CHAPTER 7
DIAMETER AND ECCENTRICITY SETS OF SPANNING TREES

7.1. INTRODUCTION

The set of diameters of all the spanning trees of a connected simple graph $G$ is called the tree-diameter set of $G$ and the elements of this set are written in the increasing order. In [11], Harary, Mokken and Plantholt proved that the tree-diameter set of any 2-connected graph is a set of consecutive integers; in other words, the tree-diameter set has the interpolation property. They also proved that if $\{d_1, d_2, \ldots, d_n\}$ is the tree-diameter set of a connected graph then $d_{i+1} \leq 2d_i - 1$. They further raised the question of characterization of tree-diameter sets for 1-connected graphs. They conjectured that if $\{s_1, s_2, \ldots, s_n\}$ is a set of positive integers with $s_{i+1} \leq 2s_i - 1$, then there exists an $s$ such that $\{s_1 + s, s_2 + s, \ldots, s_n + s\}$ is a feasible tree-diameter set.

In this work, we prove $d_{i+1} \leq 3d_i/2$ which is a better relation between the consecutive elements of a connected graph. This leads to the characterization
of all the tree-diameter sets with just two elements and
to a sufficient condition for a set of integers to be a
tree-diameter set of some connected graph. We also prove
a strengthened version of the conjecture mentioned above.
Shimizu and Shibata [18] too have given a constructive
proof for that conjecture, but their construction does
not go deep enough into the nature of the tree-diameter
sets.

In all the examples constructed by us having the
tree-diameter sets which are not sets of consecutive
integers, we found that there are at least four cut
vertices. In the case of graphs with exactly one cut
vertex, a modification of the proof given in the case of
2-connected graphs in [ll], shows that the tree diameter
set is a set of consecutive integers.

For this, the notion of \( u \)-free end line exchange
is introduced here and it is shown that this restricted
class of exchangers are sufficient in transforming any
spanning tree into any other spanning tree.

While considering the graphs with exactly two cut
vertices, it appears that the study of the eccentricities
of a vertex \( u \) in all the spanning trees becomes
necessary. Such a set of eccentricities, given in the
increasing order, is called the tree-eccentricity set.
and is studied here. The tree-eccentricity set of a vertex in a 2-connected graph is completely characterized in section 7.4. The relationship between tree-diameter set and the tree-eccentricity sets are also considered here. It is proved that in a 2-connected graph the union of a tree-eccentricity set and the tree-diameter set forms a set of consecutive integers. It is shown that this result holds good only for 2-connected graphs and not for a general graph.

7.2 TREE-DIAMETER SET

Let \( T \) be a spanning tree of \( G \). Let \( x \) be an edge in \( G-T \) and let \( y \) be an edge in the unique cycle of \( T+x \). The transformation of \( T \) into the spanning tree \( T+x-y \) is called a fundamental exchange if \( x \) and \( y \) are adjacent, it is called a neighbour exchange if \( y \) is a pendant edge of \( T \) then it is called an endline exchange. Note that an endline exchange is a neighbour exchange.

Note 7.2.1: It is known that in any graph any spanning tree \( T \) can be transformed into a spanning tree \( T^* \) by a sequence of fundamental exchanges and
2. any fundamental exchange can be split as a sequence of neighbour exchanges.

It is obvious that any spanning tree $T$ can be transformed into a spanning tree $T^*$ by a sequence of neighbour exchanges, and hence neighbour exchanges can be thought of as the building blocks for the transformation of any spanning tree into another.

Note 7.2.2: In [11], the main result is that in a 2-connected graph any neighbour exchange can be split into a sequence of endline exchanges. Since any endline exchange changes the diameter of a spanning tree by at most one, the interpolation property for the tree-diameter set follows.

Consider the neighbour exchanges transferring $T$ into $T+x-y$ in the examples of Figure 7.1. The following become obvious.

1. If $y$ is a pendant edge the maximum increase in the diameter in a neighbour exchange of the form $T+x-y$ is one, and in this case $x$ is adjacent to pendant vertex of a diametral path of $T$. 
2. If \( y \) is adjacent to a pendant edge, the maximum increase in the diameter in a neighbour exchange of the form \( T+x-y \) is two and in this case \( x \) is adjacent to a pendant vertex of a diametral path of \( T \).

Let \( v \) be the vertex of intersection of \( x \) and \( y \).

**Case 1:** The edge \( v \) is not the centre of \( T \). In any neighbour exchange with maximum increase in the diameter, a diametral path of \( T \) gets extended through \( x \). Moreover the edge \( y \) and the point \( v \) do not lie on this diametral path of \( T \). This maximum increase in the diameter can be calculated as \( 1 + (\text{eccentricity of the vertex } v, \text{ in the component containing } v \text{ in } T-y) \).

Since the above eccentricity of \( v \) can be at most \( \lfloor d(T)/2 \rfloor - 1 \), it follows that the maximum increase possible in any neighbour exchange \( T+x-y \) is \( \lfloor d(T)/2 \rfloor \) and in this case

1. \( x \) is adjacent to an end vertex of a diametral path of \( T \), and
2. y does not lie on this diametral path, but lies on another diametral path.

Case 2: The edge \( v \) is the centre of \( T \). Here \( d(T) \) is an odd integer. It is not difficult to see that when the increase in the diameter of \( T+x-y \) is maximum, \( x \) is adjacent to an end vertex of a diametral path of \( T \) and the maximum change in the diameter is \( \lfloor d(T)/2 \rfloor \).

**Theorem 7.2.3:** Let \( G \) be a connected graph and let \( \{ d_1, d_2, \ldots, d_n \} \) be the tree-diameter set of \( G \). Then \( d_{i+1} \leq 3d_i/2 \) for \( i = 1, 2, \ldots, n-1 \).

**Proof:** Let \( T_j \) be a spanning tree with \( d(T_j) = d_j \) for \( j = 1, 2, \ldots, n \). There can be more than one spanning tree with diameter \( d_j \) and we choose any one of them as \( T_j \). If \( d_{i+1} > 3d_i/2 \), then none of the trees \( T_j, j > i \) can be reached from any \( T_k, k \leq i \) by neighbour exchanges, since each such exchange can increase the diameter by at most \( \lfloor d_i/2 \rfloor \), a contradiction to Note 7.2.1. Hence \( d_{i+1} \leq \lfloor 3d_i/2 \rfloor \) for \( i = 1, 2, \ldots, n-1 \). Since \( d_{i+1} \) is an integer this is equivalent to \( d_{i+1} \leq 3d_i/2 \) for \( i = 1, 2, \ldots, n-1 \). 

By taking \( n = 2 \) in the next theorem, we can see that equality can be achieved in the relation \( d_{i+1} \leq 3d_i/2 \).
Theorem 7.2.4: If $S = \{d_1, d_2, \ldots, d_n\}$ is a set of positive integers with $d_n \leq 3d_1/2$, then $S$ is a tree-diameter set of some graph $G$, that is, $S$ is a feasible tree-diameter set.

Proof: We shall construct the graph $G$ and it can be easily verified that $S$ is the tree-diameter set of $G$.

Let $b_i = d_i - d_1$, $i \geq 2$.

Case 1: Let $d_1$ be even

Consider the graph shown in Figure 7.2(a). The vertex $u_{d_1/2}$ is joined to each $v_{b_i}$ for $i = 2, 3, \ldots, n$. A diametral path of length $d_1$ is given by the path

$u_{d_1 + 1}, u_{d_1}, u_{d_1 - 1}, \ldots, u_2, u_1$ for $i = 1$ and by

$u_{d_1 + 1}, u_{d_1}, \ldots, u_{(d_1/2) + 1}, v_1, v_2, \ldots, v_{b_1}, u_{d_1/2}, u_{(d_1/2) - 1}, \ldots, u_2, u_1$ for $i > 1$.

In the absence of the vertices $w_i$, if $b_n = d_1/2$, the path $P = u_{d_1 + 1}, u_{d_1}, \ldots, u_{d_1/2}, v_{b_n}, v_{b_n - 1}, \ldots, v_1$ will introduce a diametral path of length $d_1 + 1$. In case $d_1 + 1$ is not in $S$, this diametral path has to be
avoided. This is done by introducing the vertices \( w_2, w_3, \ldots, w_{d_1/2} \) as shown. This inclusion extends the path \( P \) to a diametral path of length \( d_n \) and moreover no new diametral path of unwanted length is introduced. Note that in the case when \( d_1 \) is odd the above situation does not arise.

**Case 2:** Let \( d_1 \) be odd.

Consider the graph shown in Figure 7.2 (b). The vertex \( u_{(d_1+1)/2} \) is joined to each \( v_{b_i}, i = 2,3,\ldots,n \).

In particular, for the tree-diameter sets \( \{ 8,9,11,12 \} \) and \( \{ 9,11,12 \} \), the graphs are given in Figure 7.3.

**Corollary 7.2.5:** \( S = \{ d_1, d_2 \} \) is a feasible tree-diameter set if and only if \( d_2 \leq \frac{3d_1}{2} \).

**Proof:** Follows from Theorems 7.2.3 and 7.2.4.

**Corollary 7.2.6:** If \( S = \{ d_1,d_2,\ldots,d_n \} \) is a tree-diameter set with \( d_n \leq \frac{3d_1}{2} \), then \( S+k = \{ d_1+k, d_2+k,\ldots, d_n+k \} \) is also a feasible tree-diameter set for any \( k \geq 0 \).

**Proof:** Follows from Theorem 7.2.4.
**Corollary 7.2.7**: Let \( S = \{d_1, d_2, \ldots, d_n\} \) be any set of positive integers with \( d_i < d_{i+1} \) for \( i = 1, 2, \ldots, n-1 \). Then there exists a non-negative integer \( K \) such that for any \( k \geq K \), \( S' = \{d_1+k, d_2+k, \ldots, d_n+k\} \) is a feasible tree diameter set.

**Proof**: If \( d_n < \frac{3d_1}{2} \), then take \( K = 0 \) and the result is true in view of Corollary 7.2.6.

So, let \( d_n \geq \frac{3d_1}{2} \). Hence \( 2d_n > 3d_1 \). Choose \( K = 2d_n-3d_1 \). Then \( S + K = \{2(d_n-d_1), 2d_n+d_2-3d_1, 2d_n+d_3-3d_1, \ldots, 3(d_n-d_1)\} \) is a feasible tree-diameter set by Theorem 7.2.4 and the corollary follows.

**Note 7.2.8**: Corollary 7.2.7 is a stronger version of the conjecture in [11]. In that conjecture it was assumed that \( d_{i+1} < 2d_i-1 \), for \( i=1, 2, \ldots, n-1 \), which turns out to be unnecessary.

### 7.3. TREE-ECCENTRICITY SET

Let \( T \) be a spanning tree of \( G \). Let \( x \) be an edge in \( G-T \) and let \( Y \) be an edge in the unique cycle of \( T+x \).

An endline exchange \( T \rightarrow T+x-y \) is said to be \( u \)-free if \( u \) is not the vertex common to \( x \) and \( y \).
In [11], the basic idea is that in a 2-connected graph, any spanning tree can be transformed into any other spanning tree, by a sequence of endline exchanges. We show that this is also possible in the case of graphs with a single cut vertex.

**Lemma 7.3.1**: Let \( G \) be a 2-connected graph and \( u \) be any vertex of \( G \). Let \( T \) be a spanning tree of \( G \). Then any neighbour exchange \( T \rightarrow T+x-y \) can be split up into a sequence of \( u \)-free endline exchanges.

The proof of the lemma is illustrated by an example given in Figure 7.4.

**Proof**: Consider the cycle \( C \) formed in \( T+x \). Let \( v_t \) be the vertex of this cycle nearest to \( u \) in \( T+x \) (\( u \) may be the same as \( v_t \)). The edge \( y \) can be reached from \( x \) along the cycle \( C \), without crossing the vertex \( v_t \). Let this path be made up of the edges

\[
x = x_1, x_2, \ldots, x_k = y.
\]

Let \( x_i \cap x_{i+1} = v_{i}, 1 \leq i \leq k-1 \). Starting from \( T \), the edge \( x_2 \) is made as a pendant edge by a sequence of endline exchanges by removing the end vertices one by one from the tree growth \( T_1 \) (in \( T \)), say, at the vertex \( v_1 \). Thus the vertices from the tree growth \( T_1 \)
get attached to some other tree growths and each endline exchange used is u-free. Now by a u-free endline exchange, delete \( x_2 \) and include \( x_1 \). By reversing the endline exchanges now, the tree growth \( T_1 \) is brought back to its original position. The above process is repeated successively to delete \( x_{i+1} \) and include \( x_i \) for \( i = 2, 3, ..., k-1 \).

**Theorem 7.3.2**: Let \( G \) be a graph with just one cut vertex. Then \( D(G) \), the tree-diameter set of \( G \), is a set of consecutive integers.

**Proof**: Let \( u \) be the cut vertex of \( G \). Let \( T_1 \) and \( T_2 \) be any two spanning trees of \( G \). We shall show that \( T_1 \) can be transformed into \( T_2 \) by a sequence of endline exchanges, which will prove the theorem.

In each block of \( G \), the portion of \( T_1 \) in that block has to be transformed into the corresponding portion of \( T_2 \) in that block. By Lemma 7.3.1, this can be achieved by u-free endline exchanges in that block. Since each u-free endline exchange in that block is an endline exchange of \( G \), we are through. (Note that an endline exchange in any block of \( G \) is an endline exchange in \( G \) if and only if it is u-free).
7.4. TREE-ECCENTRICITY SETS OF 2-CONNECTED GRAPHS

In this section, let \( E(u) \) stand for the tree-eccentricity set \( \{e_1, e_2, ..., e_m\} \) of \( u \).

Here we characterize the tree-eccentricity sets for 2-connected graphs.

**Lemma 7.4.1:** Let \( G \) be a connected graph. Let \( T \rightarrow T+x-y \) be a \( u \)-free endline exchange. Then \( e_{T+x-y}(u) \) can differ from \( e_T(u) \) by at most one.

**Proof:** If \( x \) is incident with \( u \) in \( T \), then \( y \) is not incident with \( u \). To get \( T+x-y \) from \( T \), we add a pendant edge at \( u \) and delete a pendant edge of \( T \). Hence \( e_{T+x-y}(u) = e_T(u) \) or \( e_T(u) = 1 \).

If \( x \) is not incident with \( u \), then again we get that the eccentricity of \( u \) can be changed by at most one.

**Note 7.4.2:** The above result is not true if the word \( u \)-free is omitted. For example, consider \( T \), \( x \) and \( y \) as shown in Figure 7.5, where \( x \notin E(T) \).
Here \( e_T(u) = 6 \) and \( e_{T+x-y}(u) = 4 \).

**Theorem 7.4.3:** If \( G \) is a 2-connected graph then for any \( u \in V(G) \), \( E(u) \) is a set of consecutive integers.
Proof: Follows from Lemma 7.3.1 and Lemma 7.4.1.

Note 7.4.4: It is not possible to extend the above theorem to separable graphs, even with just one cut vertex. For example in the graph of Figure 7.6, $E(u) = \{3, 4, 6, 7\}$. But it is easy to see that if $v$ is the cut vertex of $G$, then $E(v) = \{w_1, w_1 + 1, \ldots, w_2\}$ is a set of consecutive integers, where $w_1$ is the maximum of the first elements in all the tree-eccentricity sets of $v$, one set for each block of $G$. Similarly $w_2$ is the maximum of the last elements in all the tree-eccentricity sets of $v$. If we use the next theorem, it will be obvious that $w_2 \geq 2w_1 - 1$.

Theorem 7.4.5: Let $G$ be a 2-connected graph. Then for each $u \in V(G)$, there exists an integer $k > 0$ such that $\{\lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 1, \ldots, k - 1\} \subseteq E(u)$, or, in other words $e_m \geq 2e_1 - 1$ where $E(u) = \{e_1, e_1 + 1, \ldots, e_m\}$. Also, $E(u) \cup D(G)$ is a set of consecutive integers.

Proof: Let $k$ be the maximum of the lengths of the cycles passing through $u$. We show that this $k$ satisfies the required condition.

Let $C$ be a cycle of length $k$ passing through $u$. 
Consider a spanning tree $T$ of $G$ which contains the cycle $C$ except for an edge incident with $u$. Let $e_T(u) = t$. Clearly $t \geq k-1$ and $t \in E(u)$.

In $G$, every vertex is at a distance $\leq \lceil k/2 \rceil$ from $u$; for if $d(u,v) > \lceil k/2 \rceil$, then there is a cycle containing $u$ and $v$ which is of length $> k$, a contraction. Consider a spanning tree $T'$ of $G$ which contains only a shortest path from $u$ to any other vertex. Let $e_{T'}(u) = s$. Clearly $s \leq \lfloor k/2 \rfloor$ and $s \in E(u)$.

Since $G$ is a 2-connected graph, by Theorem 7.4.3, every integer between $s$ and $t$ is in $E(u)$. Hence

$$\{ \lfloor k/2 \rfloor, \lfloor k/2 \rfloor +1, \ldots, k-1 \} \subseteq E(u).$$

This is the same as $e_m \geq 2e_1 - 1$ where

$$E(u) = \{ e_1, e_1 +1, \ldots, e_m \}.$$

Since $k-1, e_m \in E(u)$, we have $k-1 \leq e_m$. That is $k \leq e_m +1$. Let $d_1$ be the diameter of $T'$.

$$d_1 \leq d_1 \leq 2e_{T'}(u) \leq 2s \leq 2\lfloor k/2 \rfloor \leq k \leq e_m +1.$$

That is $d_1 \leq e_m +1$, which implies that $D(G) \cup E(u)$ is a set of consecutive integers. \qed
We show that the necessary condition given in Theorem 7.4.5 is also sufficient to make a set of consecutive positive integers to be a tree-eccentricity set of some vertex of a 2-connected graph $G$.

**Theorem 7.4.6**: Let $S = \{a, a+1, \ldots, b\}$ be a set of consecutive integers such that $b \geq 2a-1$. Then there exists a 2-connected graph $G$ and $u \in V(G)$ such that $E(u) = S$. That is, $S$ is a feasible tree-eccentricity set.

**Proof**: Given the set $S$, we construct the required graph $G$ as follows. Consider a cycle with $b+1$ vertices named $v_0, v_1, \ldots, v_b$. Let $u = v_0$.

Introduce the edges $(v_0, v_j)$ for $j=2a, 4a-1, 6a-2, \ldots, 2(p-1)a-(p-1)$, choosing $p$ so that $2pa-(p-1) < b$ and $2(p+1)a-p \geq b$. The resulting graph is $G$. The verification is straightforward. 

For example, consider $S = \{7, 8, \ldots, 35\}$. The graph $G$ is shown in Figure 7.7. Here $j$ takes the values 14 and 27. Also $p = 2$, $a = 7$ and $b = 35$.

Consider the spanning tree $T_1$ and $T_2$ of $G$ where

$$E(T_1) = E(G) - \{(v_7, v_8), (v_{20}, v_{21}), (v_{33}, v_{34})\}$$

and $T_2$ as $E(T_2) = E(G) - \{(u,v_{14}), (u,v_{27}), (u,v_{35})\}$.

Clearly $e_{T_1}(u) = 7$ and $e_{T_2}(u) = 35$. 

Corollary 7.4.7: A necessary and sufficient condition for a set of positive integers \( \{e_1, e_2, \ldots, e_m\} \) with 
\[ e_i > e_{i-1}, \quad 2 \leq i \leq m, \]
to be a feasible tree-eccentricity set of a vertex of a 2-connected graph is that it is a set of consecutive integers and \( e_m \geq 2e_1 - 1 \).

Note 7.4.8: The statement that \( D(G) \cup E(u) \) is a set of consecutive integers is not true in general. For example in the graph given in Figure 7.8, 
\[ D(G) = \{6\} \text{ and } E(u) = \{3, 4\}. \]

7.5. TREE-ECCENTRICITY SETS OF GENERAL GRAPHS

We have seen that when \( G \) is not a 2-connected graph then \( E(u) \) need not be a set of consecutive integers. Here we find a relation between consecutive elements of \( E(u) \).

Theorem 7.5.1: Let \( \{e_1, e_2, \ldots, e_m\} \), \( m \geq 2 \), be the tree-eccentricity set of a vertex \( u \) of \( G \). Then 
\[ e_{i+1} \leq 2e_i, \quad i = 1, 2, \ldots, m-1. \]

Proof: Let \( T \) be a spanning tree of \( G \) with \( e_T(u) = e_1 \). Let \( T \rightarrow T+x-y \) be a neighbour exchange. Let us assume that in this exchange, the eccentricity of \( u \) is increased to the maximum. Clearly the eccentricity of \( u \) could increase only through the edge \( x \). The following also become clear (See Figure 7.9).
Theorem 7.5.2: If $S = \{e_1, e_2, \ldots, e_m\}$ with $e_m < 2e_1$, then $S$ is a feasible tree-eccentricity set, that is, there exist a graph $G$ and $u \in V(G)$ such that $E(u) = S$.

Proof: The proof is by construction.

Let $b_i = e_i - e_1$, $2 \leq i \leq m$.

The graph is as shown in Figure 7.10(a) where the vertex $w_1$ is joined to all $v_{b_i}$, $2 \leq i \leq m$. An easy verification proves the theorem.

Figure 7.10(b) shows the graph $G$, in the case where $S = \{6, 7, 8, 12\}$.

Corollary 7.5.3: $S = \{e_1, e_2\}$ is a feasible tree-eccentricity set if and only if $e_2 \leq 2e_1$.

Note 7.5.4: Corollary 7.5.3 shows that the relation...
\( e_{i+1} \leq 2e_i \) is the best possible, when \( E(u) \) has exactly two elements.

**Theorem 7.5.5**: If \( u \) and \( v \) are adjacent in \( G \) and \( E(u) = \{ e_1, e_2, \ldots, e_m \} \) and \( E(v) = \{ f_1, f_2, \ldots, f_s \} \) then \( |e_1-f_1| \leq 1 \) and if \( f_s \geq e_m \) then \( f_s \leq 2e_m \).

**Proof**: \( |e_1-f_1| \leq 1 \) follows from the fact that \( e_1 = e(u) \), the eccentricity of \( u \) in \( G \), and \( f_1 = e(v) \).

If \( f_s = e_m \) there is nothing to prove. Let \( f_s > e_m \).

Let \( T \) be a spanning tree of \( G \) with \( e_T(v) = f_s \).

If \( (u,v) \in E(T) \) then \( e_m \geq e_T(u) \geq f_s-1 \) and hence \( f_s \leq e_m+1 \). So let \( (u,v) \notin E(T) \). Hence \( k = d_T(u,v) > 1 \).

Let \( w \) be the vertex adjacent to \( u \) in the path of \( T \) from \( u \) to \( v \).

Let \( z \in V(G) \) be such that \( d_T(v,z) = f_s \). Let \( T_1 = T + (u,v) - (u,w) \). Clearly \( z \neq w \), since otherwise \( e_m \geq f_s \), a contradiction. The path from \( v \) to \( z \) in \( T \) contains \( u \), since otherwise \( e_m \geq e_{T_1}(u) \geq f_s+1 \), a contradiction.

Hence

\[
e_m \geq e_T(u) \geq \max\{d_T(u,v), d_T(u,z)\}
\]

\[= \max \{ k, f_s-k \} \]
Hence \( e_m \geq \lceil f_s/2 \rceil \) and \( f_s \leq 2e_m \).

The relation between the least and the largest elements of the tree-diameter set and a tree-eccentricity set can be given as

**Lemma 7.5.6**: For any \( u \), \( \lceil d_i/2 \rceil \leq e_1 \leq d_i \) and \( \lceil d_n/2 \rceil \leq e_m \leq d_n \).

**Proof**: Let \( T \) be a spanning tree of \( G \) with \( e_T(u) = e_1 \) and \( d(T) = d_i \) for some \( i \). Clearly \( \lceil d_i/2 \rceil \leq \lceil d_i/2 \rceil = \text{radius of } T \leq e_T(u) = e_1 \).

The other parts are proved by considering the spanning trees with diameter \( d_1 \), diameter \( d_n \) and eccentricity \( e_m \) respectively.
FIGURE 7.1 Some examples
FIGURE 7.2 Graphs with given tree-diameter sets
FIGURE 7.3 Two examples
FIGURE 7.4 Illustration of an algorithm
FIGURE 7.5 An endline exchange

FIGURE 7.6 Graph with $E(u) = \{3, 4, 6, 7\}$

FIGURE 7.7 Graph with $E(u) = \{7, 8, \ldots, 35\}$
FIGURE 7.8 An example
FIGURE 7.9 A part of T

FIGURE 7.10 A construction