CHAPTER 6

OPTIMUM ORDERING POLICIES WITH THE EFFECT OF WEATHER CONDITION, RANDOM LEAD TIMES AND SALVAGE COST

6.1 INTRODUCTION

Age replacement or block replacement policies have been discussed by Cleroux and Hanscom (1974) and different ordering policies have been considered by Allen and D’Esopo (1968). Kaio and Osaki (1978) have considered a one-unit model with two types of orders (emergency and regular) and with two different types of constant lead times. Kalpakam and Shahul Hameed (1981) have discussed a model in which the lead times are random variables. Sung and Park (1986) have considered an optimum ordering policy for an equipment having a sensing device with constant lead times. Apart from this, many papers in reliability theory, have been studied under the specific assumption that the atmospheric (weather) conditions are always normal and many of the electronic systems are affected by the change in the weather conditions. For example, an air conditioner is always used to support the functioning of a computer system. Hence the purpose of the present chapter is to study the optimum ordering policy for a one-unit system with two types of random lead times, taking the cost effectiveness as the objective function. The normal weather condition is acting like a sensing device which is in working condition, whereas the abnormal
condition is like a failure of the sensing device. Thus, in the abnormal weather condition if the unit fails, it cannot be detected and hence a penalty cost is incurred for the unit. Further we assume that the failure of the unit in the normal weather condition can also be detected.

Hence from the view point of the system effectiveness, the reliability characteristics are important and it is equally important to consider policies which are cost effective. The ratio (steady state availability) / (expected cost rate) serves as a good measure for the cost effectiveness, since it takes into consideration both the system effectiveness and the cost.

The plan of this chapter is as follows. In section 6.2, we give the description of the model. Section 6.3 deals with the assumptions and notation. In the penultimate section, we give the analysis of the model under consideration. Finally in section 6.5, we describe the model with a numerical example.

6.2 MODEL DESCRIPTION

We consider a one-unit system where each failed unit is scrapped without repair and a spare can only be provided after a lead time following an order. The operational status of the unit is continuously monitored by the weather condition which acts as a sensing device for the unit. The life time of the unit is a generally distributed random variable, whereas the change in the weather from normal to abnormal and vice versa., are exponentially distributed with parameters $\lambda$ and $\mu$ respectively. Lead times for the regular and emergency orders are generally distributed random variables and the planning horizon is infinite. The unit just begins to operate at time 0 and the weather condition is normal initially. If the unit does not fail up to a pre-specified time $T$, a regular order for a spare is placed at epoch $T$. After a lead time $v_s$, the spare is delivered and the unit is replaced by the spare instantaneously even if the unit is in an operable condition. If the unit fails before time $T$, the failure is detected immediately, since the weather condition is normal and an emergency order is placed
and the spare takes over the operation as soon as it is delivered after a lead time $v_s$. If, on the other hand, the weather condition is under abnormal state when the unit fails, the failure of the unit cannot be detected immediately until the weather condition becomes normal. If it becomes normal at $x(x < T)$, then an emergency order is placed at epoch $T$. Further, the state of the weather condition after time $T$ is immaterial. Once we place an order (emergency or regular), subsequent state of the weather conditions are of no significance.

6.3 ASSUMPTIONS AND NOTATION

The following assumptions and notation are associated with the model and they are

i. $Pr\{v_s \leq v\} \geq Pr\{v_r \leq v\}$. This justifies the placing of an emergency order. In addition, this implies that the mean of the lead time for an emergency order is less than or equal to that for a regular order.

ii. The failure of the unit and change in weather are mutually independent.

iii. The weather, if in normal condition, detects the failure of the unit with probability one.

iv. If the weather condition is in abnormal state and hence does not detect the failure of the unit, a penalty cost is imposed.

v. The new unit is put online immediately on arrival, even if the original unit is in working condition.

vi. A discarded operable unit fetches a salvage value proportional to the expected residual lifetime.

vii. Ordering cost associated with an emergency order is not less than that associated with a regular order.
6.4 ANALYSIS

The cost effectiveness is defined by the expression

\[
\text{steady state availability} \times \frac{\text{expected cost rate}}{\text{expected uptime in a cycle}}
\]

since the e-events are regenerative. It can be shown that the cost effectiveness \( E(T) \) is given by

\[
E(T) = \frac{\text{expected uptime in a cycle}}{\text{expected cost per cycle}}
\]

where a 'cycle' denotes the time interval between two successive e-events.

\( f(\cdot), \hat{F}(\cdot), m \)  pdf, survivor function and mean value of the lifetime of the unit
\( g_r(\cdot), m_r (g_r(\cdot), m_r) \)  pdf and mean value of the lead time corresponding to an emergency (regular) order
\( C_1 \)  down time cost per unit time
\( C_2 \)  salvage value per unit time
\( C_3 \)  penalty cost per unit time
\( C_4, (C_b) \)  ordering cost associated with an emergency (regular) order
\( P_{ww}(t) \)  \( \Pr \{ \text{weather condition is normal at time } t \mid \text{weather condition is normal at time } t=0 \} \)
\( P_{wn}(t) \)  \( \Pr \{ \text{weather condition is abnormal at time } t \mid \text{weather condition is normal at time } t=0 \} \)
e-event the time instant at which a unit begins to operate
\( E(T) \)  cost effectiveness associated with the pre-specified time parameter \( T \).
Now the expected uptime in a cycle can be written by observing that the following mutually exclusive possibilities are also exhaustive (Figure 6.1).

i. the unit fails before time $T$

ii. the unit fails after time $T$, but before the realization of the regular order

iii. the unit fails after time $T$ and also after the realization of the regular order.

In case (i) we have the following possibilities

(a) when the unit fails, the weather condition is normal

(b) when the unit fails, the weather condition is abnormal and becomes normal before time $T$

(c) when the unit fails, the weather condition is abnormal and it continues to be in the same state up to time $T$.

Hence, the expected operating time in a cycle is given by

$$U(T) = \int_0^T xf(x)dx + \int_0^\infty g_r(v)dv \left[ \int_T^{T+v} xf(x)dx + \int_{T+v}^\infty (T+v)f(x)dx \right]. \quad (6.1)$$

On simplification, equation (6.1) becomes

$$U(T) = \int_0^T \bar{F}(x)dx + \int_0^\infty g_r(v)dv \int_T^{T+v} \bar{F}(x)dx. \quad (6.1')$$

The expressions for $P_{wu}(t)$ and $P_{uw}(t)$ are easily obtained as

$$P_{wu}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$P_{uw}(t) = \frac{\lambda}{\lambda + \mu} (1 - e^{-(\lambda+\mu)t}). \quad (6.2)$$

Further, the various costs considered and their expected values in a cycle are given by (see Figure 6.2).
(i) The expected downtime cost during the interval between the failure of a unit and the realization of an order is given by

\[ C_1 \left[ \int_0^\infty v g_s(v) dv \int_0^T P_{w^0}(x) f(x) dx + \int_0^\infty g_s(v) dv \int_0^T \int_0^{T-x} (u + v) P_{w^0}(x) f(x) u e^{-n u} du \ dx + \int_0^\infty g_s(v) dv \right] \]

\[ \left[ \int_0^T (T + v - x) P_{w^0}(x) e^{-n(T-x)} f(x) dx + \int_T^{T+v} (T + v - x) f(x) dx \right] \] \hfill (6.3)

(ii) The expected salvage cost associated with the residual lifetime (from the time of arrival of the spare if the original unit has not failed by then) is given by

\[ -C_2 \int_0^\infty g_s(v) dv \int_{T+v}^\infty (x - T - v) f(x) dx. \]

On simplification, this becomes

\[ -C_2 \int_0^\infty g_s(v) dv \int_{T+v}^\infty F(x) dx. \] \hfill (6.4)

(iii) The expected penalty cost associated with the failure of the unit where the weather condition is in the abnormal state (since the weather condition is in abnormal state, failure of the unit cannot be detected) is given by

\[ C_3 \left[ \int_0^\infty g_s(v) dv \int_0^T \int_0^{T-x} u f(x) P_{w^0}(x) u e^{-n u} du \ dx + \int_0^\infty g_s(v) dv \int_0^T (T - x) f(x) P_{w^0}(x) e^{-n(T-x)} dx \right]. \] \hfill (6.5)

(iv) The expected ordering cost per cycle is

\[ C_4 \left[ \int_0^T P_{w^0}(x) f(x) dx + \int_0^T (1 - e^{-n(T-x)}) P_{w^0}(x) f(x) dx \right] + C_5 \left[ \int_0^T e^{-n(T-x)} P_{w^0}(x) f(x) dx + F(T) \right]. \]

On simplification, this becomes

\[ (C_4 - C_5) \left[ F(T) + \frac{\lambda}{\lambda + \mu} \int_0^T (e^{-\lambda x} - e^{\mu x} f(x) dx) \right] + C_5. \] \hfill (6.6)
Thus the total expected cost $K(T)$ per cycle is given by

$$
K(T) = \left[ C_1(m_e - m_r) + \frac{\lambda(C_1 + C_3)}{\mu(\lambda + \mu)} + (C_4 - C_5) \right] F(T) + \frac{\lambda e^{-\mu T}}{\lambda + \mu} \int_0^T e^{\mu x} f(x) dx \left[ C_1(m_e - m_r) - \frac{(C_1 + C_3)}{\mu} + (C_5 - C_4) \right] + \frac{\lambda e^{-\mu T}}{\lambda + \mu} \int_0^T e^{-\lambda x} f(x) dx \left[ C_1(m_e - m_r) + \frac{(C_1 + C_3)}{\mu} + (C_4 - C_5) \right] - \frac{\lambda(C_1 + C_3)}{\mu(\lambda + \mu)} \int_0^T e^{-(\lambda + \mu) x} f(x) dx + C_1 \int_0^\infty g_r(v) dv \int_T^{T+v} F(x) dx - C_2 \int_0^\infty g_r(v) dv \int_{T+v}^\infty \bar{F}(x) dx + C_5. \quad (6.7)
$$

Hence

$$
E(T) = \frac{\int_0^T \bar{F}(x) dx + \int_0^\infty g_r(v) dv \int_T^{T+v} \bar{F}(x) dx}{K(T)}, \quad (6.8)
$$

where $K(T)$ is given in equation (6.7). The following observations are also pertinent and we note the following.

(i) $U(T)$ and $K(T)$ are continuous and positive for all $T \geq 0$

(ii) $U(T) \to m$ and $K(T) \to$ a finite value as $T \to \infty$

(iii) $\lim_{T \to 0} E(T)$ is finite and equals

$$
\frac{U(0)}{C_1 m_r - C_2 m + U(0)(C_2 - C_1) + C_5}, \quad (6.9)
$$

where

$$
U(0) = \int_0^\infty g_r(v) dv \int_0^v \bar{F}(x) dx. \quad (6.10)
$$

(iv) As $T \to \infty$

$$
C(\infty) \to \frac{m}{K(\infty)},
$$

where

$$
K(\infty) = C_1(m_e - m_r) + \frac{\lambda(C_1 + C_3)}{\mu(\lambda + \mu)} \left( 1 + \int_0^\infty e^{-(\lambda + \mu) x} f(x) dx \right) + C_4 \quad (6.11)
$$
Therefore a sufficient condition for the existence of a finite $T^* (T^* > 0)$ which maximizes the cost effectiveness $E(T)$ is $C'(0) > 0$; this condition can be written as

$$U'(0)[C_1m_* - C_2m + C_5] > U(0)[(C_1(m_\epsilon - m_\tau) + C_4 - C_5)f(0) + C_1]$$

(6.12)

Equation (6.9) will be satisfied if the parameters $C_1, C_2, C_4, C_5, m_\epsilon, m_\tau, m_\sigma$ satisfy the following conditions namely

$$C_1m_* - C_2m + C_5 > 0$$
$$C_1(m_\epsilon - m_\tau) + C_4 - C_5 < 0$$
$$C_1 > 0$$
$$C_1 - C_2 < 0.$$  

(6.13)

To prove a unique optimum solution, we have

$$E(T) = \frac{\int_0^T \tilde{F}(x)dx + \int_0^\infty g_r(v)dv \int_T^{T+v} \tilde{F}(x)dx}{K(T)}.$$  

(6.14)

Clearly $E(T) > 0$ and $E(0) = \frac{\int_0^\infty g_r(v)dv \int_0^T \tilde{F}(x)dx}{K(0)}$,

where $K(0) = C_1m_* - C_2m + U(0)(C_2 - C_1) + C_5$, and

$$\lim_{T \to \infty} E(T) = E(\infty) = \frac{m}{K(\infty)},$$  

where $K(\infty)$ is given in equation (6.11).

We now prove the existence of an optimum $T$ which maximizes the cost effectiveness. For that, let us consider the numerator $n(T)$ of the derivative of $E(T)$. It is given by

$$n(T) = K(T) \left( \int_0^\infty g_r(v)F(T + v)dv - \int_0^T F(x)dx + \int_0^\infty g_r(v)dv \int_T^{T+v} F(x)dx \right)$$

$$- C_1 \int_0^\infty g_r(v)(F(T + v) - F(T))dv + (C_1(m_\epsilon - m_\tau) + C_4 - C_5)$$

$$\left\{ \frac{\lambda e^{-(\lambda + \mu)T} + \mu}{\lambda + \mu} \right\} f(T) + \frac{\lambda \mu}{\lambda + \mu} \int_0^T (e^{\mu x} - e^{-\lambda x})f(x)dx \right\}.$$  

(6.16)
Define $e(T) = \frac{n(T)}{F(T)}$ and hence, it is given by

$$e(T) = K(T) \int_0^\infty g_r(v)(1 - R_v(T))dv - \left[ \int_0^T \bar{F}(x)dx + \int_0^\infty g_r(v)dv \int_T^{T + \nu} \bar{F}(x)dx \right]$$

$$\{ \int_0^\infty [C_1R_v(T) + C_2(1 - R_v(T))]g_r(v)dv + (C_1(m_e - m_r) + C_4 - C_5) \}

$$r(T) \left\{ \frac{\lambda e^{-(\lambda + \mu)T} + \mu}{\lambda + \mu} \right\} + \frac{\lambda \mu e^{-\mu T}}{(\lambda + \mu)F(T)} \int_0^T (e^{e^T} - e^{-\lambda T})f(x)dx \right\}$$

(6.17)

where $R_v(T) = \frac{F(T + x) - F(T)}{F(T)}$ and $r(T) = \frac{f(T)}{F(T)}$.

The failure rates $R_v(T)$ and $r(T)$ are assumed to be differentiable and have the same monotone properties.

Note that

$$e(0) = K(0) \int_0^\infty g_r(v)\bar{F}(v)dv - \left[ \int_0^\infty g_r(v)dv \int_0^\nu \bar{F}(x)dx \right] \{ f(0)(C_1(m_e - m_r) + C_4 - C_5) \}

+ \int_0^\infty [C_1F(v) + C_2\bar{F}(v)]g_r(v)dv \}, \quad (6.18)$$

where $K(0)$ is given in equation (6.15) and

$$e(\infty) = K(\infty) \int_0^\infty g_r(v)(1 - R_v(\infty))dv - m(C_1(m_e - m_r) + C_4 - C_5)\frac{\mu r(\infty)}{\lambda + \mu}

- m \left\{ C_2 \int_0^\infty g_r(v)(1 - R_v(\infty))dv + C_1 \int_0^\infty g_r(v)R_v(\infty)dv \right\}. \quad (6.19)$$

Now

$$r'(T) = -K'(T) \int_0^\infty g_r(v)R_v'(T)dv - \left[ \bar{F}(T) + \int_0^\infty g_r(v)[\bar{F}(T + v) - \bar{F}(T)] \right]

\{ r'(T)\left( \frac{\mu}{(\lambda + \mu)} \right)(C_1(m_e - m_r) + C_4 - C_5) - (C_1 - C_2)

\int_0^\infty g_r(v)R_v(T)dv + (C_1(m_e - m_r) + C_4 - C_5)\frac{\lambda e^{-(\lambda + \mu)T}}{\lambda + \mu}

(r'(T) - (\lambda + \mu)r(T)) + \frac{\lambda \mu e^{-\mu T}}{(\lambda + \mu)F(T)}(e^{-\lambda T} - e^{\mu T})f(T) \}. \quad (6.20)$$
We see from the above equation that $r'(T)\geq 0$ according to whether the failure rate is decreasing or increasing.

**Case (i)**

Let $r(T)$ be a strictly increasing function. Then $\epsilon'(T) < 0$ and hence $\epsilon(T)$ is a strictly decreasing function. Three possibilities arise in this case.

(a). If $\epsilon(0) > 0$ and $\epsilon(\infty) < 0$, it implies that $\epsilon(T) = 0$ for some finite $T^*$ and this $T^*$ is unique as $\epsilon(T)$ is decreasing and continuous. This $T^*$ which maximizes the cost effectiveness $E(T)$, is the optimum ordering time and the corresponding optimum cost effectiveness is given by

$$ E(T) = \frac{\int_0^\infty g_\epsilon(v)[1 - R_\epsilon(T^*)]dv}{r(T^*)/(\lambda + \mu)}(C_1(m_e - m_r) + C_4 - C_5)$$

$$ + C_1 \int_0^\infty g_\epsilon(v)R_\epsilon(T^*)dv + (C_1(m_e - m_r) + C_4 - C_5)$$

$$\left[ \frac{\lambda e^{-(\lambda + \mu)T^*}}{\lambda + \mu} - \frac{\lambda e^{-\mu T^*}}{\lambda + \mu} \right] + \lambda \mu e^{-\mu T^*} F(T^*) \int_0^{T^*} (e^{ux} - e^{-\lambda x})f(x)dx. \quad (6.21)$$

(b). If $\epsilon(\infty) \geq 0$ (which implies $\epsilon(0) > 0$), $E(T)$ is an increasing function. Hence the optimum ordering time $T^* \rightarrow \infty$.

(c). If $\epsilon(0) \leq 0$ (which implies $\epsilon(\infty) < 0$), $E(T)$ is a decreasing function. Hence the optimum order time $T^* = 0$.

Hence we have the following theorem.

**Theorem 1** For units with strictly increasing failure rates

i. If $\epsilon(0) > 0$ and $\epsilon(\infty) < 0$, then the optimum ordering time $T^*$ is unique and finite satisfying $\epsilon(T^*) = 0$

ii. If $\epsilon(0) \leq 0$, then $T^* = 0$ (place a regular order for a spare at the same instant when a unit is put into service)
iii. If $e(\infty) \geq 0$, then $T^* = \infty$ (place an emergency order for a spare as and when a unit fails).

Case (ii)

Let $r(T)$ be a strictly decreasing function. In this case $e'(T) > 0$ and $E(T)$ is a strictly increasing function. Again we have three possibilities as given below.

(a). If $e(0) \geq 0$ (implies that $e(\infty) > 0$), then $E(T)$ is an increasing function in $(0, \infty)$. Hence $T^* \rightarrow \infty$.

(b). If $e(\infty) \leq 0$ (implies that $e(0) < 0$), then $E(T)$ is a decreasing function in $(0, \infty)$. Therefore $T^* = 0$.

(c). If $e(0) < 0$ and $e(\infty) > 0$, there exists a finite and unique $T^*$, such that $e(T^*) = 0$ and this maximizes the cost effectiveness.

Therefore in this case the optimum ordering time $T^*$ is 0 or $\infty$ according to whether $E(0) \leq E(\infty)$. Hence we have the following theorem.

**Theorem 2** For components with decreasing failure rates

i. If $e(\infty) \leq 0$, then the optimum ordering time $T^* = 0$ (place a regular order for a spare when a unit begins to operate)

ii. If $e(0) \geq 0$, then $T^* = \infty$ (place an emergency order for a spare at the instant of failure of a unit)

iii. If $e(0) < 0$ and $e(\infty) > 0$, then $T^* = 0$ or $\infty$ according to whether $E(0) \leq E(\infty)$. 
6.5 NUMERICAL EXAMPLE

For the purpose of illustration, we take the distribution of the lifetime of the unit to be an Erlangian distribution and that of the lead time corresponding to a regular order double exponential. More specifically, we assume that

\[ f(t) = te^{-t}, g_r(t) = \frac{\gamma \delta}{\delta - \gamma}(e^{-\delta t} - e^{-\gamma t}). \]

We have plotted E(T) against T for the various parametric values in Figure 6.3 and numerically obtained the optimal \( T^* \). Table 6.1 gives the relevant details.
Table 6.1 Optimum cost effectiveness Vs. time

<table>
<thead>
<tr>
<th>Graph</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$m_e$</th>
<th>$m_r$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$T^*$</th>
<th>$C(T^*)$</th>
</tr>
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<td>(a)</td>
<td>0.250</td>
<td>2.05</td>
<td>0.012</td>
<td>1.870</td>
<td>1.620</td>
<td>5</td>
<td>0.03</td>
<td>0.01</td>
<td>0.50</td>
<td>0.6</td>
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<td>157.29910</td>
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<tr>
<td>(b)</td>
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<td>3.03</td>
<td>0.011</td>
<td>1.631</td>
<td>1.588</td>
<td>5</td>
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<td>(c)</td>
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<td>3.20</td>
<td>0.012</td>
<td>1.620</td>
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<td>5</td>
<td>0.02</td>
<td>0.01</td>
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<td>91.499680</td>
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</tbody>
</table>
Figure 6.1 Possible realization corresponding to a cycle
Figure 6.2 Possible states of the model
Figure 6.3 Cost Vs. Time

\[ C_1 = 0.550 \quad m_e = 5.0 \quad \lambda = 0.02 \]
\[ C_2 = 3.200 \quad m_r = 1.00 \quad \mu = 0.01 \]
\[ C_3 = 0.012 \quad \gamma = 0.40 \]
\[ C_4 = 1.620 \quad \delta = 0.50 \]
\[ C_5 = 1.510 \]