CHAPTER 5

PACKING OF ANY SET OF GRAPHS INTO A GRACEFUL, HARMONIOUS AND ELEGANT GRAPH

5.1 INTRODUCTION

Balakrishnan et al. (1996) have shown that every graph is a subgraph of a graceful graph and an elegant graph and Liu and Zhang (1993) have shown that every graph is a subgraph of a harmonious graph. In this chapter we prove a generalization of these two results that any given set of graphs \( G_1, G_2, \ldots, G_t \) can be packed into a graceful, harmonious and elegant graph. Also, we show that every graph is a subgraph of a cordial graph.

Recall that a sequence of graphs \( G_1, G_2, \ldots, G_t \) is said to be packed into a graph \( G \), if \( G \) has edge disjoint subgraphs \( F_1, F_2, \ldots, F_t \) such that \( F_j \cong G_j \), for \( j = 1, 2, \ldots, t \).

For \( m \geq 1 \), \( mK_1 \) denotes \( m \) disjoint copies of \( K_1 \). For a given set of graphs \( G_1, G_2, \ldots, G_t, \ t \geq 1 \), we denote \( \tilde{G} = \bigcup_{i=1}^{k} G_i \) and \( H_m = \tilde{G} \cup mK_1 \).
Figure 5.1a. Graphs $C_4$, $P_3$, $K_2$.

Figure 5.1b. Graph $\tilde{G} = C_4 \cup P_3 \cup K_2$.

Figure 5.1c. Graph $H_m = \tilde{G} \cup mK_1$.

Figure 5.1d. Graph $G = H_m + K_1$.
In Section 5.2 we prove that for any set of graphs \( G_1, G_2, \ldots, G_t, \quad t \geq 1 \), the graph \( G = H_m + K_1 \) is

(i) graceful, for \( m \geq 2^{\lfloor V(G) \rfloor} - \lfloor V(G) \rfloor + |E(G)| + 1 \);

(ii) harmonious and elegant, for \( m = 2^{\lfloor V(G) \rfloor} - \lfloor V(G) \rfloor + |E(G)| - 1 \).

Also, in Section 5.3 we prove that every graph is a subgraph of a cordial graph.

5.2 THE GRAPH \( H_m + K_1 \) IS GRACEFUL, HARMONIOUS AND ELEGANT

In this section we prove that any given set of graphs can be packed into a graceful, harmonious and elegant graph.

**Theorem 5.2.1.** For any set of graphs \( G_1, G_2, \ldots, G_t, \quad t \geq 1 \), the graph \( G = H_m + K_1 \) is graceful, for \( m \geq 2^{\lfloor V(G) \rfloor} - \lfloor V(G) \rfloor + |E(G)| + 1 \).

**Proof:** For any set of graphs \( G_1, G_2, \ldots, G_t, \quad t \geq 1 \), consider the graph \( G = H_m + K_1 \) with \( m \geq 2^{\lfloor V(G) \rfloor} - \lfloor V(G) \rfloor + |E(G)| + 1 \). Then \( G \) has \( |V(G)| + m + 1 \) vertices and \( M = |E(G)| + |V(G)| + m \) edges. Let \( N = |V(G)| \). Then, for convenience we label the vertices of \( G \) as \( v_0, v_1, v_2, v_3, \ldots, v_N, u_1, u_2, \ldots, u_m \) such that where \( v_0 \) is the unique vertex of \( G \) with maximum degree \( N + m \) and \( v_1, v_2, \ldots, v_N \) are the vertices of \( \tilde{G} \) and \( u_1, u_2, \ldots, u_m \) are the remaining \( m \) pendant vertices adjacent to \( v_0 \) of \( G \).
Define \( \phi(v_0) = 0 \) \hspace{1cm} (5.1) \\
\( \phi(v_i) = 2^i - 1, \ 1 \leq i \leq N \)

For each \( i, \ 1 \leq i \leq m \), define \( \phi(u_i) \) to be any one of the (unassigned) distinct labels from the set
\[ \{1, 2, \ldots, M\} \setminus \left\{ 2^i - 1 \mid 1 \leq i \leq N \right\} \cup \left\{ 2^i(2^{j-i} - 1) \mid 1 \leq j \leq N \text{ and } u_iu_j \in E(\tilde{G}) \right\} \]

Note that any bijective mapping from the set \( \{u_1, u_2, \ldots, u_m\} \) onto the set
\[ \{1, 2, \ldots, M\} \setminus \left\{ 2^i - 1 \mid 1 \leq i \leq N \right\} \cup \left\{ 2^i(2^{j-i} - 1) \mid 1 \leq j \leq N \text{ and } u_iu_j \in E(\tilde{G}) \right\} \]
can be chosen for defining the function \( \phi \) on the vertices \( u_i \)'s. Thus \( \phi \) is injective and the edge values of \( G \) are distinct and range from 1 to \( M \).

Hence \( G \) is Graceful. \( \square \)

From the definition of \( H_m \), the following corollary is an immediate consequence of Theorem 5.2.1.

**Corollary 5.2.2.** For any set of graphs \( G_1, G_2, \ldots, G_t \), \( t \geq 1 \), there exists a graceful graph \( G \) such that \( G_1, G_2, \ldots, G_t \) can be packed into \( G \).

**Remark 5.2.3.** The corollary 5.2.2 is a generalization of the result of Balakrishnan et al. (1996) that every graph is a subgraph of a graceful graph.
Theorem 5.2.4. For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geq 1$, the graph $G = H_m + K_1$ is harmonious, for $m = 2^{|V(G)|} - (|V(G)| + |E(G)|) - 1$.

Proof. Let $v_0, v_1, v_2, \ldots, v_N$, $u_1, u_2, \ldots, u_m$ and $M$ be as in the Proof of Theorem 5.2.1.

Define

$$\phi(v_0) = 0$$

$$\phi(v_i) = 2^i - 1, \ 1 \leq i \leq N - 1$$

(5.2)

$$\phi(v_N) = 2^N - 2$$

For each $i, 1 \leq i \leq m$, define $\phi(u_i)$ to be any one of the (unassigned) distinct labels from the set

$$\{1, 2, \ldots, M - 1\} \setminus \left\{2^i - 1 | i \leq i \leq N - 1\right\} \setminus \left\{2^N - 2\right\} \cup \bigcup_{i=1}^{N-1} \left\{2^i + 2^j \beta(M) | i + 1 \leq j \leq N \text{ and } \begin{cases} v_i v_j \in E(G) & \text{where } \beta = 2 \text{ if } j \neq N, \\ \beta = 3 \text{ if } j = N \end{cases}\right\}$$

Note that any bijective mapping from the set $\{u_1, u_2, \ldots, u_m\}$ onto the set

$$\{1, 2, \ldots, M - 1\} \setminus \left\{2^i - 1 | i \leq i \leq N - 1\right\} \setminus \left\{2^N - 2\right\} \cup \bigcup_{i=1}^{N-1} \left\{2^i + 2^j \beta(M) | i + 1 \leq j \leq N \text{ and } \begin{cases} v_i v_j \in E(G) & \text{where } \beta = 2 \text{ if } j \neq N, \\ \beta = 3 \text{ if } j = N \end{cases}\right\}$$

can be chosen for defining the function $\phi$ on the vertices $u_i$'s. Thus $\phi$ is injective and the edge values of $G$ are distinct and range from 0 to $M - 1$.

Hence $G$ is harmonious. $\square$

From the definition of $H_m$, the following corollary is an immediate consequence of Theorem 5.2.4.
Corollary 5.2.5. For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geq 1$, there exists a harmonious graph $G$ such that $G_1, G_2, \ldots, G_t$ can be packed into $G$.

Remark 5.2.6. The corollary 5.2.5 is a generalization of the result of by Liu and Zhang (1993) that every graph is a subgraph of a harmonious graph.

Theorem 5.2.7. For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geq 1$, the graph $G = H_m + K_1$ is elegant, for $m = 2^{|V(G)|} - (|V(\tilde{G})| + |E(\tilde{G})|)$.

Proof. Let $v_0, v_1, v_2, \ldots, v_N$, $u_1, u_2, \ldots, u_m$ and $M$ be as in the proof of Theorem 5.2.1.

Define

$$
\phi(v_0) = 0 \quad \phi(u_i) = 2^i, \quad 1 \leq i \leq N
$$

(5.3)

For each $i$, $1 \leq i \leq m$, define $\phi(u_i)$ to be any one of the (unassigned) distinct labels from the set

$$
\{1, 2, \ldots, M\} \setminus \left\{2^i \mid 1 \leq i \leq N\right\} \setminus \bigcup_{i=1}^{N-1} \left\{2^i + 2j \pmod{M+1} \mid 1 \leq j \leq N \text{ and } v_i v_j \in E(\tilde{G})\right\}
$$

Note that any bijective mapping from the set $\{u_1, u_2, \ldots, u_m\}$ onto the set

$$
\{1, 2, \ldots, M\} \setminus \left\{2^i \mid 1 \leq i \leq N\right\} \setminus \bigcup_{i=1}^{N-1} \left\{2^i + 2j \pmod{M+1} \mid 1 \leq j \leq N \text{ and } u_i u_j \in E(\tilde{G})\right\}
$$

can be chosen for defining the function $\phi$ on the vertices $u_i$'s. Thus $\phi$ is injective and the edge values of $G$ are distinct and range from 1 to $M$.

Hence $G$ is elegant. □
From the definition of $H_m$, the following corollary is an immediate consequence of Theorem 5.2.7.

**Corollary 5.2.8.** For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geq 1$, there exists an elegant graph $G$ such that $G_1, G_2, \ldots, G_t$ can be packed into $G$.

**Remark 5.2.9.** The corollary 5.2.8 is a generalization of the result of Balakrishnan et al. (1996) that every graph is a subgraph of an elegant graph.

**Remark 5.2.10.** From the Theorems 5.2.1, 5.2.4, 5.2.7 indicate that it may not be possible to obtain any characterisation result of graceful/harmonious/elegant graphs in terms of forbidden subgraph or set of forbidden subgraphs, because there always exist graceful or harmonious or elegant graphs containing a given graph or a given set of graphs.

**Remark 5.2.11.** From the above construction it appears that in a graph $G$ if there is a vertex $v$ which is adjacent to all the other remaining vertices and its neighbourhood contains sufficiently large number of pendant vertices, then $G$ may be graceful or harmonious or elegant.
Illustrative examples of the labeling in the Proof of Theorems 5.2.1, 5.2.4 and 5.2.7 are given in the Figures 5.2, 5.3 and 5.4.

Figure 5.2. Graceful labeled $H_{495} + K_1$. 
Figure 5.3. Harmonious labeled $H_{495} + K_1$. 
Figure 5.4. Elegant labeled $H_{496} + K_1$. 
5.3 EVERY GRAPH IS A SUBGRAPH OF A CORDIAL GRAPH

In this section, it is proved that every graph is a subgraph of a cordial graph.

Let $K^*_n$ denote the graph obtained from the complete graph $K_n$, by taking a new vertex $v$ and joining $v$ with $\lceil \frac{n}{2} \rceil$ vertices of $K_n$.

**Theorem 5.3.1.** The graph $K^*_n$ is cordial.

**Proof.** Assign label 1 to $v$ and all of its $\lceil \frac{n}{2} \rceil$ adjacent vertices in $K_n$ and label 0 to all the other remaining vertices of $K_n$. Let $V_0$ and $V_1$ respectively, denote the set of vertices of $K^*_n$ assigned the label 0 and the set of vertices of $K^*_n$ assigned the label 1 and let $E_0$ and $E_1$ respectively denote the set of all edges of $K^*_n$ getting the label 0 and the set of all edges of $K^*_n$ getting the label 1. Observe that when $n$ is even, say $n = 2r$, $|V_0| = r$ and $|V_1| = r + 1$ and $|E_0| = |E_1| = r^2$, while when $n$ is odd, say $n = 2r + 1$, $|V_0| = |V_1| = r + 1$ and $|E_0| = |E_1| = r^2 + r$.

Hence $K^*_n$ is cordial. □

The following corollary immediately follows from Theorem 5.3.1.

**Corollary 5.3.2.** Every graph is a subgraph of a cordial graph.
5.4 CONCLUSION

Here in this chapter, we have proved that any set of graphs can be packed into a graceful, harmonious and elegant graphs. These results generalise the results of Balakrishnan et al (1996) that every graph is a subgraph of a graceful graph and elegant graph and the result of Liu and Zhang (1993) that every graph is a subgraph of a harmonious graph.

Also we have shown that every graph is a subgraph of a cordial graph.