CHAPTER 6

INTEGRABILITY ASPECTS OF SOME FIELD THEORETICAL EQUATIONS

6.1 INTRODUCTION

In recent years, there has been a renewal of interest in the studies of the nonlinear dynamics of the relativistic equations (nonlinear \( \sigma \) models), Ernst equation, chiral field equations (Ward 1984, Euler et al. 1990) and so on. Though many nonlinear equations like sine-Gordon equation, Pohlmeyer-Lund-Regge (PLR) equation, Ernst equation and so on have been investigated in this area, P-analysis, to our knowledge, has been reported only for the sine-Gordon and Ernst equations. This is mainly due to the complicated nonlinear structure of the governing equations. The main aim of the work reported in this chapter is to show that one such equation, namely the PLR equation, admits the P-property and the integrability properties of this equation are constructed in a systematic manner. Here, the P-analysis of the PLR equations, its bilinear equivalent equation and zoomeron equation are discussed.

Lund and Regge (1976) have shown that the dynamics of relativistic vortices (equivalently, strings) interacting through a scalar field has led to a set of two coupled, Lorentz-invariant, nonlinear equations in two independent variables. These equations are in the form
\[
\frac{\partial}{\partial X} \left[ \cot^2 \Phi \frac{\partial \Psi}{\partial X} \right] = \frac{\partial}{\partial T} \left[ \cot^2 \Phi \frac{\partial \Psi}{\partial T} \right],
\]
(6.1)

\[
\frac{\partial^2 \Phi}{\partial T^2} - \frac{\partial^2 \Phi}{\partial X^2} + c^2 \sin \Phi \cos \Phi + \frac{\cos \Phi}{\sin \Phi} \left[ \left( \frac{\partial \Psi}{\partial T} \right)^2 - \left( \frac{\partial \Psi}{\partial X} \right)^2 \right] = 0.
\]
(6.2)

These equations have a solitary wave solution and in the limit \( \Psi = 0 \), with suitable changes, reduce to the well known sine-Gordon equation

\[
\frac{\partial^2 \Phi}{\partial T^2} - \frac{\partial^2 \Phi}{\partial X^2} + \sin \Phi = 0.
\]
(6.3)

Equations (6.1) and (6.2) are the conditions for embedding a 2-dimensional surface in a three-dimensional sphere which itself is embedded in a four-dimensional Euclidean space. Equations (6.1) and (6.2) have also been found by Pohlmeyer (1977) through a study of the nonlinear \( \sigma \)-models of field theory.

Equations (6.1) and (6.2) can be derived from the Lagrangian

\[
L = \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial X} \right)^2 - \left( \frac{\partial \Phi}{\partial T} \right)^2 \right] + \frac{c^2}{2} \sin^2 \Phi + \frac{1}{2} \cot^2 \Phi \left[ \left( \frac{\partial \Psi}{\partial X} \right)^2 - \left( \frac{\partial \Psi}{\partial T} \right)^2 \right]
\]
(6.4)

which is a Lorentz scalar in \( X-T \) space time.

For further analysis, Equations (6.1) and (6.2) are transformed in terms of \( x \) and \( t \) co-ordinates in the form
\[
\Phi_x - \sin \Phi \cos \Phi - \frac{\sin \Phi}{\cos \Phi} \Psi_x \Psi_y = 0, \quad (6.5)
\]
\[
(\Psi_x \tan^2 \Phi)_x + (\Psi_y \tan^2 \Phi)_y = 0. \quad (6.6)
\]

For convenience, the following transformation (Hirota 1982) is used:

\[
q = \sin \Phi \exp(i\Psi). \quad (6.7)
\]

Under this transformation, Equations (6.5) and (6.6) are transformed into the Getmanov equation in the form

\[
q_x + \frac{q \cdot q^*}{1-|q|^2} - q (1-|q|^2) = 0. \quad (6.8)
\]

### 6.2 PAINLEVÉ ANALYSIS OF PLR AND ITS BILINEAR EQUIVALENT EQUATIONS

To apply P-analysis, the new variables \( a (= q) \) and \( b (= q^*) \) are defined. Equation (6.8), in terms of these change of variables, takes the form

\[
a_x + \frac{a \cdot a \cdot b}{1-ab} - a(1-ab) = 0. \quad (6.9)
\]
\[
b_x + \frac{b \cdot b \cdot a}{1-ab} - b(1-ab) = 0. \quad (6.10)
\]

In order to investigate the nature of the movable singularity admitted by Equations (6.9) and (6.10), the solutions for \( a \) and \( b \) may be represented by
around a non-characteristic movable singular manifold given by the analytic function $\Phi(x, t)$. For simplicity, Kruskal’s ansatz i.e., the reduced manifold concept (Jimbo et al. 1982) is used.

To find the leading orders of the solutions,

$$a = a_0 \phi, \quad b = b_0 \phi^\beta.$$  

(6.12)

are substituted into Equations (6.9) and (6.10) and balancing the most dominant terms, $\alpha = \beta = -1$ is obtained with

$$a_0 b_0 = -\phi_r.$$  

(6.13)

For finding the resonances, the full Laurent series is substituted into Equations (6.9) and (6.10) and equating the coefficients of $\phi^{j-5}$, the resonance values are obtained in the form

$$j = -1, 0, 1, 2.$$  

(6.14)

The resonance at $j = '1' \text{'}$ represents the arbitrariness of the singular manifold $\Phi(x, t) = 0$. Collecting the coefficients of $(\phi^{-3}, \phi^{-5})$, it can be clearly seen that either $a_0$ or $b_0$ is arbitrary which corresponds to the resonance at $j = '0' \text{'}$ as given by Equation (6.13). In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion is substituted into Equations (6.9) and (6.10). Collecting the coefficients of $(\phi^{-1}, \phi^{-3})$, the following result is obtained:
From Equation (6.15), it is clear that either $a_1$ or $b_1$ is arbitrary which corresponds to the resonance at $j = '1'$. In a similar way, proceeding further by collecting the coefficients of $(\phi^{-3}, \phi^{-3})$, the following results are obtained:

\[
(a_0 b_2 - b_0 a_2)\phi_i = a_{01} b_i + b_{01} a_i,
\]

\[
(b_0 a_2 - a_0 b_2)\phi_i = b_{01} a_i + a_{01} b_i.
\]

Equations (6.16), suggest that either $a_2$ or $b_2$ is arbitrary which corresponds to the resonance at $j = '2'$. Thus, the solutions $a(x, t)$ and $b(x, t)$ of Equations (6.9) and (6.10) admit sufficient number of arbitrary functions without the introduction of any movable critical manifold, thus satisfying the P-test and hence the system is expected to be integrable. Now, to establish the integrability properties, the series representation (6.11) is truncated at the constant level term ($a_j = b_j = 0$, for $j > 1$) as

\[
a = \frac{a_0}{\phi} + a_1,
\]

\[
b = \frac{b_0}{\phi} + b_1.
\]

where $a_0$ and $b_0$ satisfy Equation (6.9) and (6.10). Equation (6.17) can also be treated as auto-BT of Equation (6.8). Then, to find the bilinear form, it is assumed that $a_1 = b_1 = 0$ and then, by defining
after simplifications, the following results are obtained:

\[ D_x D_t F - F = 2GG', \]
\[ F\left[(D_x D_t - 1)G - F]\right] = \frac{1}{2} G'DxD_tD_tG. \quad (6.19) \]

The above trilinear form was first obtained by Getmanov (1977) and he obtained the two-soliton solutions for the Equations (6.9) and (6.10) and also conjectured \( N \)-soliton solutions. For completeness, the two-soliton solution is presented along the steps of Getmanov as

\[ G = \exp(z_1) + \exp(z_2) + a_{12}^* \exp(z_1 + z_2 + z_1^*) + a_{12}^* \exp(z_1 + z_2 + z_2^*) \quad (6.20) \]
\[ F = 1 + a_{11}^* \exp(z_1 + z_1^*) + a_{12}^* \exp(z_1 + z_2 + z_1^*) + a_{22}^* \exp(z_2 + z_1^*) + a_{22}^* \exp(z_2 + z_2 + z_1^*) + a_{22}^* \exp(z_2 + z_2 + z_2^*) \quad (6.21) \]

where

\[ z_i = k(i,\mu)\left(x - x_i^{(0)}\right)^\mu \]
\[ a_\mu = \left[(k(i) + k(\mu,\rho))\mu\right]; \quad a_\mu = -\left(k(i) - k(\rho,\mu)\right)^\mu \]
\[ a_{ij}^* = \left[\begin{array}{c} a_i \end{array}\right]^*; \quad a_{ij}^* = \left[\begin{array}{c} a_j \end{array}\right]^*; \quad a_{ij}^* = a_{ij} a_i a_j \]
\[ a_{ij} = a_{ij}, a_{ij} = a_{ij}, a_{ij} = a_{ij} \]
\[ k_0 = \text{msh}(\beta \pm i\alpha); \quad k_1 = \text{mch}(\beta \pm i\alpha) \]
\[ \mu = 0.1; \quad x_i = x; \quad x_0 = t \]
\[ k(i,\mu) \neq k(\rho,\mu) \text{ if } i \neq j \]
6.2.1 Bilinear Form

To construct the bilinear form from Equation (6.19), the variable $F$ is defined as

$$F^2 = ff + gg^*.$$  \hfill (6.22)

Under this transformation, Equations (6.19) are transformed into the bilinear equations

$$D_t(f^* \cdot f + g^* \cdot g) = 0,$$

$$D_xD_t(f^* \cdot g^* = 0, \hfill (6.23)$$

$$D_xD_t(f^* \cdot f - g^* \cdot g) + 2g^*g = 0.$$

Once the bilinear forms are known, the soliton solutions can be generated by expanding the dependent variables in terms of power series.

Hirota (1982) has shown that PLR equation shares the same bilinear form with the equation

$$Q_n = Q\sqrt{1 - |Q|^2}, \hfill (6.24)$$

like the NLS equation and the Heisenberg ferromagnetic system which are transformed into the same bilinear equations. For this, by defining a new field (variable)

$$Q = \frac{2D_xg \cdot f}{f^*f + g^*g} \hfill (6.25)$$
it has been shown that the PLR equation and Equation (6.24) share the same bilinear form. To establish the integrability of Equation (6.24), this equation can be rewritten in the form

\[ Q_u = Q^2(1 - |Q|^2). \]  

(6.26)

In terms of \( a \) and \( b \), Equation (6.26) can be modified as

\[
\begin{align*}
    a_u^2 &= a^2(1 - ab), \\
    b_u^2 &= b^2(1 - ab).
\end{align*}
\]

(6.27)

From the leading order analysis, it is found that \( \alpha = \beta = -2 \) and \( a_0 b_0 = -36 \phi^2 \). Substituting the full Laurent series, the resonances are found to be \( j = -1, 0, 5, 6 \).

In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansions

\[
\begin{align*}
    a &= \sum_{j=0}^\infty a_j \phi^{j+2}; \\
    b &= \sum_{j=0}^\infty b_j \phi^{j+2},
\end{align*}
\]

(6.28)

are substituted into Equation (6.27), and the existence of arbitrary functions at these resonance values is analysed by collecting the coefficients of \((\phi^7, \phi^7), (\phi^6, \phi^6), (\phi^5, \phi^5), \) and \((\phi^4, \phi^4)\). The results are shown below:
(\phi^{-7}, \phi^{-7}) : 
\begin{align*}
(a_0^2 b_1 - 84a_5 \phi_i^2) &= 24a_0 \phi_i, \\
(b_0^2 a_1 - 84b_5 \phi_i^2) &= 24b_0 \phi_i .
\end{align*}

(6.29)

(\phi^{-6}, \phi^{-6}) : 
\begin{align*}
(a_0^2 b_2 - 108b_0 a_2 \phi_i) &= 104a_7^2 \phi_i^2 - 4a_6^2 + 8a_0 a_i \phi_i + 12a_0 a_i \phi_i - 3a_0^2 a_1 b_i , \\
(b_0^2 a_2 - 108b_0 b_2 \phi_i) &= 104b_7^2 \phi_i^2 - 4b_6^2 + 8b_0 b_i \phi_i + 12b_0 b_i \phi_i - 3b_0^2 a_1 b_i .
\end{align*}

(6.30)

(\phi^{-5}, \phi^{-5}) : 
\begin{align*}
(a_0^3 b_3 - 108a_0 a_3 \phi_i^2) &= 4a_1 a_i \phi_i - 4a_0 a_i \phi_i - a_1 b_0 + 216a_1 a_2 \phi_i^2 - 3a_0^2 a_1 b_i - a_1^2 b_1 , \\
(b_0^3 a_3 - 108b_0 b_3 \phi_i^2) &= 4b_1 b_i \phi_i - 4b_0 b_i \phi_i - b_1 a_0 + 216b_1 b_2 \phi_i^2 - 3b_0^2 b_1 a_1 - a_1^2 b_1 .
\end{align*}

(6.31)

(\phi^{-4}, \phi^{-4}) : 
\begin{align*}
(a_0^4 b_4 - 84a_0 a_4 \phi_i^2) &= a_0^2 - 12a_0 a_3 \phi_i - a_1 b_0 + 216a_4 a_2 \phi_i + 108a_2^2 \phi_i^2 - 3a_0^2 a_1 b_i - a_1^2 b_i - 6a_0 b_i a_2 - 3a_0^2 a_2 b_i , \\
(b_0^4 a_4 - 84b_0 b_4 \phi_i^2) &= b_0^2 - 12b_0 b_3 \phi_i - b_1 a_0 + 216b_1 b_2 \phi_i + 108b_2^2 \phi_i^2 - 3b_0^2 b_1 a_1 - b_1^2 a_1 - 6b_0 b_i a_2 - 3b_0^2 b_2 a_2 .
\end{align*}

(6.32)

After careful analysis, from the remaining coefficients of (\phi^3, \phi^3) and (\phi^2, \phi^2), it is found that either \(a_5\) or \(b_5\) and \(a_6\) or \(b_6\) are arbitrary, respectively. From the arbitrary analysis, it is found that Equation (6.24) admits sufficient number of arbitrary functions, thus satisfying the P-test.
and hence it is expected to be integrable. As PLR equation and Equation (6.24) share the same bilinear form, from the knowledge of Equation (6.23), one can generate the soliton solutions of Equation (6.24).

6.3 PAINLEVÉ TEST FOR ZOOMERON EQUATION

In this section, the P-analysis of the zoomeron equation is discussed. The zoomeron equation is of the form:

\[
(\partial_x^3 - \partial_t^3)\frac{q_u}{q} + 2(q_x^2)_{,u} = 0. \tag{6.33}
\]

This complicated nonlinear equation is also found to display the phenomenology associated with boomerons (Calogero and Degasperis 1976a); moreover, its soliton solutions ("zoomerons") have an amplitude that changes with time along with their speed i.e., explode-decay-type solitons. For convenience, Equation (6.33) is rewritten in the following form:

\[
q^2 q_{,uu} - q^2 q_{,xxx} - q q_{,u} q_{,u} + q q_{,u} q_{,u} - 2 q q_{,u} q_{,u} + 2 q q_{,u} q_{,u} \\
+ 2 q u q_{,uu}^2 - 2 q q_{,u} q_{,u} + 4 q^4 q_{,u} q_{,u} - 4 q^4 q_{,u} = 0. \tag{6.34}
\]

To analyse the leading order behaviour, \( q = q_0 \phi^\alpha \) is substituted in Equation (6.34) and the following results are obtained:

\[
\alpha = -1, \\
q_0^2 = (1 - \phi_1^3). \tag{6.35}
\]
With the help of Laurent series, the resonances are found to be

\[ j = -1, 2, 3, 4. \]  \hspace{1cm} (6.36)

The resonance value at \( j = -1 \) corresponds to the arbitrariness of the singular manifold \( \phi(x,t) = 0 \). Collecting the coefficients of \( \phi^{-7} \), the explicit value of \( q_0 \) is obtained as given in Equation (6.35). To check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion is substituted into Equation (6.34). Then, collecting the coefficient of \( \phi^6 \), the explicit value of \( q_1 \) is obtained as

\[ q_1 = \frac{1}{10\phi_t} \left( q_{00} \phi_t^3 + 2q_{0t} \phi_t + 2q_{0\phi_t} \phi'_n \right). \]  \hspace{1cm} (6.37)

In a similar manner, proceeding further by collecting the coefficients of \( \phi^{-5} \), \( \phi^{-4} \) and \( \phi^{-3} \), it is clear that \( q_2 \), \( q_3 \) and \( q_4 \) are arbitrary, respectively. Thus, Equation (6.33) also admits the P-test and hence it is expected to be integrable.