CHAPTER 3

THE (2+1) - DIMENSIONAL DISPERSIVE LONG-WAVE EQUATIONS

3.1 INTRODUCTION

Solitons in one spatial and one temporal (1+1) dimension have been extensively studied in the last thirty years or so, and many interesting features of solitons have been discussed in the literature (Bhatnagar 1979, Ablowitz and Clarkson 1991).

The concept of soliton has also been generalised to higher-dimensional equations. The IST method has been extended to solve (2+1)-dimensional equations such as the Kadomtsev-Petviashvili (KP) equation (Manakov 1981, Ablowitz et al. 1983), Davey-Stewartson (DS) equations (Ganesan and Lakshmanan 1987, Fokas and Santini 1990), generalised K-dV equation (Boiti et al. 1986, Radha and Lakshmanan 1994), generalised Novikov-Veselov (NNV) equation (Novikov and Veselov 1986), breaking soliton equation (Konopelchenko 1993, Tian and Gao 1996), (2+1)-dimensional generalised NLS equation (Strachan 1992, 1993), (2+1)-dimensional generalised sine-Gordon equation (Konopelchenko and Rogers 1991, 1993; Nimmo 1992) and (2+1)-dimensional long dispersive wave (2LDW) equation (Chanda and Roy Chowdhury 1988, Lou 1993, Chakravarty et al. 1995). The KP equation possesses line-solitons, which do not decay as $\sqrt{x^2 + y^2} \to \infty$ in all
directions, and lump solitons which decay algebraically as \( x^2 + \frac{z}{t} \to \infty \) (Ablowitz and Clarkson 1991, Porsezian and Uthayakumar 1993). The DS equations possess several exact solutions with soliton-like behavior; in particular, dromions which decay exponentially in all directions as \( \sqrt{x^2 + y^2} \to \infty \) (Boiti et al. 1988, Fokas and Santini 1989).

In a different direction, exact higher-dimensional solutions analogous to soliton equations in (1+1)-dimensions have been derived. These are usually derived from the soliton solution of the (1+1)-dimensional NPDE and are often expressible in terms of special functions. It is remarked that such solutions of higher-dimensional equations (Roy Chowdhury and Banerjee 1987) are sometimes called solitons, yet, strictly they are not solitons unless the DE is completely integrable.

Recently, another class of solutions has been obtained for DS equations which form an intermediate state between the plane soliton and the dromion. Such solitons, termed "solitoffs" by the discoverer, decay exponentially in all directions except in a particular direction (Chow 1999).

### 3.2 PAINLEVÉ TEST FOR (2+1) - DIMENSIONAL DISPERSIVE LONG-WAVE EQUATIONS

The generalised (2+1)-dimensional dispersive long-wave equations take the form (Boiti et al. 1987; 1987a, Dubrovsky 1996),

\[
\begin{align*}
    u_t - u_{xx} - 2(\nu v)_x &= 0, \\
    \nu_{yy} + \nu_{xx} - 2\mu_{xx} - (\nu^2)_{xy} &= 0.
\end{align*}
\] (3.1)
Here, the P-analysis of Equation (3.1) is carried out by assuming the leading orders of the solutions of Equations (3.1) to have the form

\[ u = u_0 \phi^a, \quad v = v_0 \phi^b, \]  

(3.2)

where \( u_0, v_0 \) and \( \phi \) are analytic functions of \( x, y, t \) and \( \alpha \) and \( \beta \) are negative integers to be determined and then substituting Equations (3.2) into Equations (3.1) and balancing the nonlinear terms against the dominant linear terms, the following results are obtained:

\[ \alpha = -2, \quad \beta = -1, \]  

(3.3)

\[ u_0 = -\phi_x \phi_y, \quad v_0 = \phi_x. \]  

(3.4)

Considering the Laurent series expansion of the solutions in the neighborhood of the singular manifold as

\[ u = \sum_{j=0}^{\infty} u_j \phi^{j-2}, \quad v = \sum_{j=0}^{\infty} v_j \phi^{j-1}, \]  

(3.5)

the resonances can be evaluated by substituting Equations (3.5) into Equations (3.1) and comparing the coefficients of \( (\phi^{j-2}, \phi^{j-1}) \), the following matrix is obtained:

\[
\begin{bmatrix}
-j(j-3) & 2(j-3)\phi_x \\
-2(j-2)(j-3) & (j-2)(j-3)\phi_y
\end{bmatrix}
\begin{bmatrix}
u_j \\
v_j
\end{bmatrix} = 0.
\]  

(3.6)
Evaluating this equation, the resonances are calculated as

\[ j = -1, 2, 3, 4. \]

The resonance at \( j = -1 \) represents the arbitrariness of the singular manifold \( \phi(x, y, t) = 0 \).

As usual, in order to check the existence of arbitrary function at the other resonance values, the full Laurent expansions (3.5) are substitute into Equations (3.1).

From the coefficients of \( (\phi^4, \phi^4) \), the values of \( u_0 \) and \( v_0 \) are obtained explicitly as given in Equation (3.4). Collecting the coefficients of \( (\phi^3, \phi^3) \), the following matrix is obtained:

\[
\begin{bmatrix}
\phi_x & -2\phi_x\phi_y \\
2\phi_x & 2\phi_x\phi_y 
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
\phi_x\phi_y + \phi_{xx}\phi_y - \phi_y\phi_x \\
2\phi_x\phi_y - \phi_{xx}\phi_x + \phi_x\phi_y
\end{bmatrix}.
\]

(3.8)

Solving Equation (3.8), the results obtained are

\[ u_1 = \phi_y, \quad v_1 = \frac{1}{2\phi_x}(\phi_x - \phi_{xx}). \]

(3.9)

Collecting the coefficients of \( (\phi^2, \phi^2) \), the relation between \( u \) and \( v_2 \) is obtained as
From Equation (3.10), it is clear that either \( u_2 \) or \( v_2 \) is arbitrary which corresponds to the resonance at \( j = 2 \).

Proceeding further, from the coefficients of \((\phi^{-1}, \phi^{-1})\), the following equations are obtained:

\[
\phi_{\alpha x} - \phi_{\alpha y} - 2(u_x \phi_x) - 2(v_i \phi_x)^x + 2v_2 \phi_x \phi_y = 0, \tag{3.11}
\]

\[
\phi_{\alpha x} - \phi_{\alpha y} - 2(v_i \phi_x) - 2(v_x \phi_x) = 0. \tag{3.12}
\]

Equations (3.11) and (3.12) are identically satisfied from the previous results. Thus, there are two arbitrary values corresponding to the resonances at \( j = 3, 3' \). Next, collecting the coefficients of \((\phi^0, \phi^0)\), the following results are obtained:

\[
(2u_4 - v_i \phi_y) = \frac{1}{2\phi_x^2} \left\{ u_{2x} + u_x \phi_x - u_{2x} + 2u_3 \phi_x - 2(u_3 \phi_x) - 2(u \phi_x) - u \phi_x + \right. \\
- 2u_1 v_2 - 2v_2 \phi_x - 2u_2 v_3 \phi_x + 2v_3 \left( \phi_x \phi_y \right) \\
+ 2v_3 \phi_x \phi_y - 2v_3 \phi_x \phi_y \left\}, \tag{3.13a}
\]
Equations (3.13) implies that either $u_4$ or $v_4$ is arbitrary which corresponds to the resonance at $j = 4$. Thus, the general solution $(x, y, t)$ of Equations (3.1) admits the required number of arbitrary functions without the introduction of any movable critical manifold, thereby satisfying the P-test. Hence, the generalised set of integrable (2+1)-dimensional dispersive long-wave Equations (3.1) is expected to be integrable.

### 3.2.1 Bilinear Transformation

To establish the integrability properties, the Laurent series is truncated at the constant level term, i.e., $u_j = 0$ for $j \geq 3$ and $v_j = 0$ for $j \geq 2$ to give

$$u = u_0\phi^{-2} + u_1\phi^{-1} + u_2,$$

$$v = v_0\phi^{-1} + v_1,$$

where the pair of functions $(u, u_2)$ and $(v, v_1)$ satisfy Equations (3.1) and hence they may be treated as auto-BT of Equations (3.1). Using Equations (3.14) and (3.15), one can obtain the bilinear transformation by considering the values $u_2 = v_1 = 0$ in Equations (3.14) and (3.15) as
$u = u_0 \phi^{-2} + u_1 \phi^{-1} = \frac{\partial^2}{\partial x \partial y} (\ln \phi). \quad (3.16)$

$\nu = \nu_0 \phi^{-1} = \frac{\partial}{\partial x} (\ln \phi). \quad (3.17)$

Equations (3.16) and (3.17) can now be interpreted as bilinear transformations for Equations (3.1). With the help of the above bilinear transformations and the usual power series expansions the soliton solutions for Equations (3.1) can be constructed. Dubrobsky constructed the line-solitons and rational solutions of Equations (3.1) via $\bar{g}$-dressing method.

The generalised (2+1)-dimensional dispersive long-wave Equation (3.1) was first studied by Boiti et al. (1987) admits the P-test. But we are not succeeded in deriving bilinear form for the generation of soliton solutions. The reason is that the Equation (3.1) simply reduced to ordinary heat equation through truncation of Laurent series.

A similar type of (2+1) - dimensional dispersive long-wave equations has been studied by Lou (1993), and Shown that the equation fails to pass the P-test.