Chapter 7

Quasi-Static Thermal Stresses in a Thick Rectangular Plate

7.1 Introduction

In this chapter, an attempt is made to solve the quasi-static thermal stresses in a thick rectangular plate. The plate is subjected to constant heat supply \( T_0 \) on the extreme edges \( (x = a, y = b, z = c) \) where as the initial edges \( (x = 0, y = 0, z = 0) \) are thermally insulated. Initially the plate is kept at zero temperature. The governing heat conduction equation has been solved by using triple integral transform technique. The results obtained in series form in terms of circular functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

### 7.2 Formulation of the problem

Consider a thick rectangular plate occupying the space \( D: 0 \leq x \leq a, 0 \leq y \leq b \) and \( 0 \leq z \leq c \) is considered. Initially the plate is kept at zero temperature. The plate is subjected to constant heat supply at \( x = a, y = b, \) and \( z = c \) where as the edges \( x = 0, y = 0 \) and \( z = 0 \) are thermally insulated. Under these realistic prescribed conditions, the quasi-static thermal stresses in a thick rectangular plate are required to be determined.

The temperature \( T(x, y, z, t) \) of the thick rectangular plate satisfies the heat conduction equation as,

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t} \quad (7.2.1)
\]
subject to the initial condition

\[ T(x, y, z, t) = 0, \quad \text{at } t = 0, \ 0 \leq x \leq a, \ 0 \leq y \leq b \text{ and } 0 \leq z \leq c \]

(7.2.2)

and the boundary conditions are

\[ T(x, y, z, t) = T_0, \quad \text{at } x = a, \ 0 \leq y \leq b, \ 0 \leq z \leq c \quad (7.2.3) \]
\[ T(x, y, z, t) = T_0, \quad \text{at } y = b, \ 0 \leq x \leq a, \ 0 \leq z \leq c \quad (7.2.4) \]
\[ T(x, y, z, t) = T_0, \quad \text{at } z = c, \ 0 \leq x \leq a, \ 0 \leq y \leq b \quad (7.2.5) \]
\[ \frac{\partial T}{\partial x} = 0, \quad \text{at } x = 0, \ 0 \leq y \leq b, \ 0 \leq z \leq c \quad (7.2.6) \]
\[ \frac{\partial T}{\partial y} = 0, \quad \text{at } y = 0, \ 0 \leq x \leq a, \ 0 \leq z \leq c \quad (7.2.7) \]
\[ \frac{\partial T}{\partial z} = 0, \quad \text{at } z = 0, \ 0 \leq x \leq a, \ 0 \leq y \leq b \quad (7.2.8) \]

where \( k \) is the thermal diffusivity of the material of the plate.

Here the plate is assumed sufficiently thick and considered free from traction. Since the plate is in a plane stress state without bending.

Airy stress function method is applicable to the analytical development of the thermoelastic field. Airy stress function \( U(x, y, z, t) \) which satisfy the following relation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 U = -\alpha E \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T \quad (7.2.9) \]
where $\alpha$ and $E$ are linear coefficient of the thermal expansion, Youngs modulus elasticity of the material of the plate.

The displacement components $u_x$, $u_y$ and $u_z$ in the $X$, $Y$ and $Z$ direction are represented in the integral form as

$$u_x = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \quad (7.2.10)$$

$$u_y = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \quad (7.2.11)$$

$$u_z = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial z^2} \right) + \alpha T \right] dz \quad (7.2.12)$$

where $\nu$ is the poisson’s ratio of the material of the plate.

The stress components in terms of $U$ are given by

$$\sigma_{xx} = \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \quad (7.2.13)$$

$$\sigma_{yy} = \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} \right) \quad (7.2.14)$$

$$\sigma_{zz} = \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad (7.2.15)$$

Equations (7.2.1) to (7.2.15) constitute the mathematical formulation of the problem under consideration.
7.3 Solution of the problem

TEMPERATURE

To find the temperature function \( T(x, y, z, t) \), we introduce the “Triple-integral transform-I” and its corresponding “Triple-inversion formula” in [6] respectively as

\[
\overline{T}(\alpha_n, \beta_m, \gamma_p; t) = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} K(\alpha_n, x).K(\beta_m, y).K(\gamma_p, z) \\
\times T(x, y, z, t) \, dx \, dy \, dz \\
= I[T(x, y, z, t)]
\]

(7.3.1)

\[
T(x, y, z, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} K(\alpha_n, x).K(\beta_m, y).K(\gamma_p, z).\overline{T}(\alpha_n, \beta_m, \gamma_p; t)
\]

(7.3.2)

where the kernels

\[
K(\alpha_n, x) = \sqrt{\frac{2}{a}} \cos(\alpha_n x)
\]

(7.3.3)

\[
K(\beta_m, y) = \sqrt{\frac{2}{b}} \cos(\beta_m y)
\]

(7.3.4)

\[
K(\gamma_p, z) = \sqrt{\frac{2}{c}} \cos(\gamma_p z)
\]

(7.3.5)

and eigenvalues are

\( \alpha_n \) is \( n^{th} \) root of transcendental equation \( \cos(\alpha_n a) = 0 \).
\(\alpha_n = \left(\frac{2n + 1}{2a}\right)\pi\) \hspace{1cm} (7.3.6)

\(\beta_m\) is the \(m^{th}\) root of the transcendental equation \(\cos(\beta_m b) = 0\)

\(\beta_m = \left(\frac{2m + 1}{2b}\right)\pi\) \hspace{1cm} (7.3.7)

\(\gamma_p\) is the \(p^{th}\) root of the transcendental equation \(\cos(\gamma_p c) = 0\)

\(\gamma_p = \left(\frac{2p + 1}{2c}\right)\pi\) \hspace{1cm} (7.3.8)

Using

\[
I \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} \left( \sqrt{\frac{2}{a}} \cos(\alpha_n x) \right) \\
\times \left( \sqrt{\frac{2}{b}} \cos(\beta_m y) \right) \cdot \left( \sqrt{\frac{2}{c}} \cos(\gamma_p z) \right) \cdot \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \, dx \, dy \, dz
\]

\[
= \frac{2\sqrt{2}}{\sqrt{abc}} (-1)^{n+m+p} \left[ \frac{\alpha_n^2 + \beta_m^2 + \gamma_p^2}{\alpha_n, \beta_m, \gamma_p} \right] T_0 - \left( \frac{\alpha_n^2 + \beta_m^2 + \gamma_p^2}{\alpha_n, \beta_m, \gamma_p} \right) T
\]

Applying triple integral transform (7.3.1) on equation (7.2.1) for variable \(x, y\) and \(z\) one obtain, the linear differential equation

\[
\frac{dT}{dt} + k \left( \frac{\alpha_n^2 + \beta_m^2 + \gamma_p^2}{\alpha_n, \beta_m, \gamma_p} \right) T = A
\] \hspace{1cm} (7.3.9)

where \(T(\alpha_n, \beta_m, \gamma_p, t)\) is triple integral transform of \(T(x, y, z, t)\) over the variable \(x, y, z\) and
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\[
A (\alpha_n, \beta_m, \gamma_p, t) = \frac{2\sqrt{2k}}{\sqrt{abc}} (-1)^{n+m+p} \left[ \frac{\alpha_n^2 + \beta_m^2 + \gamma_p^2}{\alpha_n \beta_m \gamma_p} \right] T_0 \tag{7.3.10}
\]

On solving equation (7.3.9), one obtain

\[
\overline{T} = e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} \int_0^t A e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} dt \tag{7.3.11}
\]

Applying the inverting formula defined in (7.3.2), one obtain

\[
T = T_0 \left[ 1 - \frac{8}{abc} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{(-1)^{n+m+p}}{\alpha_n \beta_m \gamma_p} \right) \cos(\alpha_n x) \cos(\beta_m y) \right.
\]

\[
\times \cos(\gamma_p z) e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} \tag{7.3.12}
\]

which is the required expressions for temperature distribution.

**DETERMINATION OF AIRY’S STRESS FUNCTION**

Using equation (7.3.12) in equation (7.2.9), one obtains the expressions for Airy’s stress function \( U \) as

\[
U = \frac{8\alpha E T_0}{abc} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{1}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right)
\]

\[
\times \left( \frac{(-1)^{n+m+p}}{\alpha_n \beta_m \gamma_p} \right) \cos(\alpha_n x) \cos(\beta_m y) \cos(\gamma_p z) e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} \tag{7.3.13}
\]

**DETERMINATION OF DISPLACEMENT COMPONENTS**

Now using equations (7.3.12) and (7.3.13) in equations (7.2.10)–(7.2.12), one obtain the expressions for displacement as
\[
u_x = \left( \frac{8\alpha T_0}{abc} \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{(\nu - 1)\alpha_n^2 - 2\beta_m^2 - 2\gamma_p^2}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right) \right. \\
\left. \times \left( \frac{(-1)^{n+m+p}}{\alpha_n^2 \beta_m \gamma_p} \right) \sin(\alpha_n x) \cos(\beta_m y) \cos(\gamma_p z) e^{-((\alpha_n^2 + \beta_m^2 + \gamma_p^2) t)} \right\} + T_0 x
\]

\[
u_y = \left( \frac{8\alpha T_0}{abc} \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{(\nu - 1)\beta_m^2 - 2\alpha_n^2 - 2\gamma_p^2}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right) \right. \\
\left. \times \left( \frac{(-1)^{n+m+p}}{\alpha_n \beta_m^2 \gamma_p} \right) \sin(\alpha_n x) \sin(\beta_m y) \cos(\gamma_p z) e^{-((\alpha_n^2 + \beta_m^2 + \gamma_p^2) t)} \right\} + T_0 y
\]

\[
u_z = \left( \frac{8\alpha T_0}{abc} \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{(\nu - 1)\gamma_p^2 - 2\alpha_n^2 - 2\beta_m^2}{\alpha_n^2 + \alpha_m^2 + \gamma_p^2} \right) \right. \\
\left. \times \left( \frac{(-1)^{n+m+p}}{\alpha_n \beta_m \gamma_p^2} \right) \cos(\alpha_n x) \cos(\beta_m y) \sin(\gamma_p z) e^{-((\alpha_n^2 + \beta_m^2 + \gamma_p^2) t)} \right\} + T_0 z
\]

**DETERMINATION OF STRESS FUNCTIONS**

Now using equation (7.3.13) in equations (7.2.13)–(7.2.15), one obtain the expression for stresses as

\[
s_{xx} = \left( \frac{-8\alpha E T_0}{abc} \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{1}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right) \right. \\
\left. \times \left( \frac{(-1)^{n+m+p}}{\alpha_n} \right) \cos(\alpha_n x) \beta_m \cos(\beta_m y) \gamma_p \cos(\gamma_p z) e^{-((\alpha_n^2 + \beta_m^2 + \gamma_p^2) t)} \right\}
\]
\[
\sigma_{yy} = \left( -8\alpha ET_0 \right) \frac{abc}{\beta_m} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right\} \times \frac{(-1)^{n+m+p}}{\beta_m} \cdot \alpha_n \cos(\alpha_n x) \cdot \cos(\beta_m y) \cdot \gamma_p \cos(\gamma_p z) \cdot e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} \}
\]

\[
\sigma_{zz} = \left( -8\alpha ET_0 \right) \frac{abc}{\gamma_p} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{\alpha_n^2 + \beta_m^2 + \gamma_p^2} \right\} \times \frac{(-1)^{n+m+p}}{\gamma_p} \cdot \alpha_n \cos(\alpha_n x) \cdot \beta_m \cos(\beta_m y) \cdot \gamma_p \cos(\gamma_p z) \cdot e^{-(\alpha_n^2 + \beta_m^2 + \gamma_p^2) t} \}
\]

7.4 Numerical Calculations

The numerical calculations have been carried out for aluminium plate with parameters \( a = 3m, b = 2m, c = 1m \). Thermal diffusivity \( k = 0.86 \), thermal conductivity \( \lambda = 0.48 \), linear coefficient of thermal expansion \( \alpha = 25.5 \times 10^{-6} \), Young’s modulus elasticity of the material of the plate \( E = 6.9 \times 10^8 \text{ N/m}^2 \), poisson ratio \( \nu = 0.281 \) and \( t = 5 \text{ sec} \).

For convince setting

\[
A = \left( \frac{8T_0}{abc} \right) \quad B = \left( \frac{-8T_0}{abc} \right)
\]

in equations (7.3.14) to (7.3.19). In order to examine the influence of constant heat supply on the extreme ends of plate one performed numerical results in \( X, Y \) and \( Z \) direction.
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Considering

\[ \lim_{n \to \infty} \alpha_n = \lim_{m \to \infty} \beta_m = \lim_{p \to \infty} \gamma_p = \infty \]

\[ \lim_{n \to \infty} (e^{-kn^2t}) = \lim_{m \to \infty} (e^{-k\beta^2_m t}) = \lim_{p \to \infty} (e^{-k\gamma^2_p t}) = 0 \]

Also the term \( \cos(\alpha_n x) \) and \( \sin(\alpha_n x) \) are bounded.

Thus necessary condition for convergence is satisfied, by applying D-Alemberts ratio test it can be easily verify that all the series in (7.3.14) to (7.3.19) are convergent. Also the term in the expression for displacements and stresses are negligible for large value of \( n, m \) and \( p \) it converges to zero at infinity. Therefore for better accuracy numerical calculations have been performed by taking \( n = m = p = 100 \) with help of computer programme.

7.5 Concluding Remarks

In this chapter, we discussed the quasi-static thermal stresses in a thick rectangular plate which is free from traction. The plate is subjected to constant heat supply on the extreme edges \( (x = a, y = b, z = c) \). A mathematical model is constructed for aluminum plate and performed numerical calculations. The thermoelastic behavior is examined such as temperature, displacement and stress with the help of constant temperature.
Figure 7.1, Figure 7.2, Figure 7.3, shows that the displacement function $u_x$, $u_y$ and $u_z$ occur near heat source. Also displacement function $u_x$ remains constant in $Y$ and $Z$ direction, displacement function $u_y$ remains constant in $X$ and $Z$ direction and displacement function $u_z$ remains constant in $X$ and $Y$ direction.

Figure 7.4, Figure 7.5, Figure 7.6, shows that the stress function $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$ develops compressive stress near heat source in both $X$, $Y$ and $Z$ direction.

We can summaries that, due to constant heat supply on the extreme edges of thick rectangular plate ($x = a, y = b, z = c$), the displacements and stresses components occur near heated region. The both normal stress components change sharply from extreme edges to initial edges of rectangular plate where as the shear stress components approaches to from initial edges to extreme edges. Also from the figures of displacement it can be observed that, the displacements occur in both the direction, so we may conclude that the plate expands in both $X, Y$ and $Z$ direction.

The results, obtained here mainly applicable in engineering problems, particularly for industrial machines subjected to the heating such as the main shaft of a lathe, turbines, the roll of rolling mill and practical applications in air-craft structures.
Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (7.3.14)–(7.3.19).

**Figure 7.1:** The displacement function 1: $\frac{u_x}{A}$, 2: $\frac{u_y}{A}$ and 3: $\frac{u_z}{A}$ in X-direction.

**Figure 7.2:** The displacement function 1: $\frac{u_x}{A}$, 2: $\frac{u_y}{A}$ and 3: $\frac{u_z}{A}$ in Y-direction.
Figure 7.3: The displacement function 1: $\frac{u_x}{A}$, 2: $\frac{u_y}{A}$ and 3: $\frac{u_z}{A}$ in Z-direction.

Figure 7.4: The thermal stresses 1: $\frac{\sigma_{xx}}{B}$, 2: $\frac{\sigma_{yy}}{B}$ and 3: $\frac{\sigma_{zz}}{B}$ in X-direction.
Figure 7.5: The thermal stresses 1: $\frac{\sigma_{xx}}{B}$, 2: $\frac{\sigma_{yy}}{B}$ and 3: $\frac{\sigma_{zz}}{B}$ in Y-direction.

Figure 7.6: The thermal stresses 1: $\frac{\sigma_{xx}}{B}$, 2: $\frac{\sigma_{yy}}{B}$ and 3: $\frac{\sigma_{zz}}{B}$ in Z-direction.
References


