CHAPTER 3

ON THE CHARACTERISTICS OF A REPAIRABLE STANDBY HUMAN MACHINE AND PROTECTIVE TWO-UNIT SYSTEM

3.1 STATISTICAL CHARACTERISTICS OF A REPAIRABLE STANDBY HUMAN MACHINE SYSTEM

3.1.1 Introduction

Standby systems often find applications in various industrial and other setups. Goel et al (1985), Lee et al (1988), Abbas and Kuo (1990), Dhillon and Yang (1992) and reference therein have discussed the reliability and availability of human-machine systems. Most of these publications assume that the failed system repair times are exponentially distributed and in practice, the repair times of the failed unit are not exponentially distributed. This section deals with a two-unit identical system (one-unit online and other warm standby) with the repair time of the units being generally distributed. Furthermore, the system may fail either due to a common-cause failure or a critical human-error. Further the standby is activated only when the operating unit fails. At time $t=0$, the online unit starts operating, while the other unit is in its standby mode. The human-errors and common-cause failures can
occur in any of the operable states of the system. The system can fail either due to critical human-error, common-cause failure and normal hardware failures.

The organisation of this section is as follows. In section 3.1.1 we give a brief introduction about the subject matter. Sections 3.1.2 deals with the assumptions and notation. In section 3.1.3, we give the governing equations of the model where as in 3.1.4 we obtain the steady-state availability. Section 3.1.5 deals with some special cases and we illustrate the model in section 3.1.6 by giving some numerical examples. The penultimate section 3.1.7 deals with the time-dependent availability of the system. Finally in section 3.1.8, we obtain the reliability and MTTF with and without repair.

3.1.2 Assumptions and Notation

The following assumptions and symbols are associated with this model.

i. Common-cause, human-error and hardware failure rates are constant.

ii. All failures and human-errors are statistically independent.

iii. The system consists of two-identical units.

iv. The repair times of the failed units are arbitrarily distributed.

v. The occurrence of common-cause failure or critical human-error causes the entire system to fail.

vi. The repair begins as soon as a unit fails.

vii. The repaired unit is as good as new.

viii. The switching mechanism for the standby unit is instantaneous.

ix. No further failure can occur when the system is down.
i \quad \text{ith state of the system (see Figure 3.1); } i=0: \text{ online unit is operating while the other unit is in its standby mode; } 
\quad i=1: \text{ online unit has failed, the standby operating; } i=2: \text{ standby unit has failed in its standby mode, but the operating unit is still functioning; } i=3: \text{ both the units have failed due to hardware failure; } i=4: \text{ both the units have failed due to common-cause failure; } i=5: \text{ both the units have failed due to human-error}

\lambda_c, \quad \text{constant common-cause failure rates from state } i \text{ to state } 4, \quad i=0,1,2

\lambda_h, \quad \text{constant human-error rates from state } i \text{ to state } 5, \quad i=0,1,2

\lambda(\eta), \quad \text{constant hardware failure rate for online (standby) unit}

\mu_i, \quad \text{constant repair rates of a failed unit in state } i, \quad i=1,2

\mu_j(x), \quad \text{time-dependent repair rate when the system is in state } j \text{ and has an elapsed repair time of } x, \quad j=3,4,5

P_i(t), \quad \text{probability that the system is in state } i \text{ at time } t, \quad i=0,1,2

P_i(x,t), \quad \text{probability that the failed system is in state } j \text{ and has an elapsed repair time of } x, \quad j=3,4,5

P_i, \quad \text{the steady-state probability that the system is in state } i, \quad i=0,1,...,5

P_i^*(s), \quad \text{the Laplace transforms of } P_i(t)

E_j, \quad \text{the mean time to repair and that the failed system is in state } j

AV_{ss}, \quad \text{steady-state availability of the system}

UAV_{ss}, \quad \text{steady-state unavailability of the system}

AV(t), \quad \text{time-dependent system availability}

N_j(x), \quad \text{probability density function of the repair time when the system is in state } j \text{ and has an elapsed repair time of } x, \quad j=3,4,5

N_j(s), \quad \text{Laplace transforms of } N_j(x)
3.1.3 Governing Equations

The differential equation for $P_0(t)$ is given by

$$\frac{dP_0(t)}{dt} = -\left(\lambda + \lambda_\infty + \lambda_k + \eta\right)P_0(t) + \mu_1 P_1(t) + \mu_2 P_2(t) + \int_0^\infty \mu_3(x) P_3(x,t)dx + \int_0^\infty \mu_4(x) P_4(x,t)dx + \int_0^\infty \mu_5(x) P_5(x,t)dx$$

(3.1)

The above equation (3.1) is obtained by considering the following mutually exclusive and exhaustive possibilities:

i. there is no online, standby, common-cause as well as human-error failures

ii. there could be a repair completion for the online or standby unit.

iii. there could also be a repair completion when the system has failed and finally when the system is in state 4 or state 5 there could be a repair completion of common-cause or human-error failures respectively and hence state 0 occurs.

The repair facility has already elapsed $x$ units of time for repairing the failed system and hence in the interval $0$ to $\infty$, it can come to state 0 after repair completion. On similar line we can argue for state 4 and state 5.

Similarly we get

$$\frac{dP_1(t)}{dt} = \lambda P_0(t) - (\mu_1 + \lambda_c + \lambda_k + \lambda) P_1(t)$$

$$\frac{dP_2(t)}{dt} = \eta P_0(t) - (\mu_2 + \lambda_c + \lambda_k + \lambda) P_2(t)$$

$$\frac{\partial P_3(x,t)}{\partial t} + \frac{\partial P_3(x,t)}{\partial x} + \mu_3(x) P_3(x) = 0$$

$$\frac{\partial P_4(x,t)}{\partial t} + \frac{\partial P_4(x,t)}{\partial x} + \mu_4(x) P_4(x) = 0$$

$$\frac{\partial P_5(x,t)}{\partial t} + \frac{\partial P_5(x,t)}{\partial x} + \mu_5(x) P_5(x) = 0$$

(3.2)

with the boundary conditions given by

$$P_3(0,t) = \lambda [P_1(t) + P_2(t)]$$
\[ P_i(0, t) = \sum_{i=0}^{2} \lambda_i P_i(t) \]
\[ P_i(0, t) = \sum_{i=0}^{2} \lambda_{hi} P_i(t). \]  
(3.3)

At time \( t=0 \), \( P_0(0) = 1, P_i(0) = 0, i = 1,2 \) and \( P_i(x, 0) = 0, i=3,4,5. \)

### 3.1.4 Steady-state Availability Analysis

For the steady-state system, as \( t \to \infty \), we have

\[ a_0 P_0 - \mu_1 P_1 - \mu_2 P_2 = \int_0^{\infty} \mu_3(x) P_3(x) dx + \int_0^{\infty} \mu_4(x) P_4(x) dx + \int_0^{\infty} \mu_5(x) P_5(x) dx \]
\[ \lambda P_0 - a_1 P_1 = 0 \]
\[ \eta P_0 - a_2 P_2 = 0 \]
\[ \frac{\partial P_i(x)}{\partial x} = -\mu_i(x) P_i(x), \quad i = 3,4,5 \]  
(3.4)

with

\[ P_3(0) = \lambda [P_1 + P_2] \]
\[ P_4(0) = \sum_{i=0}^{2} \lambda_{ci} P_i \]
\[ P_5(0) = \sum_{i=0}^{2} \lambda_{hi} P_i \]  
(3.5)

where

\[ a_0 = \lambda + \eta + \lambda_{ao} + \lambda_{bo} \]
\[ a_1 = \lambda + \mu_1 + \lambda_{c1} + \lambda_{h1} \]
\[ a_2 = \lambda + \mu_2 + \lambda_{c2} + \lambda_{h2} \]

Now \( P_i \) is the steady-state probability that the system is in state \( i, i=0,1,...,5 \) and is given by

\[ P_i = \int_0^{\infty} P_i(x) dx, \quad i = 3, 4, 5 \]  
(3.6)
with the normalizing condition given by
\[ \sum_{i=0}^{5} P_i = 1. \] (3.7)

Solving equation (3.4) we have
\[ P_i(x) = P_i(0) \exp\left(- \int_0^x \mu_i(u) du \right), \quad i = 3, 4, 5. \] (3.8)

Thus, from equation (3.6) and (3.8), we get
\[ P_3 = \int_0^0 P_3(x) dx = \int_0^0 P_3(0) dx \exp\left(- \int_0^x \mu_i(u) du \right) dx = \lambda (P_1 + P_2) E_3(x). \]

Similarly we have
\[ P_4 = \int_0^0 P_4(x) dx = \left[ \sum_{i=0}^{2} \lambda_i P_i \right] E_4(x) \]
\[ P_5 = \int_0^0 P_5(x) dx = \left[ \sum_{i=0}^{2} \lambda_i P_i \right] E_5(x) \] (3.9)

where \( E_i = \int_0^0 \exp\left(- \int_0^x \mu_i(u) du \right) dx, \quad i = 3, 4, 5 \) and \( E_i's \) are the mean time to repair from state \( i \) to state 0, \( i=3,4,5 \). Further, for the steady-state probabilities we solve the set of equations (3.4), (3.7) and (3.9) to obtain

\[ P_i = \begin{cases} 
A_i/A_6 & 0 \leq i \leq 2 \\
A_i E_i(x)/A_6 & 3 \leq i \leq 5 
\end{cases} \] (3.10)

where
\[ A_0 = a_1 a_2, \]
\[ A_1 = \lambda a_2, \]
\[ A_2 = \eta a_1, \]
\[ A_3 = \lambda (A_1 + A_2), \]
\[ A_4 = \sum_{i=0}^{2} \lambda_i A_i, \]
\[ A_5 = \sum_{i=0}^{2} \lambda_i A_i, \]
\[ A_6 = \sum_{i=0}^{3} A_i + \sum_{j=1}^{5} A_j E_j(x). \]
Hence, the steady-state availability of the system is given by

\[ AV_{ss} = P_0 + P_1 + P_2 = \frac{A_0 + A_1 + A_2}{A_6}. \]  

(3.11)

Similarly the steady-state unavailability of the system is given by

\[ UAV_{ss} = P_3 + P_4 + P_5 = \frac{A_x E_3(x) + A_x E_4(x) + A_x E_5(x)}{A_6}. \]  

(3.12)

### 3.1.5 Special Cases

#### Case I

If the repair time \( x \) of the system is Gamma distributed with the probability density function of the repair time given by

\[ N_j(x) = \frac{\mu_j^\beta x^{\beta-1} e^{-\mu_j x}}{\Gamma(\beta)} , \quad j = 3, 4, 5; \ x \geq 0, \ \beta > 0, \ \mu_j > 0 \]  

(3.13)

(where \( \beta \) and \( \mu_j \) are the two parameters of the Gamma distribution) then the mean time to repair \( E_j \) is given by

\[ E_j = \int_0^\infty x N_j(x) dx = \frac{\beta}{\mu_j} , \quad j = 3, 4, 5. \]  

(3.14)

Substituting equation (3.14) in equation (3.11), we get the resulting steady-state availability of the system for the Gamma repair time distribution as

\[ AV_{ss} = \frac{\sum_{i=0}^2 A_i}{\sum_{i=0}^5 \frac{A_i}{\mu_j}} + \frac{\sum_{j=3}^5 A_j \beta}{\mu_j}. \]  

(3.15)

For \( \beta = 1 \), the Gamma distribution reduces to an exponential distribution. Then equation (3.15) with \( \beta = 1 \) becomes the steady-state availability for the exponential repair time distribution.

#### Case II

If the repair time \( x \) of the system is Weibull, then the probability density function of the repair time is expressed as

\[ N_j(x) = \mu_j^\beta x^{\beta-1} e^{-\mu_j x^\beta} , \quad j = 3, 4, 5; \ x \geq 0, \ \beta > 0, \ \mu_j > 0 \]  

(3.16)
(where $\beta$ and $\mu_j$ are the two parameters of the Weibull distribution) then the mean time to repair $E_j$ is given by

$$E_j = \int_0^\infty xN_j(x)dx = \frac{\Gamma(1 + 1/\beta)}{\mu_j}, \quad j = 3, 4, 5. \quad (3.17)$$

Substituting equation (3.17) in equation (3.11), we have the steady-state availability for the Weibull repair time distribution as

$$AV_{ss} = \frac{\sum_{i=0}^2 A_i}{\sum_{i=0}^2 A_i + \sum_{j=3}^5 A_j \Gamma(1 + 1/\beta)/\mu_j}. \quad (3.18)$$

**Case III**

If the repair time $x$ of the system is Rayleigh distributed and the time-dependent repair rate and probability density function of the repair time, respectively are given by

$$\mu_j(x) = \mu_j^2 x, \quad j = 3, 4, 5$$

$$N_j(x) = \mu_j^2 x \exp(-\mu_j^2 x^2/2), \quad j = 3, 4, 5; \quad x \geq 0, \mu_j > 0 \quad (3.19)$$

then the mean time to repair $E_j$ is expressed as

$$E_j(x) = \int_0^\infty xN_j(x)dx = \frac{1}{\mu_j \sqrt{\pi/2}}. \quad (3.20)$$

Substituting equation (3.20) in equation (3.11), we get the steady-state availability as

$$AV_{ss} = \frac{\sum_{i=0}^2 A_i}{\sum_{i=0}^2 A_i + \sum_{j=3}^5 A_j \sqrt{\pi/2}/\mu_j}. \quad (3.21)$$

**Case IV**

If the repair time $x$ is lognormal, then the probability density function of the repair time is given by

$$N_j(x) = \frac{1}{x \sigma_j \sqrt{2\pi}} exp \left[-\frac{(\ln x - \mu_j)^2}{2\sigma_j^2}\right], \quad j = 3, 4, 5; \quad x \geq 0 \quad (3.22)$$
where \( \mu_j \) and \( \sigma_j \) are the mean and standard deviation respectively, then the mean
time to repair \( E_j \) is given by

\[
E_j = \int_0^\infty x N_j(x) dx = \exp[\mu_j + \sigma_j^2/2], \quad j = 3, 4, 5. \tag{3.23}
\]

Substituting equation (3.23) in equation (3.11) yields the steady-state availability as

\[
AV_{ss} = \frac{\sum_{i=0}^2 A_i}{\sum_{i=0}^2 A_i + \sum_{j=3}^5 A_j \exp(\mu_j + \sigma_j^2/2)}. \tag{3.24}
\]

### 3.1.6 Numerical Example

By setting \( \lambda = 0.0001/hr, \eta = 0.001/hr, \mu_1 = 0.3/hr, \mu_2 = 0.03/hr, \mu_3 = 0.0006/hr, \mu_4 = 0.0004/hr, \lambda_{\eta_0} = 0.00002/hr, \lambda_{\mu_1} = 0.00003/hr, \lambda_{\mu_2} = 0.00005/hr, \lambda_{\mu_3} = 0.00004/hr, \lambda_{\mu_4} = 0.00006/hr \) we get the following graphs for the various statistical characteristics of the system for different distributions discussed above. Thus for the Gamma and Weibull repair time distribution, the steady-state availability \( AV_{ss} \) as a function of human-error rate \( \lambda_{h_0} \) are shown in Figures 3.2 and 3.4 for different values of \( \beta \) and \( \mu_5 = 0.0005 \). For exponential and Rayleigh repair time distributions with different values of \( \mu_5 \) the system steady-state availability decreases with respect to \( \lambda_{h_0} \) and are shown in Figures 3.3 and 3.5. In Figure 3.6 the steady-state availability as a function of \( \lambda_{h_0} \) is drawn for Lognormal repair time distribution.

Figure 3.7 shows the \( AV_{ss} \) as a function of \( \lambda_{h_0} \), for different repair time distributions like Gamma, exponential, Weibull, Rayleigh and Lognormal with \( \beta = 1, \mu_5 = 0.0001, \eta = 1, \mu_5 = 0.0001, \) and \( \sigma = 3 \) respectively.
3.1.7 Time-Dependent Availability Analysis

Solving equations (3.1-3.3) by Laplace transforms we have

\[ sP_0(s) = 1 - a_0 P_0(s) + \mu_1 P_1(s) + \mu_2 P_2(s) + \int_0^\infty \mu_3(x) P_3(x, s) dx + \int_0^\infty \mu_4(x) P_4(x, s) dx + \int_0^\infty \mu_5(x) P_5(x, s) dx \\
(\sigma + a_1)P_1(s) = \lambda P_0(s) \\
(\sigma + a_2)P_2(s) = \eta P_0(s) \\
\frac{\partial P_i(x, s)}{\partial x} + sP_i(x, s) + \mu_i(x) P_i(x, s) = 0, \quad i = 3, 4, 5 \quad (3.25) \]

with the initial and boundary conditions as \( P_0(0) = 1, \ P_1(0) = 0, \ i = 1, 2. \)

\[
P_3(0, s) = \lambda [P_1(s) + P_2(s)] \\
P_4(0, s) = \sum_{i=0}^{2} \lambda_i P_i(s) \\
P_5(0, s) = \sum_{i=0}^{2} \lambda_i P_i(s). \quad (3.26)
\]

Solving the above equation (3.25) one has

\[ P_i(x, s) = P_i(0, s) e^{-\sigma x} R_i(x) \text{ where } R_i(s) = \int_0^\infty e^{-\sigma x} R_i(x) dx, \quad i = 3, 4, 5 \quad (3.27) \]

Substituting equations (3.26) and (3.27) in (3.25), we have

\[
[\sigma + a_0 - \lambda_0 N_4(s) - \lambda_0 N_5(s)] P_0(s) - [\mu_1 + \lambda_1 N_4(s) + \lambda_1 N_5(s) + \lambda N_3(s)] P_1(s) - [\mu_2 + \lambda_2 N_4(s) + \lambda_2 N_5(s) + \lambda N_3(s)] P_2(s) = 1 \quad (3.28)
\]

where \( N_j(x) \) is the probability density function of the repair time and \( N_j(s) \) is the Laplace transforms of \( N_j(x) \).

\[
N_i(s) = \int_0^\infty e^{-\sigma x} N_i(x) dx \\
N_i(x) = \mu_i(x) exp(-\int_0^\infty \mu_i(u) du). \quad i = 3, 4, 5. \quad (3.29)
\]

But we know that

\[
P_i(s) = \int_0^\infty P_i(x, s) dx, \quad i = 3, 1, 5. \quad (3.30)
\]
Hence from equation (3.27) one obtains

\[ P_3(s) = \lambda [P_1(s) + P_2(s)] \frac{[1 - N_3(s)]}{s} \]
\[ P_4(s) = \sum_{i=0}^{2} \lambda_i P_i(s) \frac{[1 - N_4(s)]}{s} \]
\[ P_5(s) = \sum_{i=0}^{2} \lambda_i P_i(s) \frac{[1 - N_5(s)]}{s} \]

(3.31)

where

\[ \frac{[1 - N_i(s)]}{s} = \int_0^\infty e^{-sx} \exp(- \int_0^x \mu_i(u)du)dx \ (\text{for } i = 3, 4, 5). \]

Further, solving equations (3.25), (3.28) and (3.1-3.3) we obtain the following transforms solution of the state probabilities \( P_i(s), i=0,1,...,5 \) as

\[ P_0(s) = \frac{s^2 + (a_1 + a_2)s + A_0}{B_0} \]
\[ P_1(s) = \frac{\lambda s + A_1}{B_0} \]
\[ P_2(s) = \frac{\eta s + A_2}{B_0} \]
\[ P_3(s) = \frac{\lambda [A_1 + A_2 + s(\eta + \lambda)] [1 - N_3(s)]}{s B_0} \]
\[ P_4(s) = \frac{[\lambda_0 s^2 + \left( (a_1 + a_2) \lambda_0 + \lambda \lambda_{a_1} + \eta \lambda_{a_2} \right) s + A_5] [1 - N_4(s)]}{s B_0} \]
\[ P_5(s) = \frac{[\lambda_0 s^2 + \left( (a_1 + a_2) \lambda_0 + \lambda \lambda_{a_1} + \eta \lambda_{a_2} \right) s + A_6] [1 - N_5(s)]}{s B_0} \]

(3.32)

where

\[ B_0 = s^3 + B_3 s^2 + B_2 s + B_1 \]
\[ B_1 = A_0 B_4 + A_1 (B_5 + B_7) + A_2 (B_6 + B_7) \]
\[ B_2 = \sum_{i=0}^{2} A_i (a_1 + a_2) B_4 + \lambda (B_5 + B_7) + \eta (B_6 + B_7) \]
\[ B_3 = a_1 + a_2 + \lambda + \eta + B_4 \]
\[ B_4 = \lambda_0 (1 - N_4(s)) + \lambda_4 (1 - N_5(s)) \]
\[ B_5 = \lambda_{a_1} (1 - N_4(s)) + \lambda_{a_5} (1 - N_5(s)) \]
\[ B_6 = \lambda_{a_4} (1 - N_4(s)) + \lambda_{a_6} (1 - N_5(s)) \]
\[ B_7 = \lambda (1 - N_3(s)). \]
Hence the Laplace transforms of the system availability is given by
\[ AV(s) = \frac{s^2 + (a_1 + a_2 + \lambda + \eta)s + A_0 + A_1 + A_2}{B_0}. \] (3.33)

Now taking the inverse Laplace transforms of equation (3.33), one can obtain the time-dependent availability of the system. Substituting the transforms solution of the probability density function of the repair times, \(N_i(s), i=3,4,5\) in equation (3.33) and then taking the same values substituted for \(AV\), we get the time-dependent availability of the system.

As time \(t\) increases, the time-dependent availability decreases for exponential repair time distribution (for different values of \(\lambda_h\)) and are shown in Figure (3.8).

### 3.1.8 Reliability System and MTTF with and without Repair

By setting \(\mu_0(x) = \mu_4(x) = \mu_5(x) = 0\) the system of differential equations (3.1)-(3.3) become
\[
\begin{align*}
\frac{dP_0(t)}{dt} &= -a_0P_0(t) + \mu_1P_1(t) + \mu_2P_2(t) \\
\frac{dP_1(t)}{dt} &= \lambda P_0(t) - a_1P_1(t) \\
\frac{dP_2(t)}{dt} &= \eta P_0(t) - a_2P_2(t) \\
\frac{dP_3(t)}{dt} &= \lambda[P_1(t) + P_2(t)] \\
\frac{dP_4(t)}{dt} &= \sum_{i=1}^{2} \lambda_{hi} P_i(t) \\
\frac{dP_5(t)}{dt} &= \sum_{i=0}^{2} \lambda_{hi} P_i(t). \tag{3.34}
\end{align*}
\]

At time \(t=0\), \(P_0(0) = 1\) and \(P_i(0) = 0\), \(i=1,2,3,4,5\).

Solving the set of equations (3.34) by using Laplace transforms, one can get the transforms solution of the state probabilities \(P_i(s)\), \(i = 0,1,2,3,4,5\).
Further, the Laplace transforms of the system reliability with repair is given by

\[
R(s) = \frac{s^2 + (a_1 + a_2 + \lambda + \eta)s + A_0 + A_1 + A_2}{s^3 + a_4s^2 + (a_5 - \lambda\mu_1 - \eta\mu_2)s + a_6 - A_1\mu_1 - A_2\mu_2}
\]  
(3.35)

where

\[
\begin{align*}
a_4 &= a_1 + a_2 + a_0, \\
a_5 &= a_0a_1 + a_1a_2 + a_2a_0, \\
a_6 &= a_0a_1a_2.
\end{align*}
\]

Inverting equation (3.35) we get

\[
R(t) = \sum_{i=1}^{3} \frac{(s^2 + (a_1 + a_2 + \lambda + \eta)s + A_0 + A_1 + A_2)e^{s_i t}}{i \neq j \prod_{i \neq j} (s_i - s_j)}
\]  
(3.36)

where \(s_1, s_2, \text{ and } s_3\) are the roots of the cubic equation (3.36) and subsequently, the mean time to failure and the variance of time to failure with repair are respectively given by

\[
MTTF = \lim_{s \to 0} R(s) = \frac{A_0 + A_1 + A_2}{a_6 - A_1\mu_1 - A_2\mu_2}
\]  
(3.37)

\[
\sigma^2 = -2 \lim_{s \to 0} R'(s) - (MTTF)^2
\]

\[
= \frac{2[A_0^2 + A_1\mu_1a_2 + A_2\mu_2a_1] - [A_0 + A_1 + A_2]^2}{[a_6 - A_1\mu_1 - A_2\mu_2]^2}
\]  
(3.38)

For the same set of data calculated for \(AV_{ss}\), we get the following results from Figures (3.9) and (3.10).

With increase in time \(t\) (human-error), the reliability (MTTF) decreases.

Finally by putting \(\mu_1 = \mu_2 = 0\) in equations (3.36-3.38) one obtains the system reliability, MTTF and the variance of time to failure without repair.
Figure 3.1 System transition diagram
Figure 3.2 System steady-state availability Vs. $\lambda_h$ for Gamma repair time distribution
Figure 3.3 System steady-state availability Vs. $\lambda_h$, for Exponential repair time distribution
Figure 3.4 System steady-state availability Vs. $\lambda_0$ for Weibull repair time distribution
Figure 3.5 System steady-state availability Vs. $\lambda_{h_0}$ for Rayleigh repair time distribution
Figure 3.6 System steady-state availability vs. $\lambda_h$ for Lognormal repair time distribution.
Figure 3.7 The comparison of steady-state availability and \( \lambda_{ho} \) for various repair time distribution.
Figure 3.8 System Availability Vs. Time
Figure 3.9 System reliability Vs. Time
Figure 3.10 MTTF Vs. $\lambda_{ho}$
3.2 ON THE CHARACTERISTICS OF A PROTECTIVE TWO-UNIT SYSTEM

3.2.1 Introduction

Feldman (1976) considered a system which is subject to a sequence of random shocks, any of which might cause the system to fail. Many authors [Marshall and Shaked (1979) and Al Ali and Murari (1988)] have studied standby reliability systems subject to random shocks under the assumption that a component can fail only due to a shock and have obtained the life-time distribution of the systems. Murari and Al Ali (1990) formulated and studied two systems with one-unit subject to random shocks and failures were caused due to operator as well as shocks. They have also studied the effect of shocks in a two-unit cold standby system. Therefore it is essential to protect the online unit in a system from any failure. The machine under use may fail due to hardware errors or due to some other failures, such as fluctuation of voltage in power supply, change in climatic conditions or due to some internal factors such as stress and strain etc.. Hence it is necessary to make some arrangements in the system which could control such type of failures. For example, in the computer system, stabilizer is used to control the power fluctuations and whenever it is not adjusted by the stabilizer the computer faces shock.

The purpose of the present section is to investigate a model in which the online unit is protected by a single protective unit. Also due to the non-similarity of online and protective units we employ two types of repair facilities, one for the online and the other for the protective unit. They work simultaneously and independently to each other. Using Laplace transforms we obtain the availability, reliability, mean time to failure and expected profit of the system.

The organization of this section is as follows. In section 3.2.2, we deal with the assumptions and notation. Section 3.2.3 deals with the governing equations whereas in section 3.2.4, gives the profit analysis for non-repairable system is
analysed. Finally in section 3.2.5, concluding remarks are given.

### 3.2.2 Assumptions and Notation

The following assumptions and symbols are associated with this model.

i. The system consists of two-identical units with one unit operating online and the other, kept as warm standby with a protective unit for the online.

ii. The online and the protective units may fail due to shock as well as hardware failure. Whenever a shock occurs in the system, it affects the online unit with probability one.

iii. The system is in failed state when both the online as well as standby units fail or the protective unit fails.

iv. Repair and failure rates are constants and are different for the online, protective and standby units.

v. The repair starts immediately, when a unit fails.

vi. After repair the system is as good as new.

vii. No further failure can occur when the system is in failed state.

viii. The switchover mechanism is perfect and instantaneous.

\[
i \quad \text{ith state of the system} \\
\lambda_i \quad \text{constant failure rate of an online unit} \\
\lambda_s \quad \text{constant failure rate of a standby unit} \\
\lambda_p \quad \text{constant failure rate of a protective unit} \\
\lambda_c \quad \text{common failure rate} \\
p_i(t) \quad \text{probability that the system is in state } i \text{ at time } t, \quad i = 0, 1, 2, 3, 4, 5 \\
p_i^{\text{steady-state}} \quad \text{steady-state probability that the system is in state } i, \quad i = 0, 1, 2, 3, 4, 5 \\
P_i \quad \text{Laplace transforms variable} \\
\]
The following states are associated with this model and are described as follows (see Figure 3.11).

<table>
<thead>
<tr>
<th>State</th>
<th>Online unit</th>
<th>Other unit</th>
<th>Protective unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>operative</td>
<td>standby</td>
<td>operative</td>
</tr>
<tr>
<td>1</td>
<td>just fails</td>
<td>operable</td>
<td>operable</td>
</tr>
<tr>
<td>2</td>
<td>operable</td>
<td>just fails</td>
<td>operable</td>
</tr>
<tr>
<td>3</td>
<td>good</td>
<td>under repair</td>
<td>just fails</td>
</tr>
<tr>
<td>4</td>
<td>good</td>
<td>good</td>
<td>under repair</td>
</tr>
<tr>
<td>5</td>
<td>under repair</td>
<td>just fails</td>
<td>good</td>
</tr>
</tbody>
</table>

States 0, 1, and 2 are the up-states whereas states 3, 4, and 5 are the failed states of the system. After repair completion of the protective unit in state 3, it either goes to state 1 with probability $\phi_1$ or it goes to state 2 with probability $\phi_2$. 
3.2.3 The Governing Equations

The differential equation for \( P_0(t) \) is given by

\[
P_0'(t) = -[\lambda + \lambda_p + \lambda_s + f_1 + f_2]P_0(t) + [\mu_1 + \lambda_f]P_1(t) + \mu_2P_2(t) + \mu_pP_4(t)
\]

(3.39)

The above equation (3.39) is obtained as follows.

i. there could be no online or standby or protective failures and there could be no failures for both the online and protective unit due to shocks

ii. there could be a repair completion for the online or the standby units

iii. there could be a repair completion for the protective unit as well as the system.

Similarly we get

\[
\begin{align*}
P_1'(t) &= -[\lambda + \lambda_p + f_1 + f_2 + \mu_1]P_1(t) + [\lambda + f_1]P_0(t) + \phi_1\mu_pP_3(t) \\
P_2'(t) &= -[\lambda + \lambda_p + f_1 + f_2 + \mu_1]P_2(t) + \lambda_sP_0(t) + \phi_2\mu_pP_3(t) \\
P_3'(t) &= -[\mu_1 + \mu_p]P_3(t) + [\lambda_p + f_2][P_1(t) + P_2(t)] \\
P_4'(t) &= -\mu_pP_4(t) + [\lambda_p + f_2]P_0(t) + \mu_1P_3(t) \\
P_5'(t) &= -\mu_P(t) + [\lambda + f_1][P_1(t) + P_2(t)]
\end{align*}
\]

(3.40)

with \( P_0(0) = 1 \), and \( P_i(0) = 0 \), \( i = 1, 2, 3, 4, 5 \).

Using Laplace transforms in equations (3.39) and (3.40) we obtain

\[
AV(s) = P_0(s) + P_1(s) + P_2(s) = \frac{N_1}{D_1}
\]

(3.41)

where

\[
\begin{align*}
N_1 &= s^4 + T_9s^3 + T_8s^2 + T_{10}s + T_{11} \\
D_1 &= s^5 + T_{12}s^4 + T_{13}s^3 + T_{14}s^2 + T_{15}s
\end{align*}
\]
with

\[ T_1 = \lambda + \lambda p + \lambda s + f_1 + f_2; \]

\[ T_2 = T_1 - \lambda_s; \]

\[ T_3 = \lambda + f_1; \]

\[ T_4 = \lambda_p + f_2; \]

\[ T_5 = \mu + \mu_p; \]

\[ T_6 = \mu_1 + \mu_p; \]

\[ T_7 = \mu_1(T_2 + \mu_1) + \mu_p(T_3 + \mu_1); \]

\[ T_8 = T_1 + T_3 + 2T_6 + \mu; \]

\[ T_9 = T_7 + T_6(T_3 + T_5 + \lambda_s) + T_5(T_1 + T_3 + \mu_1) + \mu_p; \]

\[ T_{10} = T_5T_7 + \mu_p(T_1 + T_6 + \mu_1) + T_3T_5T_6 + \lambda_sT_5T_6; \]

\[ T_{11} = \mu_p[T_7 + T_6(T_3 + \lambda_s)]; \]

\[ T_{12} = 2(T_2 + T_6) + \mu + \lambda_s; \]

\[ T_{13} = T_7 + \mu_1[T_4 + T_5] + \mu_p[T_3 + \lambda_s + \mu] + (T_2 + T_6)(T_1 + T_5) + T_1\mu; \]

\[ T_{14} = (T_3 + \lambda_s)[\mu_1(T_2 + \mu) + \mu_p(T_2 + T_3 + \mu) + \mu(T_4 + \mu_p)(T_1 + T_3 + T_6 + \mu_1) + T_7(T_4 + T_5)]; \]

\[ T_{15} = T_6(T_3 + \lambda_s)(\mu_pT_3 + \mu T_4 + \mu_p) + T_7\mu(T_4 + \mu_p); \]

On inversion, the time-dependent availability of the system is given by

\[ AV(t) = \sum_{i=1}^{4} \frac{(s_i^4 + T_{00}s_i^3 + T_{01}s_i^2 + T_{02}s_i + T_{03})e^{s_0t}}{\prod_{i=\neq j}(s_i - s_j)} + \frac{T_{11}}{s_1s_2s_3s_4} \tag{3.42} \]

where \( s_i \)'s are the roots of the polynomial \( D_1 \) and is given in equation (3.42).

From equation (3.42), the steady-state availability of the system is given by

\[ AV_{ss} = \lim_{s \to 0} sAV(s) = \frac{T_{11}}{T_{15}}. \tag{3.43} \]

Further, the steady-state unavailability of the system is given by

\[ UAV_{ss} = 1 - AV_{ss} = \frac{(T_{15} - T_{11})}{T_{15}}. \tag{3.44} \]
Now by setting \( n = \mu_p = 0 \), we have from equations (3.39) and (3.40)

\[
\begin{align*}
P_0'(t) &= -[\lambda + \lambda_p + \lambda_s + f_1 + f_2]P_0(t) + \mu_1[P_1(t) + P_2(t)] \\
P_1'(t) &= -[\lambda + \lambda_p + f_1 + f_2 + \mu_1]P_1(t) + (\lambda + f_1)P_0(t) \\
P_2'(t) &= -[\lambda + \lambda_p + f_1 + f_2 + \mu_1]P_2(t) + \lambda P_0(t) \\
P_3'(t) &= -\mu_1 P_3(t) + [\lambda_p + f_2][P_1(t) + P_2(t)] \\
P_4'(t) &= [\lambda_p + f_2]P_0(t) + \mu_1 P_3(t) \\
P_5'(t) &= [\lambda + f_1][P_1(t) + P_2(t)]
\end{align*}
\] (3.45)

with \( P_0(0) = 1 \) and \( P_i(0) = 0, i = 1, 2, 3, 4, 5 \).

Solving equation (3.45) by Laplace transforms we have the reliability of the system with repair given by

\[
R(s) = P_0(s) + P_2(s) + P_3(s) = \frac{N_2}{D_2}, \tag{3.46}
\]

where

\[
\begin{align*}
N_2 &= s + T_2 + T_3 + \lambda_s + \mu_1 \\
D_2 &= s^2 + (2T_1 + \lambda_s + \mu_1)s + T_1T_2 + \mu_1T_4.
\end{align*}
\]

Now taking inverse Laplace transforms we get

\[
R(t) = P_0(t) + P_1(t) + P_2(t) = \sum_{i=1}^{2} \frac{(s_i + T_2 + T_3 + \lambda_s + \mu_1)e^{s_it}}{\prod_{j \neq i}(s_i - s_j)}. \tag{3.47}
\]

From equation (3.46), the MTTF with repair facility is given by

\[
MTTF = \lim_{s \to 0} R(s) = \frac{T_1 + T_3 + \mu_1}{T_1T_2 + T_4\mu_1}. \tag{3.48}
\]

Further setting \( \mu_1 = 0 \) in equation (3.45) give the reliability, MTTF of the system without repair and is given by

\[
R(t) = \frac{(T_3 + \lambda_s)e^{-T_3t} - T_3e^{-T_1t}}{\lambda_s}. \tag{3.49}
\]

with

\[
MTTF = \frac{T_1 + T_3}{T_1T_2}. \tag{3.50}
\]
3.2.4 Profit Analysis

Now let $C$ be the revenue cost per unit time without repair facility. Then the expected profit $H(t)$ is given by

$$H(t) = C \int_0^t R(t) dt = \frac{C[T_2T_3(e^{-T_2t} - 1) - T_1(T_3 + \lambda_s)(e^{-T_2t} - 1)]}{T_1T_2\lambda_s}. \quad (3.51)$$

3.2.5 Concluding Remarks

By choosing $\lambda = 0.08$, $\lambda_s = 0.04$, $\lambda_p = 0.03$, $\mu = 0.09$, $\mu_p = 0.04$, $\mu_1 = 0.05$, $f_1 = 0.01$ and $f_2 = 0.02$, we have the following remarks from the above results obtained and shown in Figures (3.12) - (3.16).

As time $t$ increases, we have from Figures (3.12) and (3.14) the availability as well as reliability of the system decreases and from Figure (3.16) the expected profit of the system increases with respect to the revenue cost. Further, as the failure rate increases we have from Figures (3.13) and (3.15) the steady-state availability and MTTF also decreases with and without repair.
Figure 3.11 System transition diagram

- Down state (3,4,5)
Figure 3.12 Availability Vs. Time
Figure 3.13 Steady-state availability Vs. Failure rate
Figure 3.14 Reliability Vs. Time

- a. Reliability (with repair)
- b. Reliability (without repair)
Figure 3.15 MTTF Vs. Failure rate
Figure 3.16 Expected profit Vs. Time (without repair)