CHAPTER 6
APPLICATIONS OF FIXED POINT THEOREMS TO BEST APPROXIMATIONS

6.1 PRELIMINARIES

Applications of fixed point theorems to best approximations are well known (See Brosowski (1969), Reich (1978), Hicks et al (1982), etc.). Subrahmanyam (1977) proved that if $T$ is a nonexpansive self-mapping of a normed linear space $X$ with a fixed point $y$ and leaving a finite dimensional subspace $K$ of $X$ invariant, then $T$ has a fixed point which is a best $K$-approximation to $y$. Theorem 3 of Meinardus (1963) is deduced from this result. A similar result was obtained by Smoluk (1981) in which the dimension of $K$ is replaced in Subrahmanyam's result by the assumption that $T$ is linear and $T|_K$ is compact. Habiniak (1989) proved that Smoluk's result remains true when $T$ is not linear.

Theorems on the existence of fixed points for nonexpansive and Banach operator in locally convex spaces are proved in section 6.2. We also prove a theorem on the existence of a fixed point for a nonexpansive mapping $T$ in a locally convex space $X$ which is a best approximation to a given fixed point of $T$ in $X$. This result is an extension of the results of Smoluk (1981) and Habiniak (1989).

Singh (1979) extended the result of Brosowski (1969) by proving that if $T$ is a nonexpansive self-mapping of a normed linear space $X$ with a fixed point $y$ and $K$ is a $T$-invariant subset of $X$ and if the set $D$ of best $K$-approximates to $y$ is nonempty compact and star-shaped, then $T$ has a fixed point which is a best $K$-approximation to $y$. This result due to Singh (1979) shows that one can relax the convexity on the set $D$ and omit the linearity of the map $T$ in the
statement of the well known theorem of Brosowski (1969). Sahab et al (1988) proved that a pair of mappings \( T \) and \( S \) have a common fixed point which is a best K-approximation to \( y \) by assuming some condition on the map \( S \) in a normed linear space.

Singh (1980) further extended the above result of Singh (1979) to locally convex spaces with the same conditions on \( T \) and \( D \). Sahney et al (1983) proved the result of Singh (1980), by assuming instead of the compactness of \( D \) that (i) \( D \) is sequentially complete and bounded and either \((I-T)(D)\) is closed (or) \( T \) is demicompact (1.7.5), and (ii) \( D \) is weakly compact and either \((I-T)(D)\) is demiclosed (2.2.4) (or) \( I-T \) is weakly pseudo-continuous (1.7.5). In section 6.3, we prove the corresponding results for asymptotically nonexpansive mappings in locally convex spaces.

In section 6.4, we introduce the notion of an asymptotically nonexpansive map \( T \) with respect to another map \( S \) and prove that for such mappings, \( T \) and \( S \) have a common fixed point in normed linear spaces \( X \) which is a best approximation to a given point in \( X \).

Mukherjee et al (1989) showed that the existence of fixed points for nonexpansive mappings \( T \) in normed linear spaces which are also best co-approximation (See Definition 1.7.1(2)) and strong best co-approximation (See Definition 1.7.1(3)) to a given fixed point of \( T \). Also, in section 6.4, the corresponding results for asymptotically nonexpansive mappings are obtained.

6.2 FOR NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

In this section, we obtain some fixed point theorems for Banach operators and nonexpansive mappings in locally convex spaces. We also prove the existence of a fixed point for a nonexpansive mapping \( T \) in a locally convex space which is also a best K-approximation to a given fixed point of \( T \).

We need the following notion to prove our theorem.
Definition 6.2.1 (Naimpally et al (1984a)). Let $K$ be a nonempty subset of a locally convex space $X$. A mapping $T$ from $K$ to $K$ is said to be a Banach operator if, for each $\alpha \in J$, there exists a real number $k_\alpha$ with $0 \leq k_\alpha < 1$ such that for all $x \in K$, we have

$$p_\alpha(T^2x - Tx) \leq k_\alpha p_\alpha(Tx - x).$$

Naimpally et al (1984a) showed that (i) every contraction map is a Banach operator but the converse may not be true and (ii) a Banach operator need not be continuous nor need its fixed points be unique.

Subrahmanyam (1974/75) proved the following fixed point theorem for Banach operators in normed linear spaces $X$.

Theorem 6.2.2 (Subrahmanyam (1974/75)). If $T$ is a continuous Banach operator of a nonempty closed subset $K$ of $X$ into itself such that $\text{cl}(T(K))$ is compact, then $T$ has a fixed point in $K$.

We extend this theorem to locally convex spaces $X$.

Theorem 6.2.3. If $T$ is a continuous Banach operator of a nonempty closed subset $K$ of $X$ into itself such that $\text{cl}(T(K))$ is sequentially complete, then $T$ has a fixed point in $K$.

Proof. Let $x \in K$ and $\alpha \in J$ be arbitrary. Then since $T$ is a Banach operator, we have

$$p_\alpha(T^nx - T^nx) \leq k_\alpha^m p_\alpha(T^n x - T^{n-1}x) \leq k_\alpha^m p_\alpha(T^{n-1} x - T^{n-2}x) \leq \ldots \leq k_\alpha^n p_\alpha(Tx - x).$$

If $m > n$, then

$$p_\alpha(T^nx - T^nx) \leq \sum_{i=n}^{m-1} k_\alpha^i p_\alpha(Tx - x) \leq \frac{k_\alpha^n}{1-k_\alpha} p_\alpha(Tx - x). \quad (6.1)$$

Since $0 \leq k_\alpha < 1$, from (6.1), given $\varepsilon > 0$, there exists $L$ such that

$$p_\alpha(T^nx - T^nx) < \varepsilon \quad \text{for all } m > n \geq L.$$ 

Therefore $\{T^nx\}$ is a Cauchy sequence in $K$. Since $\{T^nx\} \subseteq T(K)$ and...
The following theorem is due to Habiniak (1989). Theorem 6.2.4. If $T$ is a nonexpansive mapping of a nonempty closed star-shaped subset $K$ of a normed linear space $X$ into itself such that $\text{cl}(T(K))$ is compact, then $T$ has a fixed point in $K$.

Using theorem 6.2.3, we generalize the above theorem to locally convex spaces.

Theorem 6.2.5. If $T$ is a nonexpansive mapping of a nonempty closed star-shaped subset $K$ of a locally convex space $X$ into itself such that $\text{cl}(T(K))$ is sequentially compact, then $T$ has a fixed point in $K$.

Proof. Let $z$ be a star-center of $K$. Define a map $T_n$ from $K$ to $K$ by

$$T_n(x) = (1-k_n) z + k_n T x$$

for all $x \in K$, where $\{k_n\}$ is a sequence of numbers in $(0,1)$ with $k_n \to 1$ as $n \to \infty$.

Since $K$ is star-shaped, $T_n$ maps $K$ into itself. Let $x \in K$ and $\alpha \in J$ be arbitrary. Then since $T$ is nonexpansive, we have

$$p_\alpha(T_n^2 x - T_n x) = k_n p_\alpha(T(T_n x) - T x) \leq k_n p_\alpha(T x - x).$$

Therefore $T_n$ is a Banach operator on $K$. Since $\text{cl}(T(K))$ is sequentially compact, it follows that $\text{cl}(T_n(K))$ is sequentially compact and hence sequentially complete.

By Theorem 6.2.3, there is a point $x_n \in K$ such that

$$x_n = T_n x_n = (1-k_n) z + k_n T x_n.$$  \hspace{1cm} (6.2)

Since $\{T_n x_n\} \subseteq T(K)$ and $\text{cl}(T(K))$ is sequentially compact, it follows that $\{T_n x_n\}$ contains a subsequence $\{T_{n_j} x_j\}$ such that

$$T_{n_j} x_{n_j} \to x \text{ in } K \text{ as } j \to \infty.$$
From (6.2), we obtain
\[ x = (1-k_j) z + k_j T x \to x \quad \text{as} \quad j \to \infty. \]
Since \( T \) is continuous, it follows that \( T x \to T x \quad \text{as} \quad j \to \infty. \)
Therefore \( T x = x. \)

**Definition 6.2.6.** Let \( K \) be a nonempty subset of a locally convex space \( X \). For an element \( y \) in \( X \), let
\[ d_\alpha(y, K) = \inf \{ p_\alpha(y - x) : x \in K \} \quad \text{and} \]
\[ D = \{ x \in K : p_\alpha(y - x) = d_\alpha(y, K) \text{ for all } \alpha \in J \}. \]
Any point in \( D \) is called a best \( K \)-approximation to \( y \) and \( D \) is called the set of best \( K \)-approximates to \( y \).

Smoluk (1981) showed that if \( T \) is a nonexpansive self-mapping of a normed linear space \( X \), \( K \) is a \( T \)-invariant nonempty subset of \( X \) and \( y = Ty \) and if the restriction \( T|_K \) of \( T \) to \( K \) is compact, then the set \( D \) of best \( K \)-approximates to \( y \) is nonempty.

**Theorem 6.2.7.** Let \( T \) be a nonexpansive self-mapping of a locally convex space \( X \). Let \( K \) be a \( T \)-invariant nonempty subset of \( X \) and \( y = Ty \) for some \( y \in X \). Suppose that the restriction \( T|_K \) of \( T \) to \( K \) is sequentially compact (that is, \( T \) is continuous and for every bounded subset \( M \) of \( K \), \( cl(T(M)) \) is sequentially compact). Suppose that for each \( n \), the set \( K_n \) defined in the proof below is nonempty. Then the set \( D \) of best \( K \)-approximates to \( y \) is nonempty.

**Proof.** For each \( \alpha \in J \), let \( \rho_\alpha = d_\alpha(y, K) = \inf \{ p_\alpha(y - z) : z \in K \} \).

For each \( n > 0 \), let \( K_n = \{ x \in K : p_\alpha(y - x) \leq \rho_\alpha + \frac{1}{n} \text{ for each } \alpha \in J \}. \)
To show that \( K_n \) is a closed bounded and \( T \)-invariant subset of \( K \).

Since \( Ty = y \) and \( T \) is nonexpansive, \( K_n \) is a \( T \)-invariant subset of \( K \).
Now, let \( x \in K_n \). Then there exists a net \( (x_\beta) \subseteq K_n \) such that
\[ x_\beta \to x. \]
Then \( p_\alpha(y - x) = \lim \beta \ p_\alpha(y - x_\beta) \leq \rho_\alpha + (1/n) \text{ for each } \alpha \in J \text{ and hence } K_n \text{ is closed as } x \in K_n. \)
Also $K$ is bounded and $K_{n+1} \subseteq K_n$ for all $n$.

Choose $x \in K_n$. Then $T^n_x \in T^n_{K_n}$ since $K_n$ is $T$-invariant set. Since $K_n$ is bounded and $T^n_{|K_n}$ is sequentially compact, it follows that $cl(T^n(K_n))$ is sequentially compact. Therefore the sequence $\{T^n_x\}$ contains a subsequence $\{T^n_{x_j}\}$ such that

$\lim_{j \to \infty} T^n_{x_j} = z \in K_1$ as $j \to \infty$.

Therefore $p^{(z)}(x - y) = p^{(z)}(T^n_{x_j} - Ty) \leq p^{(z)}(x - y) \leq \rho^{(z)} + (1/n_j)$.

Hence $p^{(z)}(z - y) = \rho^{(z)}$.

Therefore $D \neq \emptyset$ as $z \in D$.

We do not know whether the nonemptyness of each $K_n$ will follow from the remaining hypotheses in the above theorem.

The proof of the next theorem follows from Theorem 6.2.5.

**Theorem 6.2.8.** Let $X$, $T$, $K$ and $y$ be as in Theorem 6.2.7. Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty. Then $T$ has a fixed point which is a best $K$-approximation to $y$.

**Proof.** If $u \in D$, then since $T$ is nonexpansive and $y = Ty$, we have

$p^{(y)}(y - Tu) = p^{(y)}(Ty - Tu) \leq p^{(y)}(y - u) = d^{(y)}(y, K)$ for each $a \in J$.

Therefore $D$ is $T$-invariant set as $Tu \in D$.

Clearly $D$ is closed and convex and hence $D$ is star-shaped.

Let us show that $D$ is bounded and $cl(T(D))$ is sequentially compact.

If $x \in D$, then we have

$p^{(x)}(x) = p^{(x)}(y - x) + p^{(y)}(y) = d^{(y)}(y, K) + p^{(y)}(y)$ for each $a \in J$.

Therefore $D$ is $p^{(x)}$-bounded for each $a \in J$ and hence bounded. Since $D$ is bounded and $T^n_{D}$ is sequentially compact, it follows that $cl(T^n(D))$ is sequentially compact. Therefore all the conditions of Theorem 6.2.5 are satisfied. Hence $T$ has a fixed point which is a best $K$-approximation to $y$. 
6.3 FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

In this section, we prove the theorems on the existence of fixed points for asymptotically nonexpansive mappings in locally convex spaces which are also best approximations. In this section, let $X$ denote a locally convex space.

Singh (1980), Sahney et al (1983) proved applications of fixed point theorems for nonexpansive mappings to best approximations in locally convex spaces. We now prove the corresponding results for asymptotically nonexpansive mappings in such spaces.

**Theorem 6.3.1.** Let $K$ be a nonempty subset of $X$. Suppose that $y \in X$ and $T$ is an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$. Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty sequentially complete bounded and star-shaped which is invariant under $T$. Assume further that $(I-T)(D)$ is sequentially closed. Then $T$ has a fixed point which is a best $K$-approximation to $y$.

**Proof.** The result follows from Theorem 2.2.17, by applying it to $T$ and $D$.

The Corollary 6.3.2 follows from Theorem 6.3.1 by noting that the condition (*) implies that $D$ is invariant under $T$.

**Corollary 6.3.2.** Let $K$ be a nonempty subset of $X$. Suppose that $y \in X$ and $T$ is an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$ which satisfies the following condition:

(*) \[ p_\alpha(Tx - y) \leq p_\alpha(x - y) \text{ for all } x \in K \text{ and } \alpha \in J. \]

Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty sequentially complete bounded and star-shaped. Assume further that $(I-T)(D)$ is sequentially closed. Then $T$ has a fixed point which is a best $K$-approximation to $y$. 
Remark 6.3.3. If we assume $T$ to be nonexpansive in the above theorem, then the condition of uniform asymptotic regularity of $T$ is not required.

The following example shows that if the condition that $D$ is invariant under $T$ is dropped, then the result may not be valid.

Example 6.3.4. Let $X = \mathbb{R}$ and $K = [0,1]$ with usual metric.

We define a map $T$ from $K$ to $K$ by $Tx = \frac{1}{1+x}$ for all $x \in K$.

Then $T$ is nonexpansive.

Suppose that $y = -1 \in X$. Then $D = \{0\}$ is closed convex and bounded.

Hence it is complete and star-shaped. Clearly $D$ is not invariant under $T$ and $(I-T)(D)$ is closed. Also $0$ is not a fixed point of $T$. Thus $T$ has no fixed point in $D$ which is a best $K$-approximation to $y$.

We note that a weakly compact subset of a locally convex space is both complete and bounded (p.155-156 of Kelley et al (1963)).

Theorem 6.3.5. Let $X$, $K$, $y$ and $T$ be as in Theorem 6.3.1. Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty weakly compact and star-shaped which is invariant under $T$. Assume further that $I-T$ is demiclosed. Then $T$ has a fixed point which is a best $K$-approximation to $y$.

Proof. It can be shown as in Theorem 2.2.12 that $T$ has a fixed point, say, $x$ in $D$. It follows as in Theorem 6.3.1 $x$ is a best $K$-approximation to $y$.

Corollary 6.3.6. Let $X$, $K$, $y$ and $T$ be as in Corollary 6.3.2. Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty weakly compact and star-shaped. Assume further that $I-T$ is demiclosed. Then $T$ has a fixed point which is a best $K$-approximation to $y$.

These results have appeared in the paper of Vijayaraju (1991).
6.4 FOR ASYMPTOTICALLY NONEXPANSIVE MAPS WITH RESPECT TO ANOTHER MAP IN NORMED LINEAR SPACES

In this section, we prove common fixed point theorems for a pair of mappings \( T \) and \( S \) (when \( T \) is asymptotically nonexpansive and \( S \) is continuous) in normed linear spaces which are also best approximations.

We introduce the following concept.

**Definition 6.4.1.** Let \( K \) be a nonempty subset of a normed linear space \( X \). A mapping \( T \) from \( K \) to \( K \) is said to be asymptotically nonexpansive mapping with respect to another mapping \( S \) from \( K \) to \( K \), if there is a sequence \( \{k_n\} \) of real numbers with \( k_n \to 1 \) as \( n \to \infty \) such that

\[
\|T^n x - T^n y\| \leq k_n \|Sx - Sy\| \quad \text{for all } x, y \in K \text{ and } n \in \mathbb{N}.
\]

We note that if \( S \) is the identity map, then \( T \) is an asymptotically nonexpansive mapping (See Definition 1.5.1) of \( K \) into itself.

**Example 6.4.2.** Let \( X = \ell_2 \) with the usual norm and \( K \) be the closed unit ball in \( X \). Define a map \( T \) from \( K \) to \( K \) as in Example 3.2.8.

We define a map \( S \) from \( K \) to \( K \) by

\[
S y = (0, \eta_1, \eta_2, \ldots) \quad \text{for all } y = (\eta_1, \eta_2, \ldots) \in K
\]

Let \( a = (1,0, \ldots), \quad b = (1,0, \ldots) \in K, \) where \( i^2 = -1 \).

Then \( \|Ta - Tb\| = 2, \quad \|a - b\| = |1| = 1, \quad \|Sa - Sb\| = 2^{1/2} = \|Sa - Sb\| \).

Therefore \( \|Ta - Tb\| > \|Sa - Sb\| \) and hence \( T \) is not nonexpansive with respect to \( S \).

Now, let \( x = (\xi_1, \xi_2, \ldots), \quad y = (\eta_1, \eta_2, \ldots) \in K \). Then we have

\[
\|Tx - Ty\|^2 = \|\xi_1^2 - \eta_1^2\|^2 + \sum_{j=2}^{\infty} |A_j \xi_j - A_j \eta_j|^2
\]
\[ = |\xi_1 - \eta_1|^2 |\xi_1 + \eta_1|^2 + \sum_{j=2}^{\infty} A_j^2 |\xi_j - \eta_j|^2 \]
\[ \leq 2^2 |\xi_1 - \eta_1|^2 + \sum_{j=2}^{\infty} A_j^2 |\xi_j - \eta_j|^2 \]
\[ \leq 2^2 \|x - y\|^2, \text{ since } 0 < A_i < 1 \text{ for all } i. \]

Therefore \( \|Tx - Ty\| \leq 2 \|x - y\| = 2 \|Sx - Sy\| \).

Let \( x \in K \). Then we have
\[ T^n x = (0, \ldots, 0, \Pi A_{i_1}^2 \xi_1, \Pi A_{i_2}^2 \xi_2, \ldots, \Pi A_{i_k}^2 \xi_k, \ldots). \]

It follows by induction that
\[ \|T^n x - T^n y\| \leq k_n \|x - y\| = k_n \|Sx - Sy\| \text{ where } k_n = \prod_{l=2}^{n} A_l. \]

Therefore \( T \) is asymptotically nonexpansive with respect to \( S \).

The following well known theorem was proved by Jungck (1976).

**Theorem 6.4.3.** (Jungck (1976)). Let \((X, d)\) be a compact metric space. Suppose that \( T \) and \( S \) are commuting self-mappings of \( X \) such that \( T(X) \subseteq S(X) \), \( S \) is continuous and \( d(Tx, Ty) < d(Sx, Sy) \) for all \( x, y \) in \( X \) whenever \( Sx \neq Sy \). Then \( T \) and \( S \) have a unique common fixed point in \( X \).


**Theorem 6.4.4** (Sahab et al (1988)). Let \( X \) be a normed linear space. Suppose that \( T \) and \( S \) are self-mappings of \( X \). Let \( K \) be a nonempty subset of \( X \) such that \( T(\partial K) \subseteq K \) and \( y \) is a common fixed point of \( T \) and \( S \). Suppose that \( D \) is the set of best \( K \)-approximates to \( y \). Assume that \( T \) and \( S \) satisfy
\[ \|Tx - Ty\| \leq \|Sx - Sy\| \text{ for all } x, y \text{ in } D \text{ whenever } Sx \neq Sy. \] (that is, \( T \) is nonexpansive with respect to \( S \)). Suppose that \( S \) is a linear continuous mapping on \( D \) and \( STx = TSx \) for all \( x \) in \( D \). If \( D \) is nonempty
compact and star-shaped with respect to a point \( z \) in \( F(S) \), the set of fixed points of \( S \), and if \( S(D) = D \), then \( T \) and \( S \) have a common fixed point which is also a best \( K \)-approximation to \( y \).

The following theorem is an extension of Sahab et al's Theorem 6.4.4. for nonexpansive mapping \( T \) with respect to continuous mapping \( S \), to asymptotically nonexpansive mapping \( T \) with respect to continuous mapping \( S \), in normed linear spaces, by assuming the uniform asymptotic regularity of \( T \).

**Theorem 6.4.5.** Let \( K \) be a nonempty subset of \( X \) and \( y \in X \). Suppose that \( T \) and \( S \) are self-mappings of \( K \) such that \( T \) is asymptotically nonexpansive with respect to \( S \). Suppose that \( F(S) \) is nonempty. Suppose that the set \( D \) of best \( K \)-approximates to \( y \) is nonempty compact and star-shaped with respect to an element \( z \) in \( F(S) \), and \( D \) is invariant under \( T \). Assume further that \( T \) is uniformly asymptotically regular on \( D \), \( S \) is an affine continuous mapping on \( D \) such that \( S(D) = D \) and \( STx = TSx \) for all \( x \in D \). Then \( T \) and \( S \) have a common fixed point which is also a best \( K \)-approximation to \( y \).

**Proof.** Define a map \( T_n \) from \( D \) to \( D \) by

\[
T_n(x) = a_n T^n x + (1 - a_n) z \quad \text{for all } x \in D,
\]

where \( a_n = \frac{1 - 1/n}{k_n} \) and \( \{k_n\} \) is as in Definition 6.4.1.

Since \( T(D) \subseteq D \) and \( D \) is star-shaped with respect to \( z \), it follows that \( T_n \) maps \( D \) into itself.

Now, let \( a \in D \). Then we have

\[
T_n(Sa) = a_n T^n(Sa) + (1 - a_n) Sz = a_n S(T^n a) + (1 - a_n) Sz = S(T_n a), \text{ since } S \text{ is affine.}
\]

Therefore \( T_n \) and \( S \) commute for each \( n \).

Since \( T_n(D) \subseteq D \) and \( S(D) = D \), \( T_n(D) \subseteq S(D) \).

Suppose that \( a, b \in D \) and \( Sa \neq Sb \). Then we have

\[
\|T_n a - T_n b\| = a_n \|T^n a - T^n b\| \leq \frac{1}{n} \|a - b\| < \|a - b\|.
\]
Also D is compact and S is continuous on D. By Jungck's Theorem 6.4.3, there is a point \( x_n \) in D such that
\[
x_n = T^n x_n = S x_n.
\]
Therefore \( x_n = T^n x_n = (1 - a_n) (z - T^n x_n) \to 0 \) as \( n \to \infty \), since \( a_n \to 1 \) as \( n \to \infty \) and D is bounded.

Since \( T \) is uniformly asymptotically regular on D, it follows that
\[
x_n - T^{n+1} x_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, for all \( x_n \in D \),
\[
\| x_n - T x_n \| \leq \| x_n - T^{n+1} x_n \| + \| T^{n+1} x_n - T x_n \|
\]
\[
= \| x_n - T^{n+1} x_n \| + k \| S(T^n x_n) - S x_n \|
\]
\[
= \| x_n - T^{n+1} x_n \| + k \| T^n x_n - x \|, \quad \text{since} \quad S \text{ commutes with} \quad T^n \quad \text{and} \quad x_n = T^n x_n.
\]

Therefore \( (I - T) x_n \to 0 \) as \( n \to \infty \).

Since D is compact and \( \{ x_n \} \) is in D, there is a subnet \( \{ x_{\beta_n} \} \) of \( \{ x_n \} \) such that \( x_\beta \xrightarrow{\beta} x \in D \).

Since \( I - T \) is continuous, it follows that
\[
(I - T) x_\beta \xrightarrow{\beta} (I - T) x.
\]

But \( (I - T) x_\beta \xrightarrow{\beta} 0 \). Therefore \( (I - T) x = 0 \).

Since S is continuous, it follows that
\[
S x = S \lim_{\beta} x_\beta = \lim_{\beta} S x_\beta = \lim_{\beta} x_\beta = x
\]
Thus \( T \) and \( S \) have a common fixed point \( x \) which is also a best K-approximation to y.

The Corollary 6.4.6 follows from Theorem 6.4.5 by noting that the condition (i) implies that D is invariant under T.

**Corollary 6.4.6.** Let \( X, K, y, T \) and \( S \) be as in Theorem 6.4.5. Assume further that \( T \) satisfies the following condition:
Suppose that the set $D$ of best $K$-approximates to $y$ is nonempty compact and star-shaped with respect to $z \in F(S)$. Then $T$ and $S$ have a common fixed point which is also a best $K$-approximation to $y$.

**Remark 6.4.7.** Linearity of $S$ of Theorem 3 of Sahab et al (1988) is not necessary. The result is also true for an affine mapping $S$.

Mukherjee et al (1989) used the concepts of best co-approximation and strong best co-approximation in normed linear space, to obtain theorems (1.7.8 - 1.7.9) on the existence of fixed points for nonexpansive mappings $T$ which are also best co-approximation and strong best co-approximation to a given fixed point of $T$.

**Theorem 6.4.8.** Let $K$ be a nonempty subset of a normed linear space $X$ and $y \in X$. Suppose that $T$ is an asymptotically nonexpansive, uniformly asymptotically regular self-mapping of $K$ such that $T$ satisfies the following condition:

$$
\|Tx - z\| \leq \|x - z\| \text{ for all } x, z \in X.
$$

If the set $D$ of best $K$-co-approximates to $y$ is nonempty compact and star-shaped, then $T$ has a fixed point which is also a best $K$-co-approximation to $y$.

**Proof.** The proof of this theorem is similar to that of Corollary 2.2.11, by applying it to $T$ and $D$.

**Theorem 6.4.9.** Let $K, X, y$ and $T$ be as in Theorem 6.4.8. If the set $D$ of strong best $K$-co-approximates to $y$ is nonempty compact and star-shaped, then $T$ has a fixed point which is also a strong best $K$-co-approximation to $y$.

**Proof.** The proof of this theorem is similar to that of Corollary 2.2.11, by applying it to $T$ and $D$. 