6.1 INTRODUCTION

Chang, Hsu and Rogers (1981) introduced elegant labeling as a variation of harmonious labeling. The significant remark is that in contrast to the definition of harmonious labeling, it is not necessary to make an exception for trees. There are class of graphs which are harmonious but not elegant. Of course, other way is also true. It is interesting to note that there are families of graphs which possess more than one graph labeling, like they are graceful and harmonious or harmonious and elegant etc. Here we consider such families of graphs. For an exhaustive survey about results on elegant labeling refer Gallian (2003).

Recall that $P_n^k$, the $k$th power of $P_n$, is the graph obtained from $P_n$ by adding edges between all vertices $u$ and $v$ of $P_n$ with $d(u,v) = k$. Grace (1983) has shown that the graph $P_n^2$ is harmonious, Kang, et al. (1996) have shown that $P_n^2$ is graceful. Chang et al. (1981) have shown that the graph $S_m + K_1$ is harmonious, where $S_m$ is a star graph on $m$ vertices. Graham and Sloane (1980) have proved that the graphs $P_n + K_1$ and $P_n + K_2$ are graceful and harmonious. Koh and Punnim (1983) have proved the cycles with 3-consecutive chords is graceful.

Here, in this chapter, in section 6.2 we prove that the graph $P_n^2$ is elegant, for all $n \geq 1$. In Section 6.3, we prove that the graphs $P_m^2 + \overline{K}_n, S_m + S_n$ and $S_m + \overline{K}_n$ are elegant, for all $m, n \geq 1$. Finally, in Section 6.4, we show that every even cycle $C_{2n} : a_0a_1, a_2 \cdots a_{2n-2}a_0$ with $2n - 3$ chords $a_0a_2, a_0a_3, \cdots, a_0a_{2n-2}$ is elegant, for all $n \geq 2$. 
6.2 THE GRAPH $P_n^2$ IS ELEGANT FOR ALL $n \geq 1$

In this section we prove that the graph $P_n^2$ is elegant, for all $n \geq 1$

**Theorem 6.2.1.** The graph $P_n^2$ is elegant, for all $n \geq 1$.

**Proof.** Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices of the graph $G = P_n^2$. By the definition of $G$, $v_i$ and $v_{i+1}$ are adjacent for $0 \leq i \leq n - 2$ and $v_i$ and $v_{i+2}$ are adjacent for $0 \leq i \leq n - 3$. Note that $G$ has $n$ vertices and $2n - 3$ edges. Let $M = |E(G)| = 2n - 3$.

Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, M\}$ by

$$
f(v_0) = 0 \quad (6.1)
$$

$$
f(v_i) = 2(n - 1) - i, \text{ for } 1 \leq i \leq n - 1
$$

It is clear from the above labeling that the labels of the vertices of $G$ are all distinct. From the above definition it follows that the edges $v_i v_{i+1}$ get the labels $(4n - 2i - 5)(\mod M + 1) = M - 2i$, for $0 \leq i \leq n - 2$, and the edges $v_i v_{i+2}$ get the labels $(4n - 2i - 6)(\mod M + 1) = M - 2i - 1$, for $0 \leq i \leq n - 3$. Observe that the edge values are distinct and range from 1 to $M$. Hence the graph $G$ is elegant. □

Illustrative example of the labeling provided in the Proof of Theorem 6.2.1, is given in the Figure 6.1.

![Figure 6.1. Elegant labeled $P_{12}^2$.](image-url)
6.3 THE GRAPHS $P_m^+K_n, S_m + S_n$ AND $S_m + K_n$ ARE ELEGANT, FOR ALL $m, n \geq 1$.

In this section we prove that the graphs $P_m^+K_n, S_m + S_n$ and $S_m + K_n$ are elegant, for all $m, n \geq 1$.

**Theorem 6.3.1.** The graph $P_m^+K_n$ is elegant, for all $m, n \geq 1$.

**Proof.** Let $G = P_m^+K_n$, for all $m, n \geq 1$. Let $u_1, u_2, \ldots, u_m$ be the vertices of $P_m$ and let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$. Note that $G$ has $m + n$ vertices and $m(n + 2) - 3$ edges. Let $M = |E(G)| = m(n + 2) - 3$.

Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, M\}$ by

$$f(u_1) = 0$$
$$f(u_i) = m(n + 2) - (i + 1), \text{ for } 2 < i \leq m$$
$$f(v_j) = jm, \text{ for } 1 \leq j \leq n$$

From the above labeling it is clear that the labels of the vertices of $G$ are all distinct. It follows that the edges $u_iu_{i+1}$ get the labels $[2m(n + 2) - (2i + 3)](mod M + 1) = M - 2(i - 1)$, for $1 \leq i \leq m - 1$, the edges $u_iu_{i+2}$ get the labels $[2m(n + 2) - (2i + 4)](mod M + 1) = M - 2i + 1$, for $1 \leq i \leq m - 2$, the edges $u_iu_{i+2}$ get the labels $jm$, for $1 \leq j \leq n$ and the edges $u_iv_j$ get the labels $[m(n + j + 2) - (i + 1)](mod M + 1)$, for $2 \leq i \leq m$ and $1 \leq j \leq n$. Observe that the edge values are distinct and range from 1 to $M$.

Hence the graph $G$ is elegant. \square
Illustrative example of the labeling provided in the Proof of Theorem 6.3.1, is given in the Figure 6.2.

Figure 6.2. Elegant labeled $P_5 + \overline{K}_5$. 
Theorem 6.3.2. The graph \( G = S_m + S_n \) is elegant, for all \( m, n \geq 1 \).

Proof. Let \( G = S_m + S_n \), for all \( m, n \geq 1 \). Let \( u_1, u_2, \ldots, u_m \) be the vertices of \( S_m \) and let \( v_1, v_2, \ldots, v_n \) be the vertices of \( S_n \) of \( G \). Note that \( G \) has \( m + n \) vertices and \( m(n + 1) + n - 2 \) edges. Let \( M = |E(G)| = m(n + 1) + n - 2 \).

Define \( f : V(G) \to \{0, 1, 2, \ldots, M\} \) by

\[
\begin{align*}
  f(u_1) &= 0 \\
  f(u_i) &= M - (i - 2), \text{ for } 2 \leq i \leq m \\
  f(v_1) &= n(m + 1) - 1 \\
  f(v_j) &= (j - 1)(m + 1), \text{ for } 2 \leq j \leq n
\end{align*}
\]  

It is clear from the above labeling that the labels of the vertices of \( G \) are distinct. From the above definition it follows that the edges \( u_1v_i \) get the labels \( M - (i - 2) \), for \( 2 \leq i \leq m \), the edge \( u_1v_1 \) get the label \( n(m + 1) - 1 \), the edges \( u_1v_j \) get the labels \( (j - 1)(m + 1) \), for \( 2 \leq j \leq n \), the edges \( u_iv_1 \) get the labels \( (2n(m + 1) + m - (i + 1))(M + 1) \), for \( 2 \leq i \leq m \), the edges \( u_iv_j \) get the labels \( (m + 1)(n + j) - (i + 1)(M + 1) \), for \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \), and the edges \( v_1v_j \) get the label \( [m(n + j - 1) + n + j - 2](M + 1) \), for \( 2 \leq j \leq n \).

Observe that the edge values are distinct and range from \( 1 \) to \( M \).

Hence the graph \( G \) is elegant. \( \square \)
Illustrative example of the labeling provided in the Proof of Theorem 6.3.2, is given in the Figure 6.3.

Figure 6.3. Elegant labeled $S_5 + S_4$.
Theorem 6.3.3. The graph $S_m + \overline{K}_n$ is elegant, for all $m, n \geq 1$.

Proof. Let $G = S_m + \overline{K}_n$, for $m, n \geq 1$. Let $v_1, v_2, \cdots, v_m$ be the vertices of $S_m$ of $G$ and let $v_1, v_2, \cdots, v_n$ be the vertices of $\overline{K}_n$ of $G$. Note that $G$ has $m + n$ vertices and $m(n + 1) - 1$ edges. Let $M = |E(G)| = m(n + 1) - 1$.

Define $f : V(G) \to \{0, 1, 2, \cdots, M\}$ by

\[
\begin{align*}
  f(u_1) &= 0, \\
  f(u_i) &= M - (i - 2), \text{ for } 2 \leq i \leq m, \\
  f(v_j) &= jm, \text{ for } 1 \leq j \leq n.
\end{align*}
\]  

(6.4)

It is clear from the above the labeling that the labels of the vertices of $G$ are all distinct. From the above definition it follows that the edges $u_1u_i$ get the labels $M - (i - 2)$, for $2 \leq i \leq m$, the edges $u_1v_j$ get the labels $jm$, for $1 \leq j \leq n$, and the edges $u_iv_j$ get the labels $[m(n + j + 1) - (i - 1)](\text{mod } M + 1)$, for $2 \leq i \leq m$ and $1 \leq j \leq n$. Observe that the edge values are distinct and range from 1 to $M$. Hence the graph $G$ is elegant. \(\square\)
Illustrative example of the labeling provided in the Proof of Theorem 6.3.3, is given in the Figure 6.4.

Figure 6.4. Elegant labeled $S_{10} + \overline{K}_4$. 
6.4 EVERY EVEN CYCLE $C_{2n}$ WITH $2n - 3$ CHORDS AND THE GRAPH $C_3 \times P_m$ ARE ELEGANT FOR ALL $m \geq 1$ AND $n \geq 2$

Here we prove that every even cycle $C_{2n} : d_0a_1 \cdots a_{2n-1}d_0$ with $2n - 3$ chords $a_0a_2, a_0a_3, \cdots, a_0a_{2n-2}$ is elegant, for all $n \geq 2$ and also we prove that graph $C_3 \times P_m$ is elegant, for all $m \geq 1$.

**Theorem 6.4.1.** Every even cycle $C_{2n} : d_0a_1 \cdots a_{2n-1}d_0$ with $2n - 3$ chords $a_0a_2, a_0a_3, \cdots, a_0a_{2n-2}$ is elegant, for all $n \geq 2$.

**Proof.** Let $G$ be an even cycle $C_{2n} : d_0a_1 \cdots a_{2n-1}d_0$ with $2n - 3$ chords $a_0a_2, a_0a_3, \cdots, a_0a_{2n-2}$, for $n \geq 2$. Note that $G$ has $2n$ vertices and $4n - 3$ edges. Let $M = |E(G)| = 4n - 3$.

Define $f : V(G) \to \{0, 1, 2, \cdots, M\}$ by

$$f(a_0) = 0$$

$$f(a_i) = 2i - 1, \text{ for } 1 \leq i \leq 2n - 1$$

It is clear from the above labeling that the labels of the vertices of $G$ are all distinct. From the above definition it follows that the edges $a_0a_i$ get the labels $2i - 1$, for $1 \leq i \leq 2n - 1$, the edges $a_ia_{i+1}$ get the labels $4i$ for $1 \leq i \leq n - 1$ and $4i \pmod{M + 1}$, for $n \leq i \leq 2n - 2$. Observe that the edge values are distinct and range from 1 to $M$. Hence the graph $G$ is elegant. \hfill \Box
Theorem 6.4.2. The graph $C_3 \times P_m$ is elegant, for all $m \geq 1$.

Proof. Let $G = C_3 \times P_m$ for all $m \geq 1$ and let $v_{11}, v_{12}, v_{13}, \ldots, v_{1m}$, $v_{21}, v_{22}, \ldots, v_{2m}, v_{31}, v_{32}, \ldots, v_{3m}$ be the vertices of the graph $G$.

Let $M = |E(G)| = 3(2m - 1)$.

Define $f : V(G) \to \{0, 1, 2, \ldots, M\}$ by

$$f(v_{ki}) = 3(2m - i) + \delta_{ki},$$

where

$$\delta_{ki} = \frac{1 + (-1)^i}{2},$$

if $k = 1$ and $1 \leq i \leq m$

$$\delta_{ki} = (-1)^i,$$

if $k = 2$ and $1 \leq i \leq m$ (6.6)

$$\delta_{ki} = \frac{1 - (-1)^i}{2},$$

if $k = 3$ and $2 \leq i \leq m$

and $f(v_{31}) = 0$.

It is clear from the above labelings that the labels of the vertices of $G$ are all distinct. From the above definition it follows that the edges $v_1v_2$ get the labels $[6(2m - i) - \alpha](\text{mod } (M + 1))$, for $1 \leq i \leq m$, where $\alpha = 1$ or $0$ depends on $i$ is odd or even, the edge $v_{21}v_{31}$ get the label $6m - 4$, the edges $v_{3i}v_{3(i+1)}$ get the labels $[6(2m - i) + \alpha](\text{mod } (M + 1))$, for $2 \leq i \leq m$, where $\alpha = 0$ or $1$ depends on $i$ is odd or even, the edge $v_{31}v_{11}$ get the label $6m - 3$, the edges $v_{2i}v_{3i}$ get the labels $[6(2m - i) - \alpha](\text{mod } (M + 1))$, for $2 \leq i \leq m$, where $\alpha = -1$ or $1$ depends on $i$ is odd or even, the edges $v_{1i}v_{1(i+1)}$ get the labels $6(m - i) - 2$, for $1 \leq i \leq m - 1$, the edges $v_{2i}v_{2(i+1)}$ get the labels $6(m - i) - 1$, for $1 \leq i \leq m - 1$, the edge $v_{3i}v_{3(i+1)}$ get the labels $6(m - i)$, for $1 \leq i \leq m - 1$. Observe that the edge values are distinct and range from $1$ to $M$. Hence the graph $G$ is elegant. □
Illustrative example of the labeling provided in the Proof of Theorem 6.4.2, is given in the Figure 6.6.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6_6}
\caption{Elegant labeled $C_3 \times P_6$.}
\end{figure}
6.5 CONCLUSION

Here in this chapter, we have proved that a few new families of graphs are elegant, more precisely the following results are proved.

1. The graph $P_n^2$ is elegant, for all $n \geq 1$.

2. The graphs $P_m^2 + \overline{K}_n, S_m + S_n$ and $S_m + \overline{K}_n$ are elegant, for all $m, n \geq 1$.

3. Every even cycle $C_{2n}$ with $2n - 3$ chords and the graph $C_3 \times P_m$ are elegant, for all $m \geq 1$ and $n \geq 2$. 