11) as

\[ w(r, t) = \sum_{m=1}^{\infty} C_m(t) \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right] \] \[ (10.3.14) \]

where \( \beta_1, \beta_2, \beta_3, \ldots \) are the positive root of transcendental equation

\[ \frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)} = 0. \]

It can be easily shown that

\[ w = 0 \quad \text{at } r = a \quad \text{and } r = b, \] \[ (10.3.15) \]

Hence the solution (10.3.14) satisfies the condition (10.2.11).

Now,

\[ \nabla^2 \nabla^2 w = \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)^2 \sum_{m=1}^{\infty} C_m(t) \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right] \]

\[ (10.3.16) \]
Using the well-known result
\[
\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r) 
\] (10.3.17)

\[
\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r) 
\] (10.3.18)

in equation (10.3.16), one obtains
\[
\nabla^2 \nabla^2 w = \sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right] 
\] (10.3.19)

Using equation (10.3.13) in equation (10.2.10), one obtains
\[
M_T = -\sqrt{\frac{h}{2}} a_t E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h) + 1]}{\eta_p} K_0(\beta_m, r) e^{-\alpha(\beta_m^2 + \eta_p^2) t} 
\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r'. K_0(\beta_m, r'). K(\eta_p, z'). F(r', z'). dr'. dz' 
+ \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2) t'} \left[ \frac{\alpha}{k} \int_{r'=a}^{b} \int_{z'=0}^{h} r'. K_0(\beta_m, r'). K(\eta_p, z'). g(r', z', t'). dr'. dz' 
+ \alpha a K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z'). f_1(z', t'). dz' 
- \alpha b K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z'). f_2(z', t'). dz' 
+ \sqrt{\frac{2}{\pi}} \alpha h. \int_{r'=a}^{b} r'. K_0(\beta_m, r'). f_3(r', t'). dr' 
+ \sqrt{\frac{2}{\pi}} \alpha h. \cos(\eta_p h). \int_{r'=a}^{b} r'. K_0(\beta_m, r'). f_4(r', t'). dr' \right] dt' \right\} 
\] (10.3.20)

\[
\nabla^2 M_T = -\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) \sqrt{\frac{h}{2}} a_t E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h) + 1]}{\eta_p} K_0(\beta_m, r) e^{-\alpha(\beta_m^2 + \eta_p^2) t} 
\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r'. K_0(\beta_m, r'). K(\eta_p, z'). F(r', z'). dr'. dz' \right\} 
\]
+ \int_{r' = 0}^{b} \int_{z' = 0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').g(r', z', t').dr'.dz' \\
+ \alpha.a.K_1(\beta_m, a) \int_{z' = 0}^{h} K(\eta_p, z').f_1(z', t').dz' \\
- \alpha.b.K_1(\beta_m, b) \int_{z' = 0}^{h} K(\eta_p, z').f_2(z', t').dz' \\
+ \sqrt{2 \over \pi} \alpha.\eta_p \int_{r' = a}^{b} r'.K_0(\beta_m, r').f_3(r', t').dr' \\
+ \sqrt{2 \over \pi} \alpha.\eta_p . \cos(\eta_p h) \int_{r' = a}^{b} r'.K_0(\beta_m, r').f_4(r', t').dr' \right) \right) dt' \right) \\
(10.3.21)

solving equation (10.3.21), one obtains

\[ \nabla^2 M_T = \sqrt{\frac{h}{2}} a_i E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h) + 1]}{\eta_p} \beta_m^2 K_0(\beta_m, r).e^{-\alpha(\beta_m^2 + \eta_p^2)t} \]

\[ \times \left\{ \int_{r' = a}^{b} \int_{z' = 0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').F(r', z').dr'.dz' \\
+ \int_{t' = 0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \int_{r' = a}^{b} \int_{z' = 0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').g(r', z', t').dr'.dz' \\
+ \alpha.a.K_1(\beta_m, a) \int_{z' = 0}^{h} K(\eta_p, z').f_1(z', t').dz' \\
- \alpha.b.K_1(\beta_m, b) \int_{z' = 0}^{h} K(\eta_p, z').f_2(z', t').dz' \\
+ \sqrt{2 \over \pi} \alpha.\eta_p \int_{r' = a}^{b} r'.K_0(\beta_m, r').f_3(r', t').dr' \\
+ \sqrt{2 \over \pi} \alpha.\eta_p \cos(\eta_p h) \int_{r' = a}^{b} r'.K_0(\beta_m, r').f_4(r', t').dr' \right) \right) dt' \right) \\
(10.3.22)

Substituting equation (10.3.19) and (10.3.22) into equation (10.2.7), one obtains

\[ \sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right] \]
Solving equation (10.3.23), one obtains

\[
C_m(t) = -\sqrt{\frac{h}{2}} \frac{a_t E h}{(1 - \nu) D} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h) + 1]}{\eta_p} \beta_m^2 \\
\times \left[ 1 - \frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)} \right]^{1/2} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \\
\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').F(r', z').dr'.dz' \\
+ \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left[ \frac{\alpha}{k} \int_{r'=a}^{b} \int_{z'=0}^{h} r'.K_0(\beta_m, r').K(\eta_p, z').g(r', z', t').dr'.dz' \\
+ \alpha_a K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z').f_1(z', t').dz' \\
- \alpha_b K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z').f_2(z', t').dz' \\
+ \sqrt{\frac{2}{\pi}} \alpha \eta_p \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_3(r', t').dr' \\
+ \sqrt{\frac{2}{\pi}} \alpha \eta_p \cos(\eta_p h) \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_4(r', t').dr' \right\} dt' \right\}
\] (10.3.23)
\[ w(r, t) = -\sqrt{\frac{h}{2} a_t E h} \frac{1}{(1 - \nu)D} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos(\eta_p h) + 1}{\eta_p \beta_m^2} \left( e^{-\alpha(\beta_m^2 + \eta_p^2) t} \times \right. \]
\[ \left. \times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' \cdot K_0(\beta_m, r') \cdot K(\eta_p, z').F(r', z').dr'.dz' \right. \right. \]
\[ + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2) t'} \left[ \frac{\alpha}{k} \int_{r'=a}^{b} \int_{z'=0}^{h} r' \cdot K_0(\beta_m, r').K(\eta_p, z').g(r', z', t').dr'.dz' \right. \right. \]
\[ + \alpha a K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z').f_1(z', t').dz' \]
\[ - \alpha b K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z').f_2(z', t').dz' \]
\[ + \sqrt{\frac{2}{\pi}} \alpha \eta_p \int_{r'=a}^{b} r' \cdot K_0(\beta_m, r').f_3(r', t').dr' \]
\[ + \sqrt{\frac{2}{\pi}} \alpha \eta_p \cos(\eta_p h) \int_{r'=a}^{b} r' \cdot K_0(\beta_m, r').f_4(r', t').dr' \] \( \right\} \]
\[ (10.3.25) \]

Finally, substituting equation (10.3.24) in equation (10.3.14), one obtains the expression for the quasi-static thermal deflection \( w(r, t) \) as

The kernel \( K_0(\beta_m, r) \) for use in the equations (10.3.13)-(10.3.25) and it’s derivative are
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\[
K_1(\beta_m, a) = \left. \frac{dK_0(\beta_m, r)}{dr} \right|_{r=a} \\
= -\frac{1}{\sqrt{N}} \frac{1}{J_0(\beta_m b)Y_0(\beta_m b)} \frac{1}{S} \frac{2}{\pi \beta_m a}
\]

\[
K_1(\beta_m, b) = \left. \frac{dK_0(\beta_m, r)}{dr} \right|_{r=b} \\
= -\frac{1}{\sqrt{N}} \frac{1}{J_0(\beta_m b)Y_0(\beta_m b)} \frac{2}{\pi \beta_m b}
\]

where

\[
N = \frac{2}{\pi^2} \frac{1}{\beta_m^2 J_0^2(\beta_m b)Y_0^2(\beta_m b)} \left( 1 - \frac{J_0^2(\beta_m b)}{Y_0^2(\beta_m a)} \right)
\]

\[
S = \frac{J_0(\beta_m a)}{J_0(\beta_m b)} \frac{Y_0(\beta_m a)}{Y_0(\beta_m b)}
\]

### 10.4 Special Case and Numerical Calculations

Setting

\[
f_1(z, t) = f_2(z, t) = (z^2 - h^2)^2 e^{-At}
\]

\[
f_3(r, t) = f_4(r, t) = (r^2 - a^2)^2(r^2 - b^2)^2 e^{-At}
\]

\[
F(r, z) = (r^2 - a^2)^2(r^2 - b^2)^2(z^2 - h^2)^2
\]

\[
g(r, z, t) = g_i \delta(r - r_1) \delta(z - z_1) \delta(t - \tau)
\]

where \( r \) is the radius measured in feet, \( \delta \) is the Derac-delta function and \( A > 0 \).

The heat source \( g(r, z, t) \) is an instantaneous line heat source of strength \( g_i = 50 \text{ Btu/hr.ft}^3 \), situated at center of the hollow circular disk along the radial direction and axial direction and releases its
instantaneously at the time \( t = \tau = 2 \) hr.

Notice that

\[
\int_{z'=0}^{h} K(\eta_p, z'). f_{1,2}(z', t'). dz' = \sqrt{\frac{2}{\pi}} \left\{ \frac{h^4}{\eta_p^3} \cos(\eta_p h) + \frac{4h^2}{\eta_p^3} (2 \cos(\eta_p h) + 1) + \frac{24}{\eta_p^5} (1 - \cos(\eta_p h)) \right\}
\]

(10.4.1)

\[
\int_{r'=a}^{b} r'. K_0(\beta_m, r'). f_{3,4}(r', t'). dr' =
\frac{\pi \sqrt{N} \beta_m^1 J_1(\beta_m a) J_1(\beta_m b) Y_1(\beta_m b) b(40a^2 b^2 \beta_m^4 - 32b^4 \beta_m^4 - 8a^2 \beta_m^4 + 2304b^2 \beta_m^2 - 576a^2 \beta_m^2 - 18432) J_1(\beta_m a)}{8 \left\{ \begin{array}{l}
- b(40a^2 b^2 \beta_m^4 - 32b^4 \beta_m^4 - 8a^2 \beta_m^4 + 2304b^2 \beta_m^2 - 576a^2 \beta_m^2 - 18432) J_1(\beta_m a) \\
\end{array} \right\}}
\]

(10.4.2)

\[
\int_{r'=a}^{b} \int_{z'=0}^{h} r'. K(\beta_m, r'). K(\eta_p, z'). g(r', z', t'). dr'. dz' =
\sqrt{\pi} r_1 \frac{\beta_m J_0(\beta_m b). Y_0(\beta_m b)}{1 - \left( \frac{J_0(\beta_m b)}{J_0(\beta_m a)} \right)^{1/2}} \left[ \frac{J_0(\beta_m r_1)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r_1)}{Y_0(\beta_m b)} \right] \times \sin(\eta_p z_1)
\]

(10.4.3)

**DIMENSION**

The constants associated with the numerical calculation are taken as

Inner radius of a circular disk \( a = 1 \) ft,

Outer radius of a circular disk \( b = 2 \) ft,

Thickness of circular disk \( h = 0.2 \) ft,

Central circular path of disk in radial and axial directions: \( r_1 = 1.5 \) ft, and \( z_1 = 0.2 \) ft

\( t = 2 \) in hours.
MATERIAL PROPERTIES

The numerical calculation has been carried out for a thin hollow circular disk with the material properties as,

<table>
<thead>
<tr>
<th>Material</th>
<th>$k$, Btu/hr ft $^\circ F$</th>
<th>$c_p$, Btu/lb$^\circ F$</th>
<th>$\rho$, lb/ft$^3$</th>
<th>$\alpha$, ft$^2$/hr</th>
<th>$\lambda$, 1/F</th>
<th>$E$, GPa</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum(Al)</td>
<td>117</td>
<td>0.208</td>
<td>169</td>
<td>3.33</td>
<td>$12.84 \times 10^{-6}$</td>
<td>70</td>
<td>0.35</td>
</tr>
<tr>
<td>Copper(Cu)</td>
<td>224</td>
<td>0.091</td>
<td>558</td>
<td>4.42</td>
<td>$9.3 \times 10^{-6}$</td>
<td>117</td>
<td>0.36</td>
</tr>
<tr>
<td>Iron(Fe)</td>
<td>36</td>
<td>0.104</td>
<td>491</td>
<td>0.70</td>
<td>$6.7 \times 10^{-6}$</td>
<td>193</td>
<td>0.21</td>
</tr>
<tr>
<td>Silver(Ag)</td>
<td>242</td>
<td>0.056</td>
<td>655</td>
<td>6.60</td>
<td>$10.7 \times 10^{-6}$</td>
<td>83</td>
<td>0.37</td>
</tr>
</tbody>
</table>

TRANSCENDENTAL ROOTS

The transcendental roots of \( \left( \frac{J_0(b\alpha)}{J_0(b\beta)} - \frac{Y_0(b\alpha)}{Y_0(b\beta)} = 0 \right) \) as defined in [7] are $\beta_1 = 3.1965$, $\beta_2 = 6.3123$, $\beta_3 = 9.4445$, $\beta_4 = 12.5812$, $\beta_5 = 15.7199$.

For convenience, we set

\[
A = 10^4, \quad B = \frac{10^4 a_t Eh}{(1 - \nu) D}
\]

in equations (10.3.13)–(10.3.25).

Numerical variations in radial directions are shown in the figures with help of a computer programme.
10.5 Discussion

In this chapter, we discussed a nonhomogeneous heat conduction problem in a thin hollow circular disk and it’s thermal deflection under unsteady-state temperature field due to internal heat generation within it. As an illustration, we carried out numerical calculations for a thin hollow circular disk made up of different metals viz. Aluminium, Copper, Iron, Silver and examined the thermoelastic behavior in the state for the temperature and thermal deflection in radial direction.

Figure 10.2, shows the variation of temperature $T$ versus radius $r$, it is clear that temperature is maximum at the inner boundary surface ($r = 1$) and decreases from outer boundary surface with the increase of radius $r$. It becomes zero at the ($r = 1.4$) of the circular disk.

Figure 10.3, shows the variation of thermal deflection versus radius, it is seen that deflection decreases from inner boundary surface to the outer boundary surface. It becomes maximum at the inner boundary surface ($r = 1$) and zero at the outer boundary surface ($r = 2$).

It means we may find out that, temperature and thermal deflection occurs near heat source, due to internal heat generation in a thin
hollow circular disk. The numerical values of the temperature and thermal deflection for the disc of metals Steel, Iron, Aluminum and Copper are in the proportion and follows relation $Iron \leq Aluminum \leq Copper \leq Silver$. From the figures, the copper metal has high thermal conductivity, its deflection is low, where as the Iron metal has low thermal conductivity, its deflection is high. Hence, these values are inversely proportional to their thermal conductivity.

### 10.6 Conclusion

The temperature and thermal deflection of a nonhomogeneous heat conduction problem in a thin hollow circular disk under unsteady-state temperature field due to internal heat generation is presented. The present method is based on the direct method, using the finite Hankel transform and the generalized finite Fourier transform. The numerical results are compared with different metal disks. We conclude that, due to internal heat generation in a thin hollow circular disks, temperature and thermal deflection are inversely proportional to their thermal conductivity.

The results presented here will be useful in engineering problems, particularly in aerospace engineering for stations of a missile body not influenced by nose tapering.
Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (10.3.13)–(10.3.25).

Figure 10.2: Variations of $\frac{T}{A}$ versus $r$.

Figure 10.3: Variations of $\frac{w(r,t)}{B}$ versus $r$. 

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References


