CHAPTER 2

SOME FIXED POINT THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

2.1 PRELIMINARIES

Theorem 1.2.2 on the existence of fixed points for nonexpansive mappings in uniformly convex Banach spaces was proved independently by Browder (1965) and Göhde (1965). Goebel (1969) gave an elementary proof of Browder's and Göhde's Theorem 1.2.2. The notion of asymptotically nonexpansive mapping in a Banach space was first introduced by Goebel and Kirk (1972) and they proved a fixed point theorem (1.5.2) for such mappings in the setting of uniformly convex Banach spaces.

Dotson (1972) generalized Browder's Theorem 1.2.2 to nonexpansive mappings on a nonempty compact star-shaped subset of a Banach space. Taylor (1972) established theorems on the existence of fixed points and approximations of fixed points for nonexpansive mappings in locally convex spaces. These theorems extend those of Dotson (1972), Browder et al (1966). Tarafdar (1974) generalized the Banach's contraction principle in a complete metric space to a complete Hausdorff uniform space, to obtain some theorems on the existence of fixed points and approximations of fixed points for nonexpansive mappings in locally convex spaces. The theorems of Taylor (1972) and Tarafdar (1974) on the existence of fixed points for nonexpansive self- mappings were extended by Su and Sehgal (1974) to nonexpansive non-self mappings T of a nonempty compact star-shaped subset K of a locally convex space X into X by assuming T(∂K) ⊆ K, where ∂K denotes the boundary of K.
Tarafdar (1975) assumed the weaker condition of sequential completeness on the set $K$ in the place of completeness on the set $K$ of Taylor's Lemma 1.2.9 and Lemma 3.1 of Tarafdar (1974). Taylor's Theorem 1.2.10 to nonexpansive mappings in locally convex spaces leads one to wonder what analogous results can be obtained for asymptotically nonexpansive mapping and the purpose of our results is to provide some partial answers to this question.

In section 2.2, the notions of asymptotically nonexpansive and uniformly asymptotically regular mappings in locally convex spaces have been formulated. Examples are also constructed. We extend Taylor's Theorems 1.2.9 - 1.2.11 and Theorems of Tarafdar (1974, 1975) to asymptotically nonexpansive mappings $T$ in locally convex spaces by assuming the uniform asymptotic regularity of $T$. In section 2.3, we extend Taylor's Theorems 1.2.9, 1.2.11 and Theorem 3 of Su and Sehgal (1974) to asymptotically nonexpansive non-self mappings $T$ in locally convex spaces by assuming that $T$ is uniformly asymptotically regular. In section 2.4, we extend Taylor's Theorem 1.2.12 and Theorem 3.2 of Tarafdar (1974) for nonexpansive mappings to asymptotically nonexpansive mappings in locally convex spaces.

Demarr (1963), Belluce et al (1966) established common fixed point theorems for a commuting family of nonexpansive mappings in Banach spaces. Tarafdar (1975) extended the result of Belluce et al (1966) (which includes the result of Demarr (1963)), to quasi-complete locally convex spaces. In section 2.5, we prove the result corresponding to that of Demarr (1963) result for a commuting family of asymptotically nonexpansive mappings in locally convex spaces.

Several authors (Dotson (1970), Rhoades (1974), etc.,) have shown that, for a mapping $T$ satisfying a certain contractive condition in Hilbert spaces and in Banach spaces, if the sequence of Mann iterates converges, then it converges to a fixed point of $T$. Khan (1988) has extended some results of these authors to the case of metrizable linear topological spaces. More recently, Pathak et al
(1990) have shown that, for a pair of mappings \(T_1\) and \(T_2\) satisfying a certain contractive condition in normed linear spaces, if the sequence of \(G\)-iterates converges, then it converges to a common fixed point of \(T_1\) and \(T_2\). We also prove in section 2.5 that the corresponding result holds good in metrizable linear topological spaces.

In sections 2.2 - 2.4 of this chapter, \(X\) denotes a locally convex Hausdorff linear topological space with a family \((P_\alpha)_{\alpha \in J}\) of seminorms which defines the topology on \(X\).

### 2.2 Existence of Fixed Points for Self-Mappings

In this section, we extend Taylor's results 1.2.9 - 1.2.11, and Theorems of Tarafdar (1974, 1975) for nonexpansive mappings in locally convex spaces \(X\) to asymptotically nonexpansive, uniformly asymptotically regular mappings in \(X\).

We introduce the following concepts.

**Definition 2.2.1.** Let \(K\) be a nonempty subset of \(X\). If \(T\) maps \(K\) into itself, then

(i) \(T\) is said to be asymptotically nonexpansive if, there is a sequence \((k_n)\) of real numbers with \(k_n \to 1\) as \(n \to \infty\), such that

\[
p_\alpha(T^n x - T^n y) \leq k_n p_\alpha(x - y) \quad \text{for all } x, y \in K, \ n \in \mathbb{N} \text{ and } \alpha \in J.
\]

It is assumed that \(k_n \geq 1\) and \(k_n \leq k_{n+1}\) for \(n \in \mathbb{N}\).

(ii) \(T\) is said to be uniformly asymptotically regular on \(K\) if, for each \(\alpha \in J\) and for each \(\eta > 0\), there exists \(L(\alpha, \eta) = L, \text{ say}\) such that

\[
p_\alpha(T^n x - T^{n+1} x) \leq \eta \quad \text{for all } n \geq L \text{ and for all } x \in K.
\]

(that is, \(\delta((I - T)T^n(K), \alpha) < \eta\) for all \(n \geq L\)).

The following example shows that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.
in locally convex spaces.

**Example 2.2.2.** Let $X$ be the space of all sequences of complex numbers whose topology is defined by the family of seminorms $p_n$ defined by

$$
p_n(x) = \max_{1 \leq i \leq n} |x_i| \quad \text{for } x = (x_1, x_2, \ldots) \in X \text{ and } n \in \mathbb{N}.
$$

Let $K = \left\{ x = (x_1, x_2, \ldots) \in X : |x_i| \leq 1/2 \text{ and } |x_j| \leq 1 \text{ for } j = 2, \ldots \right\}$.

Define a map $T$ from $K$ to $K$ by

$$
T x = (0, 2\xi_1, A_2 \xi_2, \ldots, A_k \xi_k, \ldots) \text{ for all } x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots) \in K,
$$

where $(A_i)$ is a sequence of real numbers in $(0, 1)$ such that

$$
\prod_{i=2}^{\infty} A_i = 1/2.
$$

Let $a = (1/2, 0, \ldots), b = (0, \ldots) \in K$. Then we have

$$
p_2(Ta - Tb) = 1 > 1/2 = p_2(a - b) \quad \text{and hence } T \text{ is not nonexpansive.}
$$

Now, let $x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots), y = (\eta_1, \eta_2, \ldots, \eta_k, \ldots) \in K$.

Then

$$
p_n(Tx - Ty) \leq 2 \cdot p_n(x - y) \quad \text{for } n \in \mathbb{N},
$$

and

$$
T^m(x) = (0, \ldots, 0, 2 \prod_{i=2}^{m+1} A_i \xi_1, \prod_{i=2}^{m+2} A_i \xi_2, \ldots, \prod_{i=2}^{m+k-1} A_i \xi_k, \ldots).
$$

Therefore

$$
p_n(T^m(x) - T^n(y)) = 0 \quad \text{for } m > n.
$$

If $m < n$, then $m = n - k$, where $k > 0$ and $n > k$ and therefore

$$
p_n(T^m(x) - T^n(y)) = \max \left\{ \left( \prod_{i=2}^{m+1} A_i \right) |\xi_1 - \eta_1|, \left( \prod_{i=2}^{m+2} A_i \right) |\xi_2 - \eta_2|, \ldots, \left( \prod_{i=2}^{m+k-1} A_i \right) |\xi_k - \eta_k| \right\}
$$

$$
\leq \max \left\{ \left( \prod_{i=2}^{m+1} A_i \right) \prod_{i=2}^{m+2} A_i \prod_{i=2}^{m+3} A_i \ldots \prod_{i=2}^{m+k-1} A_i \right\} \cdot p_k(x - y)
$$

$$
\leq 2 \prod_{i=2}^{m+1} A_i \cdot p_n(x - y) = k \cdot p_n(x - y),
$$

where $k = 2 \prod_{i=2}^{m+1} A_i \rightarrow 1$ as $m \rightarrow \infty$. 

Hence \( T \) is asymptotically nonexpansive. Also \( T \) is uniformly asymptotically regular on \( K \).

We show in the following example that uniform asymptotic regularity is stronger than asymptotic regularity.

**Example 2.2.3.** Let \( X = l_p \), \( 1 < p < \infty \) and \( K \) denote the closed unit ball in \( X \). Define a map \( T \) from \( K \) to \( K \) by

\[
Tx = (\xi_2, \xi_3, \ldots) \text{ for all } x = (\xi_1, \xi_2, \ldots) \in K.
\]

Let \( x = (\xi_1, \xi_2, \ldots), y = (\eta_1, \eta_2, \ldots) \in K \). Then we have

\[
\|Tx - Ty\| = \left( \sum_{n=2}^{\infty} |\xi_n - \eta_n|^p \right)^{1/p} \leq \left( \sum_{n=2}^{\infty} |\xi_n - \eta_n|^p \right)^{1/p} = \|x - y\|.
\]

Therefore \( T \) is nonexpansive and hence it is an asymptotically nonexpansive mapping of \( K \) into itself.

For each \( x \in K \), \( T^nx \to 0 \) as \( n \to \infty \) and hence \( T^{n+1}x - T^nx \to 0 \) as \( n \to \infty \). Therefore \( T \) is asymptotically regular on \( K \).

Suppose \( e_n = (0, \ldots, 0, 1, 0, \ldots) \in K \). Then we have

\[
\|T^{n+1}(e_{2n}) - T^n(e_{2n})\| = 2^{1/p} \text{ for } n \geq 2.
\]

Choose \( \varepsilon = 2^{(1/p)-1} > 0 \). Then there exists no integer \( L \geq 0 \) such that

\[
\|T^{n+1}x - T^nx\| < \varepsilon \text{ for all } n \geq L \text{ and for all } x \in K.
\]

Hence \( T \) is not uniformly asymptotically regular on \( K \).

We need the following definitions which are used to prove our theorems.

**Definition 2.2.4** (Taylor (1972)). A mapping \( T \) from a nonempty subset \( K \) of \( X \) to \( X \) is said to be demiclosed in \( K \) if, for every net \( (x_n) \) in \( K \) such that \( (x_n) \) weakly converges to \( x \) in \( K \) and \( (Tx_n) \) converges to \( y \) in \( X \), we have \( Tx = y \).
Definition 2.2.5 (See Halpern (1969)). A nonempty subset $K$ of $X$ is said to be star-shaped provided that there is at least one point $x$ in $K$ such that if $y$ is any element of $K$ and $t \in (0,1)$, then $(1-t)x + ty$ in $K$. Such a point $x$ is called a star-center of $K$.

Every convex set is a star-shaped set but the converse need not be true.

Taylor (1972) defined the following notions which are used to prove the following lemma.

Definition 2.2.6 (Taylor (1972)). Let $u$ be the uniformity of a uniform space $(X,u)$ and let $\beta$ be a basis for $u$. If $T$ maps $X$ into itself, then
(a) $T$ is said to be a $\beta$-contraction on $X$ if for each $U \in \beta$, there is a $V \in \beta$ such that $(x,y) \in U \cdot V$ implies $(Tx,Ty) \in U$;
(b) $T$ is said to be a $\beta$-nonexpansion on $X$ if $(x,y) \in U$ implies $(Tx,Ty) \in U$ for each $U \in \beta$.

Definition 2.2.7 (See Taylor (1972)). A uniform space $(X,u)$ is well-chained if, for each $U \in u$ and pair of points $x$ and $y$ in $X$, there is a positive integer $n$ such that $(x,y) \in U^n$.

Lemma 2.2.8 (Taylor (1972)). Let $(X,u)$ be a complete well chained uniform space. Assume that $\beta$ is a basis for $u$ such that $T$ is a $\beta$-contraction on $X$. Then $T$ has a unique fixed point $a$ in $X$ such that $T^n x \rightarrow a$ for each $x \in X$.

Remark. A topological vector space is a connected uniform space and therefore it is well chained (See Taylor (1972)).

The family $u$ of zero neighbourhoods for a linear topological space $X$ induces a unique translation invariant uniformity on $X$ (and hence any nonempty subset of $X$) given by the collection of all sets of the form
\[ \{(x,y) \in X \times X : x - y \in U \} , \text{ where } U \in u. \]
Therefore Definition 2.2.6 can be written in the following form (Taylor (1972)).

Let \( u \) be the family of zero neighbourhoods for a linear topological space \( X \) and \( \beta \) a basis for \( u \). If \( T \) maps a nonempty subset \( K \) of \( X \) into itself, then

(a) \( T \) is said to be a \( \beta \)-contraction on \( K \) if for each \( U \in \beta \), there is a \( V \in \beta \) such that

\[
\text{If } x - y \in U + V \text{ implies } Tx - Ty \in U \text{ for each } x, y \in K;
\]

(b) \( T \) is said to be a \( \beta \)-nonexpansion on \( K \) if \( x - y \in U \) implies

\[
Tx - Ty \in U \text{ for each } U \in \beta \text{ and } x, y \in K.
\]

If \( \beta_p \) is the family of all finite intersections

\[
\bigcap_{i=1}^{n} r_i V(p_i), \ r_i > 0, \ p_i \in P, \text{ where } P \text{ denotes the family of seminorms on a locally convex space } X \text{ and } V(p_i) = \{ x \in X : p_i(x) < 1 \}, \text{ then } \beta_p \text{ is a neighbourhood basis at } 0 \text{ for } u.
\]

Hence a contraction (a nonexpansive mapping) is a \( \beta_p \)-contraction (a \( \beta_p \)-nonexpansion).

Taylor (1972) and Tarafdar (1974) proved Lemma 1.2.9 for \( \beta \)-nonexpansions in locally convex spaces. We now prove the corresponding lemma for asymptotically nonexpansive mappings in locally convex spaces.

**Lemma 2.2.9.** Let \( K \) be a nonempty complete bounded star-shaped subset of \( X \). Let \( T \) be an asymptotically nonexpansive self-mapping of \( K \). Then there is a sequence \( \{ x_n \} \) in \( K \) such that

\[
x_n - T^n x_n \to 0 \text{ as } n \to \infty.
\]

If further, \( T \) is a uniformly asymptotically regular self-mapping of \( K \), then \( x_n - T x_n \to 0 \text{ as } n \to \infty \), whence \( 0 \in \text{cl}(I-T)(K) \).

**Proof.** Let \( y \) be a star-center of \( K \). Define a map \( T_n \) from \( K \) to \( K \) by

\[
T_n(x) = a_n T^n x + (1 - a_n) y \text{ for all } x \in K, \ n \in N,
\]

where \( a_n = (1 - 1/n)/k_n \) and \( \{ k_n \} \) is as in Definition 2.2.1(1).
Since $y$ is a star-center of $K$, $T_n$ maps $K$ into itself.

Let $x, z \in K$. Then since $T$ is asymptotically nonexpansive, we have

$$p_n(Tx - Tz) = a_n p_n(T^n x - T^n z) \leq (1 - 1/n)p_n(x - z).$$

Therefore $T_n$ is a contraction on $K$ and hence a $\beta_p$-contraction on $K$.

By Lemma 2.2.8, $T_n$ has a unique fixed point $x_n$ in $K$.

That is, $x_n = T^n x = a_n T^n x + (1 - a_n)y$.

Therefore $x_n - T^n x_n = (1 - a_n)(y - T^n x_n) \to 0$ as $n \to \infty$.

Since $K$ is bounded and $a_n \to 1$ as $n \to \infty$. (2.1)

Since $T$ is uniformly asymptotically regular on $K$, it follows that

$$T^n x_n - T^{n+1} x_n \to 0 \text{ as } n \to \infty. \quad (2.2)$$

Therefore $x_n - T^{n+1} x_n \to 0$ as $n \to \infty$. (2.3)

Now,

$$p_n(x_n - Tx_n) \leq p_n(x_n - T^{n+1} x_n) + p_n(T^{n+1} x_n - Tx_n)$$

$$\leq p_n(x_n - T^{n+1} x_n) + k_n p_n(T^n x_n - x_n).$$

From (2.1) and (2.3) we obtain

$$p_n(x_n - Tx_n) \to 0 \text{ as } n \to \infty.$$ 

Taylor (1972) and Tarafdar (1974) proved Theorems on the existence of fixed points for $\beta$-nonexpansive mappings in locally convex spaces. Using Lemma 2.2.9, we prove the corresponding theorem for asymptotically nonexpansive, uniformly asymptotically regular mappings in such spaces.

**Theorem 2.2.10.** Let $K$ be a nonempty complete bounded star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$. Assume further that $(I-T)(K)$ is closed. Then $T$ has a fixed point in $K$.

**Proof.** It follows from Lemma 2.2.9 that there is a sequence $\{x_n\}$ in $K$ such that $(I-T)x_n \to 0$ as $n \to \infty$.

Since $(I-T)(K)$ is closed, $0 \in (I-T)(K)$ and hence there is a point $x$ in
K such that \((I - T)x = 0\). Thus \(x\) is a fixed point of \(T\) in \(K\).

**Corollary 2.2.11.** Let \(K\) be a nonempty compact star-shaped subset of \(X\). Let \(T\) be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of \(K\). Then \(T\) has a fixed point in \(K\).

**Proof.** Suppose that \(K\) is compact. Then \(K\) is bounded and complete (p.61 of Kelley et al (1963)). Since \(T(K) \subseteq K\) and \(I-T\) is continuous, \((I-T)(K)\) is compact and hence it is closed (p.192 of Kelley (1955)). Therefore the proof of this corollary follows from the above theorem.

Note that any weakly compact subset of a locally convex Hausdorff linear topological space is complete and bounded (pp.155-156 of Kelley et al (1963)). The following theorem shows that if we assume the subset \(K\) of \(X\) to be weakly compact and \(I-T\) to be demiclosed in \(K\) in the place of closedness of \((I-T)(K)\) of Theorem 2.2.10, the result remains true for asymptotically nonexpansive mappings \(T\). This result is an extension of Taylor's Theorem 1.2.11.

**Theorem 2.2.12.** Let \(K\) be a nonempty weakly compact star-shaped subset of \(X\). Let \(T\) be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of \(K\). Assume further that \(I-T\) is demiclosed in \(K\). Then \(T\) has a fixed point in \(K\).

**Proof.** Let \(\{x_n\}\) be a sequence in \(K\) such that

\[
x_n - Tx_n \longrightarrow y \quad \text{as} \quad n \longrightarrow \infty.
\]  

(2.4)

Since \(K\) is weakly compact and \(\{x_n\} \subseteq K\), there is a subnet \(\{x_{n_p}\}\) of \(\{x_n\}\) such that

\[
x_{n_p} \longrightarrow x \quad \text{for some} \quad x \in K.
\]  

(2.5)

From (2.4) we obtain

\[
x_{n_p} - Tx_{n_p} \longrightarrow y.
\]  

(2.6)

Since \(I-T\) is demiclosed, it follows from (2.5) and (2.6) that

\((I - T)x = y\). Hence \((I-T)(K)\) is closed. The proof of this theorem now follows from Theorem 2.2.10.
Remark. The results 2.2.3, 2.2.9, and 2.2.12 have appeared in the paper of Vijayaraju (1988).

We prove Theorems 2.2.17 and 2.2.18 by using the method of Tarafdar (1975), which uses directly the Banach’s contraction principle. For this, we need the following.

Let $K$ be a nonempty bounded subset of $X$ and $U = \{ x : p_\alpha(x) \leq 1 \}$. Then the family of scalar multiples $\rho U$, $\rho > 0$, of finite intersections $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ of the $U_{\alpha_i}$ form a base of closed absolutely convex neighbourhoods of $0$. Since $K$ is bounded, there exists a number $\lambda_\alpha > 0$ for each $\alpha \in J$ such that $K \subseteq \lambda_\alpha U_{\alpha}.$

Let $B = \bigcap_{\alpha \in J} \lambda_\alpha U_{\alpha}$. Then $B$ is bounded closed and absolutely convex and $K \subseteq B$. We know that the linear subspace of $X$ generated by $B$ (or) the linear span of $B$ is simply $\bigcup_{\alpha \in J} B = X_B$, say) and that on this subspace the Minkowski functional of $B$ is a norm $\| \cdot \|_B$. Thus $X_B$ is a normed space with the norm $\| \cdot \|_B$ and the closed unit ball $B$. It is known that the norm topology on $X_B$ is finer than the topology on $X_B$ induced by the topology on $X$ (p.252 of Köthe (1969)). Since $p_\alpha$ is the Minkowski functional of $U_{\alpha}$ and $\| \cdot \|_B$ is the Minkowski functional of $B$ and $B \subseteq \lambda_\alpha U_{\alpha}$, it follows that for each $x \in X_B$, we have $p_\alpha(x) \leq \lambda_\alpha \| x \|_B$.

Therefore for each $\alpha \in J$, we have $p_\alpha(x/\lambda_\alpha) \leq \| x \|_B$. \hspace{1cm} (2.7)

To show that sup $p_\alpha(x/\lambda_\alpha) = \| x \|_B$ for each $x \in X_B$. \hspace{1cm} (2.8)

Let $x \in X_B$ and sup $p_\alpha(x/\lambda_\alpha) < \| x \|_B$.

Suppose that $\lambda = \sup p_\alpha(x/\lambda_\alpha)$.

Then from(2.8) we have $\lambda < \| x \|_B$. \hspace{1cm} (2.9)

Now, $p_\alpha(x/\lambda_\alpha) \leq \lambda$ and hence $x/\lambda \in \lambda_\alpha U_{\alpha}$ for each $\alpha \in J.$
Therefore \( x/\lambda \in B \). But by (2.9), \( x/\lambda \notin B \).

This contradiction shows that

\[
\sup_{\alpha} p_{\alpha}(x/\lambda) = \|x\|_{B} \quad \text{for each } x \in X_{B}.
\] (2.10)

The following theorem generalizes Theorem 1.1 of Tarafdar (1975) for a nonexpansive mapping to an asymptotically nonexpansive mapping in a locally convex space.

**Theorem 2.2.13.** Let \( K \) be a nonempty bounded subset of \( X \). Let \( T \) be an asymptotically nonexpansive self-mapping of \( K \). Then \( T \) is an asymptotically nonexpansive self-mapping of \( K \) with respect to \( \|\cdot\|_{B} \), where \( \|\cdot\|_{B} \) is defined as above.

**Proof.** Since \( T \) is asymptotically nonexpansive, it follows that

\[
p_{\alpha}(T^{n}x - T^{n}y) \leq k_{n} p_{\alpha}(x - y) \quad \text{for all } x, y \in K, \alpha \in J \text{ and } n \in \mathbb{N}.
\]

Therefore

\[
\sup_{\alpha \in J} p_{\alpha}
(\frac{T^{n}x - T^{n}y}{\lambda_{\alpha}})
\leq
\sup_{\alpha \in J} \left[ k_{n} p_{\alpha}
(\frac{x - y}{\lambda_{\alpha}})
\right].
\]

From (2.10) we obtain

\[
\|T^{n}x - T^{n}y\|_{B} \leq k_{n} \|x - y\|_{B} \quad \text{for all } x, y \in K \text{ and } n \in \mathbb{N}.
\]

Hence \( T \) is an asymptotically nonexpansive self-mapping of \( K \) with respect to norm \( \|\cdot\|_{B} \).

**Definition 2.2.14.** Let \( \{x_{n}\} \) be any sequence in a locally convex space \( X \). Then \( \{x_{n}\} \) is said to be a Cauchy sequence in \( X \) if for each \( \alpha \in J \),

\[
p_{\alpha}(x_{n} - x_{m}) \to 0 \text{ as } n, m \to \infty.
\]

\( X \) is said to be sequentially complete if every Cauchy sequence in \( X \) converges to some point in \( X \), and it is said to be quasi-complete if every bounded closed subset of \( X \) is complete.

**Remark 2.2.15.** It is shown in p.210 of Köthe (1969) that every complete space is quasi-complete, and every quasi-complete space is sequentially complete.
Tarafdar (1975) has relaxed the completeness on the set $K$ of Taylor's Lemma 1.2.9 by the weaker condition of sequential completeness on the set $K$ and proved a corresponding result for nonexpansive mappings.

The following lemma shows that the completeness on the set $K$ of Lemma 2.2.9 can be relaxed.

**Lemma 2.2.16.** Let $K$ be a nonempty sequentially complete bounded star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive self-mapping of $K$. Then there is a sequence $(x_n)$ in $K$ such that

$$x_n - T^nx_n \to 0 \quad \text{as} \quad n \to \infty$$

in $\|\cdot\|_B$ and therefore

$$x_n - T^n x_n \to 0 \quad \text{as} \quad n \to \infty.$$

If further, $T$ is a uniformly asymptotically regular self-mapping of $K$, then

$$x_n - Tx_n \to 0 \quad \text{as} \quad n \to \infty.$$

**Proof.** Define $T$ from $K$ to $K$ as in the proof of Lemma 2.2.9.

Let $x, z \in K$. Then since $T$ is asymptotically nonexpansive, $T$ is asymptotically nonexpansive with respect to $\|\cdot\|_B$ and therefore

$$\|T^n x - T^n z\|_B \leq k_n \|x - z\|_B$$

(by Theorem 2.2.13).

Hence

$$\|T^n x - T^n z\|_B \leq \left(1 - \frac{1}{n}\right) \|x - z\|_B.$$

Thus $T$ is a contraction mapping of $K$ into itself with respect to the norm $\|\cdot\|_B$.

Since the norm topology on $X_B$ has a base of neighbourhoods of 0 consisting of closed sets, namely the scalar multiples of $B$ and $K$ is sequentially complete, it follows that $K$ is $\|\cdot\|_B$ sequentially complete in $X_B$ (apply 4.4 (b), p.210 of Köthe (1969) to the topology on $X_B$ induced by the topology on $X$ and the $\|\cdot\|_B$ topology on $X_B$) and hence $\|\cdot\|_B$ complete.

By Banach's contraction principle, $T^n$ has a unique fixed point say, $x_n$, in $K$.

Since $K$ is bounded and $a_n \to 1$ as $n \to \infty$, it follows that

$$\|x_n - T^n x\|_B \to 0 \quad \text{as} \quad n \to \infty.$$
Since the norm topology on $X_B$ is finer than the topology induced on $X_B$ by the topology of $X$, it follows that

$$x_n - T^n x_n \to 0 \text{ as } n \to \infty.$$  

The remaining part follows as in the proof of Lemma 2.2.9.

Tarafdar (1975) has relaxed the condition of compactness on the set $K$ (closedness of $(I-T)(K)$) of Taylor's Theorem 1.2.10 by a weaker condition of sequential compactness on the set $K$ (sequential closedness of $(I-T)(K)$) and proved a fixed point theorem for a nonexpansive mapping in a locally convex space. The corresponding theorems, which are extensions of Theorem 2.2.10 and Corollary 2.2.11, are obtained for asymptotically nonexpansive mappings.

**Theorem 2.2.17.** Let $K$ be a nonempty sequentially complete bounded star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$. Assume further that $(I-T)(K)$ is sequentially closed. Then $T$ has a fixed point in $K$.

**Proof.** The proof follows by using Lemma 2.2.16 first, as the proof of Theorem 2.2.10 follows by using Lemma 2.2.9.

**Theorem 2.2.18.** Let $K$ be a nonempty sequentially compact star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$. Then $T$ has a fixed point in $K$.

**Proof.** Since $K$ is sequentially compact, it follows that it is sequentially complete and bounded.

By Lemma 2.2.16, there is a sequence $(x_n) \subseteq K$ such that

$$x_n - T x_n \to 0 \text{ as } n \to \infty. \quad (2.11)$$

Since $K$ is sequentially compact, there is a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \to x \in K$ as $k \to \infty$.

From (2.11) we obtain $x_{n_k} - T x_{n_k} \to 0$ as $k \to \infty$.

Since $I-T$ is continuous, it follows that

$$(I - T)x_{n_k} \to (I - T)x \text{ as } k \to \infty.$$
Since $X$ is Hausdorff, it follows that $x = Tx$.

**Remark 2.2.19.** By Cain and Nashed's Theorem 3.3.3 for contraction mappings in locally convex spaces, we see that the mapping $T_n$, used in the proof of Lemma 2.2.16, has a unique fixed point in $K$. Hence Theorems 2.2.17 and 2.2.18 can also be proved without resorting to Tarafdar's construction of the normed space $X_B$.

### 2.3 Existence of Fixed Points for Non-self Mappings

In this section, we prove a theorem on the existence of fixed points for asymptotically nonexpansive, uniformly asymptotically regular non-self mappings in locally convex spaces.

Su and Sehgal (1971) defined a contractive mapping as follows:

**Definition 2.3.1 (Su and Sehgal (1974)).** Let $K$ be a nonempty subset of $X$. A mapping $T$ from $K$ to $X$ is said to be contractive if

$$p_\alpha(Tx - Ty) < p_\alpha(x - y) \text{ if } p_\alpha(x - y) \neq 0$$

$$= 0 \quad \text{otherwise}$$

for all $x, y$ in $K$ and for each $\alpha \in J$.

The following lemma due to Su and Sehgal (1974) is stated without proof.

**Lemma 2.3.2 (Su and Sehgal (1974)).** Let $K$ be a nonempty compact subset of $X$. If $T$ is a contractive mapping of $K$ into $X$ and $T(\partial K) \subseteq K$, then $T$ has a unique fixed point in $K$.

Taylor (1972) proved a result on the existence of fixed points for nonexpansive self-mapping $T$ of a nonempty compact star-shaped subset $K$ of a locally convex space $X$. This result was extended by Su and Sehgal (1974) to nonexpansive non-self mapping $T$ of $K$ into $X$ by assuming the condition that $T(\partial K) \subseteq K$. We extend the corresponding theorem for asymptotically nonexpansive, uniformly asymptotically
regular mappings. The following theorem is an extension of Corollary 2.2.11 of this chapter and it is new even in the case of Banach spaces.

**Theorem 2.3.3.** Let $T$ be a mapping of $X$ into itself. Let $K$ be a nonempty compact star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$ into $X$ such that $T^n(\partial K) \subseteq K$ for every $n \in \mathbb{N}$. Then $T$ has a fixed point in $K$.

**Proof.** Let $y$ be a star center of $K$. Define a map $T^n$ from $K$ to $X$ as in the proof of Lemma 2.2.9. Since $T^n(X) \subseteq X$, $T^n$ maps $K$ into $X$ and $T^n$ is a contraction of $K$ into $X$ and hence a contractive mapping of $K$ into $X$. Since $T^n(\partial K) \subseteq K$, $T^n(\partial K) \subseteq K$. Therefore by Lemma 2.3.2, $T^n$ has a unique fixed point, say, $x_n$ in $K$. It follows as in the proof of Lemma 2.2.9 that $x_n - T^n x_n$ converges to 0 as $n \to \infty$, when $T$ is uniformly asymptotically regular in $K$. Now it seen from the proof of Corollary 2.2.11 that $x = Tx$.

2.4 CONVERGENCE OF ITERATION TO FIXED POINTS FOR SELF-MAPPINGS

Taylor (1972) proved for $\beta$-nonexpansions in locally convex spaces that the sequence of iterates converges to a fixed point. In this section, we generalize Taylor's Theorem 1.2.12 to asymptotically nonexpansive mappings.

We need the following definition to prove our theorem.

**Definition 2.4.1** (p.71 of Kelley (1955)). A point $x$ of a topological space $X$ is called a cluster point of a net $S$ if and only if $S$ is frequently in every neighbourhood of $x$.

**Theorem 2.4.2.** Let $K$ be a nonempty closed bounded subset of $X$. Let $T$ be a continuous, asymptotically regular self-mapping of $K$. Assume that $I-T$ maps closed subsets of $K$ into closed subsets of $X$. Then, for each $x \in K$, the sequence of iterates $(T^n x)$ clusters at a fixed point of $T$ and each such cluster point is fixed by $T$. If, in addition, $T$ is an
asymptotically nonexpansive self-mapping of $K$, then every sequence $(T^n x)$ converges to a fixed point of $T$.

**Proof.** Let $T$ be a continuous, asymptotically regular self-mapping of $K$. Let $x \in K$ and $M$ denote the closure of $(T^n x)$. Since $T$ is asymptotically regular, it follows that

$$T^n x - T^{n+1} x \to 0 \text{ as } n \to \infty.$$ 

Therefore $0$ lies in the closure of $(I-T)(M)$. Since $M$ is closed and $I-T$ maps closed subsets of $K$ into closed subsets of $X$, it follows that $(I-T)(M)$ is closed. Therefore $0 \in (I-T)(M)$ and hence there is a point $y$ in $M$ such that $(I-T)(y) = 0$.

Since $y \in M$, either $y \in (T^n x)$ or $y$ is a cluster point of $(T^n x)$.

If $y = T^m x$ for some $m$, then

$$T^{n+m}(x) = T^n(T^m x) = T^n y = y \quad \text{for } n \in \mathbb{N}.$$ 

Therefore $T^k x = y$ if $k > m$. Hence $y$ is a cluster point of $(T^n x)$.

Let $z$ be any cluster point of $(T^n x)$. We know that a point $b$ in a topological space $X$ is a cluster point of a net $S$ if and only if some subnet of $S$ converges to $b$ (p. 71 of Kelley (1955)).

Therefore there is a subnet $(T^\beta x)$ of $(T^n x)$ such that $T^\beta x \to z$.

Hence $(I-T)z = (I-T) \lim T^\beta x = \lim (I-T)(T^\beta x)$, since $(I-T)$ is continuous.

Thus $z$ is a fixed point of $T$.

Assume further that $T$ is an asymptotically nonexpansive self-mapping of $K$. We already know that $y$ is a cluster point of $(T^n x)$.

Therefore for each $\alpha \in J$ and $\delta > 0$, there exists an integer $m$ such that

$$p_\alpha (T^m x - y) < \delta \quad (2.12)$$

Since $T$ is asymptotically nonexpansive , it follows that

$$p_\alpha (T^{n+m} x - T^{n+m} y) \leq k_{n-m} p_\alpha (T^m x - y) \text{ for } n \geq m.$$ 

From (2.12) we obtain

$$p_\alpha (T^n x - y) < k_{n-m} \delta.$$
Therefore \[ \limsup_{n \to \infty} p_\alpha(T^n x - y) \leq \limsup_{n \to \infty} k = \delta. \]

Hence \[ \lim_{n \to \infty} p_\alpha(T^n x - y) = 0. \]

That is, the sequence \((T^n x)\) converges to a fixed point of \(T\).

### 2.5 Common Fixed Point Theorem for Self-Mappings

In this section, we prove a common fixed point theorem for a commuting family of asymptotically nonexpansive self-mappings of a nonempty compact convex subset of a locally convex space. We also prove a common fixed point theorem for a pair of certain nonlinear mappings in metrizable linear topological spaces by using the G-iterative process.

To prove Theorem 2.5.3, we need the following lemma and definition.

**Lemma 2.5.1** (Tarafdar (1975)). Let \(M\) be a nonempty compact subset of \(X\) and \(K = \text{clco}(M)\). If, for any \(\beta \in J\), the \(p_\beta\)-diameter \(\delta(M, \beta)\) of \(M\) is greater than 0, then there is an element \(u \in K\) such that

\[ \sup \{ p_\beta(x-u) : x \in M \} < \delta(M, \beta). \]

**Definition 2.5.2.** Suppose that \(T\) is a self-mapping of \(X\). Then a nonempty subset \(K\) of \(X\) is said to have the property (1) under \(T\) if, for each \(x \in K\), the limit of every convergent subnet of \(\{T^n x\}\) belongs to \(K\).

**Theorem 2.5.3.** Let \(K\) be a nonempty compact convex subset of \(X\). Let \(F\) be a commuting family of asymptotically nonexpansive mappings of \(K\) into itself. Then the family \(F\) has a common fixed point in \(K\).

**Proof.** Let \(A_1 = \{ L : L \subseteq K \text{ is nonempty compact convex and has } \text{ property (1) under each } T \in F \}\). Since \(T(K) \subseteq K\) for each \(T \in F\) and \(K\) is compact, \(K\) has the property (1) under each \(T \in F\). Therefore \(A_1 \neq \emptyset\) as \(K \in A_1\).
Using Zorn's Lemma, $A_1$ has a minimal element, say, $L$.

Suppose that $A_2 = \{ M : M \subseteq L \text{ is nonempty compact and has the } (i) \text{ under each } T \in F \}$.

Again by using Zorn's Lemma, we obtain $A_2$ has a minimal element, say, $M$. Let $S \in F$ be fixed and $N = M \cap S(M)$.

To prove that $N$ is nonempty compact and has the property (1) under each $T$ in $F$. Since $S$ is continuous and $M$ is compact, $S(M)$ is compact and hence $N$ is compact.

Now, let $x \in N$. Since $K$ is compact and $\{ S^n x \} \subseteq K$, $\{ S^n x \}$ has a subnet converging to an element in $K$. Suppose that $w \in K$ is the limit of a convergent subnet $\{ S^\lambda x \}$ of $\{ S^n x \}$.

Since $x \in M$ and $M$ has the property (1) under $S$, $w \in M$.

Therefore $Sw \in S(M)$. Since $S$ is continuous, it follows that

$$Sw = S \lim_{\lambda} S^\lambda x = \lim_{\lambda} S^{\lambda + 1} x.$$ 

Since $M$ has the property (1) under $S$, $Sw \in M$. Therefore $Sw \in M$.

Hence $N$ is nonempty.

Let $T \in F$ be arbitrary.

Now, let $x \in N$. Suppose that $z \in K$ is the limit of a convergent subnet $\{ T^\lambda x \}$ of $\{ T^n x \}$.

Since $x \in M$ and $M$ has the property (1) under each $T$ in $F$, $z \in M$. Since $x \in S(M)$, $x = Sy$ for some $y \in M$.

Hence $S(T^\lambda y) = T^\lambda (Sy) \xrightarrow{\lambda} z$.

Since $L$ is compact and $\{ T^\lambda y \} \subseteq L$, there is a subnet $\{ T^\gamma y \}$ of $\{ T^\lambda y \}$ such that $T^\gamma y \xrightarrow{\gamma} v \in L$.

Since $y \in M$ and $M$ has the property (1) under $T$, $v \in M$. Since $S$ is continuous, it follows that $S(T^\gamma y) \xrightarrow{\gamma} Sv$.

Therefore $Sv = z$ and hence $z \in S(M)$. Therefore $z \in N$. Hence $N$ has the property (1) under each $T$ in $F$.

By the minimality of $M$, $N = M$. Therefore $M \subseteq S(M)$. Since $S$ was arbitrary, $M \subseteq T(M)$ for each $T \in F$.

If $\delta(M, \alpha) = 0$ for each $\alpha \in J$, then the theorem is proved.

Suppose that $\delta(M, \beta) > 0$ for some $\beta \in J$. Then by Tarafdar's Lemma 2.5.1, there is an element $u \in L$ such that
0 < r = \sup \{ p_\beta(u - y): y \in M \} < \delta(M, \beta).

Let \( D = \bigcap_{y \in M} B_\beta(y, r) \cap L \), where \( B_\beta(y, r) = \{ a : p(y - a) < r \} \).

To show that \( D \) is a nonempty compact convex proper subset of \( L \) and has the property (1) under each \( T \) in \( F \). Since \( u \in D \), \( D \) is nonempty. Since \( L \) compact convex and \( B_\beta(y, r) \) is closed convex, it follows that \( D \) is compact convex.

Since \( M \) is compact, there exist \( x_1, x_2 \in M \) such that
\[
P_\beta(x_1 - x_2) = \delta(M, \beta) > r.
\]
Therefore \( x_2 \notin B_\beta(x_1, r) \) and hence \( x_2 \notin D \). Thus \( D \) is a proper subset of \( L \).

Finally to show that \( D \) has the property (1) under each \( T \in F \).

Let \( x \in K \), \( z \in D \). Suppose that \( w \in K \) is the limit of a convergent subnet \( (T^\lambda w) \) of \( (T^\epsilon z) \). It is enough to show that \( w \in D \).

Let \( \epsilon > 0 \) be given. Since \( T^\lambda w \to w \), there exists \( \mu \) such that
\[
p_\beta(w - T^\lambda z) < \epsilon, \text{ whenever } \lambda \geq \mu \tag{2.13}
\]
Since \( k_1 \to 1 \) as \( \lambda \to \infty \), \( k_\lambda \to 1 \), and since \( K \) is compact, there exists \( \eta \geq \mu \) such that
\[
(k_\lambda - 1) p_\beta(x - z) < \epsilon, \text{ whenever } \lambda \geq \eta \geq \mu \tag{2.14}
\]
Since \( T \) is asymptotically nonexpansive, it follows that
\[
p_\beta(T^\lambda x - T^\lambda z) \leq k_\lambda p_\beta(x - z).
\]
Since \( M \subseteq T(M) \) for each \( T \in F, M \subseteq T^\lambda(M) \).

Let \( y \in M \subseteq T^\lambda(M) \). Then for each \( \lambda \), there is a point \( a_\lambda \in M \) such that
\[
y = T^\lambda(a_\lambda) \quad \text{and} \quad p_\beta(w - y) \leq p_\beta(w - T^\lambda z) + p_\beta(T^\lambda z - y)
\]
\[
= p_\beta(w - T^\lambda z) + p_\beta(T^\lambda z - T^\lambda(a_\lambda)), \quad \text{since } y = T^\lambda(a_\lambda).
\]
\[
\leq p_\beta(w - T^\lambda z) + k_\lambda p_\beta(z - a_\lambda),
\]
\[
\text{since } T \text{ is asymptotically nonexpansive.}
\]
\[
< \epsilon + (k_\lambda - 1) p_\beta(z - a_\lambda) + p_\beta(z - a_\lambda), \text{ by (2.13) whenever } \lambda \geq \mu
\]
\[
< \epsilon + \epsilon + p_\beta(z - a_\lambda), \text{ by (2.14), whenever } \lambda \geq \eta \geq \mu
\]
\[
\leq 2 \epsilon + r, \text{ since } z \in D.
\]
Since ε was arbitrary, \( p^\beta(w-y) \leq r \) and therefore \( w \in B^\beta(y,r) \).

Hence \( w \in \bigcap_{y \in M} B^\beta(y,r) \). Since \( z \in D \subseteq L \) and \( L \) has the property (1) under each \( T \in F \), \( w = \lim_{\lambda} T^\lambda z \in L \). Therefore \( w \in D \).

Hence \( D \) has the property (1) under each \( T \in F \), which contradicts the minimality of \( L \). Therefore \( \delta(M, \alpha) = 0 \) for each \( \alpha \in J \). Hence \( M \) consists of a single point. Since \( M \subseteq T(M) \) for each \( T \in F \) and \( M \) consists of a single point, say, \( x \), it follows that \( x = Tx \) for each \( T \in F \).

Khan (1988) has shown that, for a mapping \( T \) satisfying a certain contractive condition in metrizable linear topological spaces, if the sequence of Mann iterates converges, then it converges to a fixed point of \( T \). More recently, Pathak et al (1990) have shown that, for a pair of mappings \( T_1 \) and \( T_2 \) satisfying a certain contractive condition in normed linear spaces, if the sequence of G-iterates (2.17 - 2.18) converges, then it converges to a common fixed point of \( T_1 \) and \( T_2 \). The following theorem shows that the corresponding result holds good in metrizable linear topological spaces.

Let \( X \) be a metrizable linear topological space. The topology of \( X \) is generated by a \( F \)-norm \( q \) which has the following properties (Köthe(1969)):

1. \( q(x) \geq 0 \) and \( q(x) = 0 \) if and only if \( x = 0 \)
2. \( q(x + y) \leq q(x) + q(y) \)
3. \( q(\lambda x) \leq q(x) \) for all (real or complex) scalars \( \lambda \) with \( |\lambda| \leq 1 \)
4. \( q(\lambda x_n) \rightarrow 0 \) if \( q(x_n) \rightarrow 0 \) for all scalars \( \lambda \)
5. \( q(\lambda x_n) \rightarrow 0 \) for all \( x \in X \).

We remark that \( q \) is continuous on \( X \), the equation \( d(x,y) = q(x - y) \) defines a metric on \( X \), \( q(-x) = q(x) \) and \( q(\lambda x) \leq q(\mu x) \) for all scalars \( \lambda, \mu \) with \( |\lambda| \leq |\mu| \).

Suppose that \( T \) is a self- mapping of a nonempty subset \( K \) of \( X \). Then we recall that the G-iterative process (2.15) associated with \( T \) is defined as follows:
where \( \{\mu_n\} \) and \( \{\lambda_n\} \) satisfy (i) \( \lambda_0 = \mu_0 = 1 \), (ii) \( 0 < \lambda_n < 1, 0 \leq \mu_n \leq 1 \) such that \( \lambda_n \geq \lambda_{n+1}, n > 0 \), (iii) \( \lim_{n \to \infty} \lambda_n = h > 0 \) and (iv) \( \lim_{n \to \infty} \mu_n = 1 \).

If \( \mu_n = 1 \) in (2.15), then G-iterative process reduces to the Mann iteration.

**Theorem 2.5.4.** Let \( K \) be a nonempty closed bounded subset of a metrizable linear topological space \( X \). Let \( T_1 \) and \( T_2 \) be any two mappings of \( K \) into itself satisfying

\[ q(T_1 x - T_2 y) \leq \delta \max \{ c q(x - y), [q(x - T_1 x) + q(y - T_2 y)], [q(x - T_2 y) + q(y - T_1 x)] \} \]

for all \( x, y \in K \), where \( 0 \leq \delta < 1 \) and \( c > 0 \).

Suppose that the sequence \( \{x_n\} \) is defined in accordance with G-iterates associated with \( T_1 \) and \( T_2 \) as follows:

\[ x_0 \in K, \]

\[ x_{n+1} = (\mu_{n+1} - \lambda_{n+1}) x_n + \lambda_{n+1} T_1 x_n + (1 - \mu_{n+1}) T_2 x_n, \quad \text{for } n \in \mathbb{N} \cup \{0\} \]

for \( n \in \mathbb{N} \cup \{0\} \), where \( \{\lambda_n\} \) and \( \{\mu_n\} \) satisfy (i) - (iv).

If \( \{x_n\} \) converges to \( z \) in \( K \), then \( z \) is a common fixed point of \( T_1 \) and \( T_2 \). If further, \( \max \{c \delta, 2 \delta\} < 1 \), then the mappings \( T_1 \) and \( T_2 \) have a unique common fixed point in \( K \).

**Proof.** Suppose that \( z \) is an element of \( K \) such that \( \lim_{n \to \infty} x_n = z \).

To show that \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

Now,

\[ q(z - T_1 z) \leq q(z - x_n) + q(x_n - T_1 x_n) + q(T_1 x_n - T_2 z) \]

\[ \leq q(z - x_n) + q(x_n - T_1 x_n) + \delta \max \{ c q(x_n - z), [q(x_n - T_1 x_n) + q(z - T_2 z)] \} \]

by (2.16)
\[ \leq q(z-x_{2n}) + q(x_{2n} - T x_{2n}) + \delta \max \left\{ c \cdot q(x_{2n} - z), [q(x_{2n} - T x_{2n}) + q(z - T z)] \right\} \]

From (2.17) we obtain

\[ x_{2n+1} = x_{2n} = \lambda^{-1}(1 - \mu) (T x_{2n-1} - x_{2n}) + \lambda^{-1}(x_{2n} - x_{2n+1}). \]

Using (2.20) in (2.19) we get

\[ q(z - T z) \leq q(z - x_{2n}) + q(\lambda^{-1}(1 - \mu) (T x_{2n-1} - x_{2n}) + q(\lambda^{-1}(x_{2n} - x_{2n+1}))+ q(z - T z)] \]

Taking limit as \( n \to \infty \) on both sides in (2.21) we get

\[ q(z - T z) \leq \delta q(z - T z). \]

Therefore \( z \) is a fixed point of \( T \) in \( K \).

Similarly

\[ q(T z - z) \leq \delta q(T z - z). \]

Therefore \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

Suppose that \( u (u \neq z) \) is another common fixed point of \( T_1 \) and \( T_2 \) in \( K \). Then we have

\[ q(u - z) = q(T_1 u - T_2 z) \leq \max \{c \delta, 2 \delta\} q(u - z) \]

by (2.16)

\[ < q(u - z), \text{ since } \max \{c \delta, 2 \delta\} < 1, \text{ which is a contradiction.} \]

Hence \( z \) is a unique common fixed point of \( T_1 \) and \( T_2 \) in \( K \).

The above result is an extension of Theorem 2 of Khan (1988) to a pair of mappings in metrizable linear topological spaces by the use of the \( G \)-iterative process. If we assume that \( T_1 = T_2 = T \) and \( c = 1 \) in the above theorem, then we obtain the following corollary.
Corollary 2.5.5 (Khan (1988)). Let $K$ be a nonempty closed bounded subset of $X$. Let $T$ be a mapping of $K$ into itself satisfying

$$q(Tx-Ty) \leq \delta \max\{q(x-y), [q(x-Tx) + q(y-Ty)], [q(x-Ty) + q(y-Tx)]\}$$

for all $x, y \in K$, where $0 \leq \delta < 1$.

Suppose that the sequence $\{x_n\}$ is defined as in (2.15). If $\{x_n\}$ converges to $z$ in $K$, then $z$ is a unique fixed point of $T$ in $K$.

Remark 2.5.6. Pathak et al (1990) showed by an example that the boundedness of $K$ is a necessary condition in the above results in order to have the sequence $\{x_n\}$ converge.