Chapter 5

An Inverse Quasi-Static Thermal Stresses in a Thick Annular Disc

5.1 Introduction

the inverse transient thermoelastic problem for a composite circular disc constructed of transversely isotropic layer. Also, Kulkarni et al. [6] studied quasi-static thermal stresses in a thick circular plate subjected to arbitrary initial temperature on the upper surface. Recently, Dange et al. [7] studied two dimensional transient problems for a thick annular disc in thermoelasticity.

In this chapter, an attempt is made to solve an inverse quasi-static thermoelastic problem in a thick annular disc. The upper surface of a thick annular disc subjected to an arbitrary known interior temperature under a steady state field is considered. The fixed circular edges are thermally insulated and the lower surface is kept at zero temperature. The governing heat conduction equation has been solved by using the Hankel transform technique. The results are obtained in series form in terms of Bessel’s functions. The results for unknown temperature, displacement and thermal stresses have been computed numerically and illustrated graphically.

5.2 Formulation of the problem

Consider a thick annular disc of thickness $2h$ occupying the space $D : a \leq r \leq b, -h \leq z \leq h$. Initially the disc is at zero temperature. Let the disc be subjected to an arbitrary known temperature $f(r)$
within region $-h \leq \xi \leq h$. The circular edges ($r = a$ and $r = b$) are thermally insulated and the lower surface ($z = -h$) is kept at zero temperature. Assume that the boundary of the annular disc is free from traction.

Under these more realistic prescribed conditions, the unknown temperature $g(r)$ which is at the upper face of the annular disc ($z = h$) and the quasi-static thermal stresses due to unknown temperature $g(r)$ need to be determined.

The differential equation governing the displacement potential function $\phi(r, z)$ is given in [8] as,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K\tau$$

(5.2.1)

where $K$ is the restraint coefficient and the temperature change is given by $\tau = T - T_i$, where $T_i$ is the initial temperature. The displacement function $\phi$ is known as Goodier’s thermoelastic potential.

The steady-state temperature of the disc satisfies the heat condition equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

(5.2.2)

subject to the boundary conditions,

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = a, -h \leq z \leq h$$

(5.2.3)
\[ \frac{\partial T}{\partial r} = 0 \quad \text{at } r = b, -h \leq z \leq h \] (5.2.4)  
\[ \frac{\partial T}{\partial z} + h_{s1} T = f(r) \text{ (known) at } z = \xi, -h \leq \xi \leq h, \ a \leq r \leq b \] (5.2.5)  
\[ \frac{\partial T}{\partial z} - h_{s2} T = 0 \quad \text{at } z = -h, a \leq r \leq b \] (5.2.6)  

and  
\[ T = g(r) \text{ (unknown) at } z = h, a \leq r \leq b, -h \leq z \leq h \] (5.2.7)  

where \( h_{s1} \) and \( h_{s2} \) are relative heat transfer coefficients on the upper and the lower surface of the thick annular disc.

The displacement function in the cylindrical coordinate system are represented by the Michell’s function defined in [8] as,

\[ u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z}, \] (5.2.8)  
\[ u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2}. \] (5.2.9)  

The Michell’s function \( M \) must satisfy

\[ \nabla^2 \nabla^2 M = 0 \] (5.2.10)  

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r \partial r} + \frac{\partial^2}{\partial z^2}. \] (5.2.11)
The components of the stresses are represented by the thermoelastic displacements potential \( \phi \) and Michell’s function \( M \) as

\[
\sigma_{rr} = 2G \left[ \frac{\partial^2 \phi}{\partial r^2} - K \tau + \frac{\partial}{\partial z} \left( \nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right) \right]
\]

(5.2.12)

\[
\sigma_{\theta \theta} = 2G \left[ \frac{1}{r} \frac{\partial \phi}{\partial r} - K \tau + \frac{\partial}{\partial z} \left( \nu \nabla^2 M - \frac{1}{r} \frac{\partial M}{\partial r} \right) \right]
\]

(5.2.13)

\[
\sigma_{zz} = 2G \left[ \frac{\partial^2 \phi}{\partial z^2} - K \tau + \frac{\partial}{\partial z} \left( (2 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]

(5.2.14)

and

\[
\sigma_{r z} = 2G \left[ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial z} \left( (1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]

(5.2.15)

where \( G \) and \( \nu \) are the Shear modulus and Poisson’s ratio respectively.

The boundary conditions on the traction free surfaces of the thick annular disc are

\[
\sigma_{rr} = \sigma_{r z} = 0 \quad \text{at } r = a \text{ and } r = b
\]

(5.2.16)

\[
\sigma_{zz} = \sigma_{r z} = 0 \quad \text{at } z = \pm h.
\]

Equations (5.2.1) to (5.2.16) constitute the mathematical formulation of the problem under consideration.
5.3 Solution of the problem

TEMPERATURE
To obtain the expression for temperature $T(r, z)$ we introduce the finite Hankel transform over the variable $r$ and its inverse transform defined in [9] as

$$
\mathcal{T}(\lambda_n, z) = \int_a^b rK_0(\lambda_n, r)T(r, z)dr
$$

(5.3.1)

$$
T(r, z) = \sum_{n=1}^{\infty} \mathcal{T}(\lambda_n, z)K_0(\lambda_n, r)
$$

(5.3.2)

where

$$
K_0(\lambda_n, r) = \frac{R_0(\lambda_n, r)}{\sqrt{N}}
$$

(5.3.3)

$$
R_0(\lambda_n, r) = \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{\lambda_n Y_1(\lambda_n b)}
$$

(5.3.4)

The normality constant

$$
N = \frac{b^2}{2} R_0^2(\lambda_n, b) - \frac{a^2}{2} R_0^2(\lambda_n, a)
$$

(5.3.5)

and $\lambda_1, \lambda_2, \ldots$ are the roots of the transcendental equation,

$$
\frac{J_1(\lambda a)}{J_1(\lambda b)} - \frac{Y_1(\lambda a)}{Y_1(\lambda b)} = 0
$$

(5.3.6)

with $J_n(x)$ is Bessel function of the first kind of the order $n$ and $Y_n(x)$ Bessel’s function of the second kind of the order $n$. 
This transform satisfies the relation

\[ H \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = -\lambda_n^2 \mathcal{T}(\lambda_n, z) \]  

(5.3.7)

and

\[ H \left[ \frac{\partial^2 T}{\partial z^2} \right] = \frac{d^2 \mathcal{T}}{dz^2}. \]  

(5.3.8)

On applying the finite Hankel transform defined in equation (5.3.1) to equation (5.2.2), one obtains

\[ \frac{d^2 \mathcal{T}}{dz^2} - \lambda_n^2 \mathcal{T} = 0 \]  

(5.3.9)

where \( \mathcal{T} \) is the Hankel transform of \( T \).

On solving equation (5.3.9) under the conditions given in equation (5.2.5) and (5.2.6), one obtains

\[ \mathcal{T} = \sum_{n=1}^{\infty} \mathcal{J}(\lambda_n) \times \left[ \frac{\lambda_n \cosh [\lambda_n(z + h)] + h_{s2} \sinh [\lambda_n(z + h)]}{(\lambda_n^2 + h_{s1}h_{s2}) \sinh [\lambda_n(\xi + h)] + \lambda_n(h_{s1} + h_{s2}) \cosh [\lambda_n(\xi + h)]} \right]. \]  

(5.3.10)

where

\[ \mathcal{J}(\lambda_n) = \int_a^b \left( \frac{r}{\lambda_n \sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) f(r) dr. \]  

(5.3.11)

On applying the inverse Hankel transform defined in equation (5.3.2), one obtains the expression for the temperature as
\[ T = \sum_{n=1}^{\infty} \left( \frac{f_n}{\sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \left[ \frac{\lambda_n \cosh [\lambda_n(z+h)] + hs_2 \sinh [\lambda_n(z+h)]}{(\lambda_n^2 + hs_1 hs_2) \sinh [\lambda_n(\xi + h)] + \lambda_n(h_{s1} + hs_2) \cosh [\lambda_n(\xi + h)]} \right] \]

(5.3.12)

Since \( T_i = 0 \), the temperature change is \( \tau = T - T_i = T \).

(5.3.13)

**UNKNOWN TEMPERATURE** \( g(r) \)

The unknown temperature can be obtained by substituting \( z = h \) into equation (5.3.12) such that,

\[ g(r) = \sum_{n=1}^{\infty} \left( \frac{f_n}{\sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \left[ \frac{\lambda_n \cosh [2\lambda_n h] + hs_2 \sinh [2\lambda_n h]}{(\lambda_n^2 + hs_1 hs_2) \sinh [\lambda_n(\xi + h)] + \lambda_n(h_{s1} + hs_2) \cosh [\lambda_n(\xi + h)]} \right] \]

(5.3.14)

**MICHELL’S FUNCTION** \( M \)

A suitable form of \( M \) satisfying equation (5.2.10) is given by

\[ M = (K) \sum_{n=1}^{\infty} \left( \frac{f_n}{\sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \{ B_n(\cosh [\lambda_n(z+h)] + \lambda_n \sinh [\lambda_n(z+h)]) \}
+ C_n\lambda_n(\cosh [\lambda_n(z+h)] + \lambda_n \sinh [\lambda_n(z+h)]) \}
\]

(5.3.15)

where \( B_n \) and \( C_n \) are arbitrary constants, which can be determined finally by using conditions (5.2.16).
GOODIER’S THERMOELASTIC DISPLACEMENT POTENTIAL FUNCTION $\phi$

Assuming that the displacements function $\phi(r, z)$ has the form and satisfying the equation (5.2.1) as

$$
\phi(r, z) = \sum_{n=1}^{\infty} D_n \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \left[ \frac{(z + h)(h_s h_2 \cosh \lambda_n (z + h) + \lambda_n \sinh \lambda_n (z + h))}{(\lambda_n^2 + h_s h_2) \sinh \lambda_n (\xi + h) + \lambda_n (h_s h_1 + h_s h_2) \cosh \lambda_n (\xi + h)} \right]
$$

and using $\phi$ in equation (5.2.1), one obtains

$$
D_n = \frac{K\overline{T}(\lambda_n)}{2\sqrt{N}\lambda_n}.
$$

Thus the equation (5.3.16) becomes

$$
\phi(r, z) = \left( \frac{K}{2} \right) \sum_{n=1}^{\infty} \left( \frac{\overline{T}(\lambda_n)}{\lambda_n \sqrt{N}} \right) \left[ \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right] \times \left[ \frac{(z + h)(h_s h_2 \cosh \lambda_n (z + h) + \lambda_n \sinh \lambda_n (z + h))}{(\lambda_n^2 + h_s h_2) \sinh \lambda_n (\xi + h) + \lambda_n (h_s h_1 + h_s h_2) \cosh \lambda_n (\xi + h)} \right].
$$

DETERMINATION OF DISPLACEMENT FUNCTION

Now using equations (5.3.15) and (5.3.18) in equation (5.2.8) and (5.2.9), one obtains the expressions for displacements as

$$
u_r = (K) \sum_{n=1}^{\infty} \left( \frac{\overline{T}(\lambda_n)}{\sqrt{N}} \right) \left[ \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right] \times \left[ \frac{-(z + h)(h_s h_2 \cosh \lambda_n (z + h) + \lambda_n \sinh \lambda_n (z + h))}{(\lambda_n^2 + h_s h_2) \sinh \lambda_n (\xi + h) + \lambda_n (h_s h_1 + h_s h_2) \cosh \lambda_n (\xi + h)} \right].$$
CHAPTER 5. INVERSE QUASI-STATIC THERMAL STRESSES IN A THICK ANNULAR DISC

Now using equations (5.3.13), (5.3.15) and (5.3.18) in equation (5.2.12)–(5.2.15), one obtains the expressions for stresses as,

$$\sigma_{rr} = (2GK) \sum_{n=1}^{\infty} \left( \frac{f(\lambda_n)}{\sqrt{N}} \right) \left\{ \frac{\partial}{\partial r} \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \right\} \times \left[ \frac{(z+h)\langle h_s \sinh[\lambda_n(z+h)] + \lambda_n \cosh[\lambda_n(z+h)] \rangle}{(\lambda_n^2 + h_s h_s) \sinh[\lambda_n(\xi+h)] + \lambda_n (h_s + h_s) \cosh[\lambda_n(\xi+h)]} \right]$$

$$+ B_n \lambda_n^2 \partial \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right)$$

$$+ C_n \lambda_n^2 \langle (h_s \sinh[\lambda_n(z+h)] + \lambda_n \cosh[\lambda_n(z+h)]) \rangle$$

$$- B_n \lambda_n^2 \langle h_s \cosh[\lambda_n(z+h)] + \lambda_n \sinh[\lambda_n(z+h)] \rangle$$

$$- C_n \lambda_n^2 \langle 2(1-2\nu) (h_s \cosh[\lambda_n(z+h)] + \lambda_n \sinh[\lambda_n(z+h)]) \rangle$$

$$- \lambda_n (z+h) \langle h_s \sinh[\lambda_n(z+h)] + \lambda_n \cosh[\lambda_n(z+h)] \rangle \right\}.$$
\[ \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ + C_n \lambda_n^2 \left[ 2\nu \lambda_n \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \right] \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ + \frac{\partial}{\partial r} \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ + \lambda_n(z+h) \langle \langle h_s \cosh [\lambda_n(z+h)] + \lambda_n \sinh [\lambda_n(z+h)] \rangle \rangle \] 

\[ \sigma_{\theta \theta} = (2GK) \sum_{n=1}^{\infty} \left( \frac{f(\lambda_n)}{\sqrt{N}} \right) \left\{ \left( -\frac{1}{r} \right) \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \right\} \]
\[ \times \left( -\frac{1}{r} \right) \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ - \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ - \frac{1}{r} \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ + C_n \lambda_n^2 \left[ 2\nu \lambda_n \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \right] \times \langle h_s \sinh [\lambda_n(z+h)] + \lambda_n \cosh [\lambda_n(z+h)] \rangle \]
\[ + \lambda_n(z+h) \langle \langle h_s \cosh [\lambda_n(z+h)] + \lambda_n \sinh [\lambda_n(z+h)] \rangle \rangle \]
\[
\sigma_{zz} = (2GK) \sum_{n=1}^{\infty} \left( \frac{f(\lambda_n)}{\sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \\
\times \left\{ \frac{\lambda_n(z + h)\langle h s_2 \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)] \rangle}{(\lambda_n^2 + h s_1 h s_2) \sinh[\lambda_n(\xi + h)] + \lambda_n(h s_1 + h s_2) \cosh[\lambda_n(\xi + h)]} \right\} \\
- B_n \lambda_n^3 \langle h s_2 \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)] \rangle \\
+ C_n \lambda_n^3 \left\{ (1 - 2\nu) \langle h s_2 \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)] \rangle \\
- \lambda_n(z + h) \langle h s_2 \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)] \rangle \right\} \\
\] (5.3.23)

\[
\sigma_{rz} = (K) \sum_{n=1}^{\infty} \left( \frac{f(\lambda_n)}{\sqrt{N}} \right) \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \\
\times \left\{ \frac{-\langle h s_2 \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)] \rangle}{(\lambda_n^2 + h s_1 h s_2) \sinh[\lambda_n(\xi + h)] + \lambda_n(h s_1 + h s_2) \cosh[\lambda_n(\xi + h)]} \right\} \\
- \lambda_n(z + h) \langle h s_2 \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)] \rangle \\
+ B_n \lambda_n^3 \langle h s_2 \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)] \rangle \\
+ C_n \lambda_n^3 \left\{ 2\nu \langle h s_2 \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)] \rangle \\
+ \lambda_n(z + h) \langle h s_2 \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)] \rangle \right\} \right\} \\
\] (5.3.24)

Now in order to satisfy the boundary conditions given in the equation (5.2.16), we use equations (5.3.21), (5.3.23) and (5.3.24) for \( B_n \) and \( C_n \), one obtains

\[
B_n = \frac{(1 - 2\nu)}{2\lambda_n^3 \langle (\lambda_n^2 + h s_1 h s_2) \sinh[\lambda_n(\xi + h)] + \lambda_n(h s_1 + h s_2) \cosh[\lambda_n(\xi + h)] \rangle} \\
\]

\[
C_n = \frac{1}{2\lambda_n^3 \langle (\lambda_n^2 + h s_1 h s_2) \sinh[\lambda_n(\xi + h)] + \lambda_n(h s_1 + h s_2) \cosh[\lambda_n(\xi + h)] \rangle}. 
\]
Using these values of $B_n$ and $C_n$ in equations (5.3.19) to (5.3.24), one obtains the expression for displacements and stresses as,

\[
u_r = K(1 - \nu) \sum_{n=1}^{\infty} \left( \frac{\overline{f}(\lambda_n)}{\lambda_n \sqrt{N}} \right) \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \frac{h_{s2} \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)]}{(\lambda_n^2 + h_{s1} h_{s2}) \sinh[\lambda_n(\xi + h)] + \lambda_n(h_{s1} + h_{s2}) \cosh[\lambda_n(\xi + h)]}
\]

(5.3.25)

\[
u_z = K(1 - \nu) \sum_{n=1}^{\infty} \left( \frac{\overline{f}(\lambda_n)}{\lambda_n \sqrt{N}} \right) \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \frac{h_{s2} \cosh[\lambda_n(z + h)] + \lambda_n \sinh[\lambda_n(z + h)]}{(\lambda_n^2 + h_{s1} h_{s2}) \sinh[\lambda_n(\xi + h)] + \lambda_n(h_{s1} + h_{s2}) \cosh[\lambda_n(\xi + h)]}
\]

(5.3.26)

\[
\sigma_{rr} = -2GK(1 - \nu) \sum_{n=1}^{\infty} \left( \frac{\overline{f}(\lambda_n)}{\lambda_n \sqrt{N}} \right) \left( \frac{1}{r} \right) \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) \times \frac{h_{s2} \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)]}{(\lambda_n^2 + h_{s1} h_{s2}) \sinh[\lambda_n(\xi + h)] + \lambda_n(h_{s1} + h_{s2}) \cosh[\lambda_n(\xi + h)]}
\]

(5.3.27)

\[
\sigma_{\theta\theta} = 2GK(1 - \nu) \sum_{n=1}^{\infty} \left( \frac{\overline{f}(\lambda_n)}{\lambda_n \sqrt{N}} \right) \left\{ \left[ \frac{1}{\lambda_n r} \left( \frac{J_1(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n b)} \right) - \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) \right] \times \frac{h_{s2} \sinh[\lambda_n(z + h)] + \lambda_n \cosh[\lambda_n(z + h)]}{(\lambda_n^2 + h_{s1} h_{s2}) \sinh[\lambda_n(\xi + h)] + \lambda_n(h_{s1} + h_{s2}) \cosh[\lambda_n(\xi + h)]} \right\}
\]

(5.3.28)

\[
\sigma_{zz} = 0
\]

(5.3.29)

and

\[
\sigma_{rz} = 0.
\]

(5.3.30)
5.4 Special Case and Numerical Calculations

Setting

\[ f(r) = e^h(r^2 - a^2)^2(r^2 - b^2)^2, \quad (5.4.1) \]

Using equation (5.4.1) in equation (5.3.11), one obtains

\[ \overline{f}(\lambda_n) = e^h \int_a^b \frac{1}{\lambda_n \sqrt{N}} \left( \frac{J_0(\lambda_n r)}{J_1(\lambda_n b)} - \frac{Y_0(\lambda_n r)}{Y_1(\lambda_n b)} \right) r(r^2 - a^2)^2(r^2 - b^2)^2 dr \]

\[ \overline{f}(\lambda_n) = e^h \left\{ \frac{b(40a^2b^2\beta_m^4 - 32b^4\beta_m^4 - 8a^4\beta_m^4 + 2304b^2\beta_m^2 - 576a^2\beta_m^2 - 18432)J_1(\beta_m a)}{8 \pi \sqrt{N} \beta_m^{11} J_1(\beta_m a)J_1(\beta_m b)Y_1(\beta_m b)} \right\} \]

(5.4.2)

(5.4.3)

The numerical calculations have been carried out for a steel \((SN \ 50C)\) plate with parameters chosen \(a = 1m, \ b = 2m, \ h = 0.4m\). Thermal diffusivity \(k = 15.9 \times 10^6(m^2s^{-1})\) and Poisson ratio \(\nu = 0.281\). Relative heat transfer coefficients \(h_{s1}=10\) and \(h_{s2}=5\).

**TRANSCENDENTAL ROOTS**

The transcendental roots of \(\left( \frac{J_1(\lambda a)}{J_1(\lambda b)} - \frac{Y_1(\lambda a)}{Y_1(\lambda b)} \right) = 0\) as in [9] are \(\lambda_1 = 3.1965, \ \lambda_2 = 6.3123, \ \lambda_3 = 9.4445, \ \lambda_4 = 12.5812, \ \lambda_5 = 15.7199\).

For convenience, setting

\[ \alpha \left( \frac{-8}{\pi} \right), \quad \beta = \left( \frac{8(1 - \nu)K}{\pi} \right) \text{ and } \gamma = \left( \frac{16(1 - \nu)GK}{\pi} \right) \]
in equations (5.3.14) and (5.3.25)–(5.3.28).

In order to examine the influence of an unknown temperature on the upper surface of annular disc, we performed numerical calculations $z = \frac{h}{2}$, $r = 1, 1.2, 1.4, 1.6, 1.8, 2m$ and $\xi = -0.2, -0.1, 0, 0.1, 0.2m$.

Numerical variations in radial directions are shown in the figures with help of a computer programme.

### 5.5 Concluding Remarks

In this chapter, we discussed an inverse quasi-static thermoelastic problem in a thick annular disc which is free from traction subjected to an arbitrary known interior temperature and determined the expressions for unknown temperature, displacement and stress functions.

As a special case mathematical model is constructed for

$$f(r) = e^h(r^2 - a^2)^2(r^2 - b^2)^2$$

and numerical calculations were performed.

The thermoelastic behavior such as an unknown temperature, displacement and stresses with the help of arbitrary known interior temperature is examined.
Figure 5.1, The unknown temperature $g(r)$ decreases from the upper surface to the lower surface in the annular region $0.6 \leq r \leq 1$ in the radial direction.

Figure 5.2, The radial displacement function $u_r$ shows the normal curve. Also it is zero at both the circular boundaries of an annular disc.

Figure 5.3, The axial displacement function $u_z$ decreases from inner circular surface to outer circular surface in the radial direction.

Figure 5.4, The radial stress function $\sigma_{rr}$ shows the normal curve. Also it is zero at both the circular boundaries of an annular disc and it develops the compressive stresses in the radial direction.

Figure 5.5, The stress function $\sigma_{\theta\theta}$ increases from the upper surface to the lower surface in the annular region $1 \leq r \leq 1.3$. whereas it decreases from the upper surface to the lower surface in the annular region $1.3 \leq r \leq 2$ and it develops the compressive stresses in the radial direction.

We can summaries that, the displacement and stress components occur near heat source. With an increases the temperature in the annular disc will tend to expand in the radial direction as well as in the axial direction. In the traction free surfaces the stress components $\sigma_{zz}$ and $\sigma_{rz}$ are zero. Also from the figures of displacement it
can observe that displacement occurs around the center towards in the downward direction. We concluded that due to unknown temperature the annular disc expands in the axial direction and bends concavely at the center. This expansion is inversely proportional to the thickness of the annular disc.

The results, obtained here mainly applicable in engineering problems, particularly for industrial machines subjected to the heating such as the main shaft of a lathe, turbines, the roll of rolling mill and practical applications in air-craft structures.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (5.3.14) and (5.3.25)–(5.3.28).
Figure 5.1: The unknown temperature $\frac{g(r)}{\alpha}$ in radial direction.

Figure 5.2: The radial displacement function $\frac{u_r}{\beta}$ in radial direction.
Figure 5.3: The axial displacement function $\frac{u_z}{\beta}$ in radial direction.

Figure 5.4: The radial stress function $\frac{\sigma_{rr}}{\gamma}$ in radial direction.
References


