CHAPTER 3

GRACEFULNESS OF UNION OF CYCLE WITH PARALLEL CHORDS AND COMPLETE BIPARTITE GRAPHS OR PATHS

3.1 INTRODUCTION

From the excellent survey of Gallian (2003), it is evident that the tendency to have gracefulness in disconnected graphs is relatively less than that of connected graphs. Choudum and Kishore (1999) investigated the gracefulness of union of cycles with paths/stars. The gracefulness of union of cycle with complete bipartite graphs was studied by Seoud and Youssef (2000). Bhat-Nayak and Deshmukh (1996) investigated the gracefulness of $C_{2x+1} \cup P_{x-2\theta}$, where $1 \leq \theta \leq \lfloor(x - 2)/2\rfloor$.

For detailed survey on gracefulness of disconnected graphs refer Gallian (2003).

In this chapter, we study the gracefulness of union of cycle with parallel chord and complete bipartite graph or paths.

In Section 3.2, we prove that $H_n \cup K_{p,q}$ is graceful, for $p, q \geq 1$ and for all odd positive integer $n \geq 7$, here $H_n$ denotes an $n$-cycle with parallel chords.

In Section 3.3, we prove that $H_n \cup P_m$ is graceful, for all $n \geq 6$,

where $m = \begin{cases} 
  n & \text{if} \quad n \equiv 1 \pmod{4} \\
  n - 2 & \text{if} \quad n \equiv 3 \pmod{4} \\
  n - 1 & \text{if} \quad n \equiv 0 \pmod{4} \\
  n - 3 & \text{if} \quad n \equiv 2 \pmod{4}. 
\end{cases}$
3.2 GRACEFULNESS OF UNION OF $H_n$ WITH COMPLETE BIPARTITE GRAPHS

In this section we prove that $H_n \cup K_{p,q}$ is graceful, for $p, q \geq 1$ and for all odd positive integer $n \geq 7$.

**Theorem 3.2.1.** For $p, q \geq 1$ and for all odd positive integer $n \geq 7$, $H_n \cup K_{p,q}$ is graceful.

**Proof.** Let $H_n$ be a cycle $C_n$ with parallel chords with $n$ odd and $n \geq 7$. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of $H_n$ and let $v_1, v_2, \ldots, v_p$ and $w_1, w_2, \ldots, w_q$ be the vertices of a complete bipartite graph $K_{p,q}$. Let $G = H_n \cup K_{p,q}$. Observe that $G$ has $n + p + q$ vertices and $M = \frac{3(n-1)+3pq}{2}$ edges.

Define $f(u_0) = 0$

$$f(u_{2i-1}) = M - 2(i - 1), \text{ for } 1 \leq i \leq \delta, \text{ where } \delta = \left(\frac{n-1}{4}\right), \text{ when } n \equiv 1(\text{mod } 4)$$

or $\delta = \left(\frac{n+1}{4}\right), \text{ when } n \equiv 3(\text{mod } 4)$

$$f(u_{2i}) = 4i - 1, \text{ for } 1 \leq i \leq \delta, \text{ where } \delta = \left(\frac{n-1}{4}\right), \text{ when } n \equiv 1(\text{mod } 4)$$

or $\delta = \left(\frac{n-3}{4}\right), \text{ when } n \equiv 3(\text{mod } 4)$

$$f(u_{n-2i}) = 4i - 3, \text{ for } 1 \leq i \leq \delta, \text{ where } \delta = \left(\frac{n-1}{4}\right), \text{ when } n \equiv 1(\text{mod } 4)$$

or $\delta = \left(\frac{n+1}{4}\right), \text{ when } n \equiv 3(\text{mod } 4)$

$$f(u_{n-2i-1}) = M - 2i + 1, \text{ for } 1 \leq i \leq \delta, \text{ where } \delta = \left(\frac{n-1}{4}\right), \text{ when } n \equiv 1(\text{mod } 4)$$

or $\delta = \left(\frac{n-3}{4}\right), \text{ when } n \equiv 3(\text{mod } 4)$

$$f(u_i) = (n - 3) + q(i - 1), \text{ for } 1 \leq i \leq p$$

$$f(w_j) = (n - 1) + pq - q + j - 1, \text{ for } 1 \leq j \leq q \quad (3.1)$$
From the above vertex labeling, the sets

\[
\left\{ f(u_{2i-1}) \mid 1 \leq i \leq \delta \right\}, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n+1}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4},
\]

\[
\left\{ f(u_{n-2i}) \mid 1 \leq i \leq \delta \right\}, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n-3}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4}.
\]

form a monotonically decreasing sequence and the sets

\[
\left\{ f(u_{2i}) \mid 1 \leq i \leq \delta \right\}, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n-3}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4},
\]

\[
\left\{ f(u_{n-(2i-1)}) \mid 1 \leq i \leq \delta \right\}, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n+1}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4},
\]

as well as \( \left\{ f(v_i) \mid 1 \leq i \leq p \right\} \) and \( \left\{ f(w_j) \mid 1 \leq j \leq q \right\} \)

also form a monotonically increasing sequence.

Also, observe that

\[
\max \left\{ f(u_{2i}) \mid 1 \leq i \leq \delta, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n-3}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4} \right\}
\]

\[
\cup \left\{ f(u_{n-(2i-1)}) \mid 1 \leq i \leq \delta, \quad \text{where} \quad \delta = \left( \frac{n-1}{4} \right), \quad \text{when} \quad n \equiv 1 \pmod{4} \\
\text{or} \quad \delta = \left( \frac{n+1}{4} \right), \quad \text{when} \quad n \equiv 3 \pmod{4} \right\}
\]

(3.2)
\[< \min \left\{ \left\{ f(u_i) \right\} \text{for } 1 \leq i \leq p \right\} \cup \left\{ f(w_j) \right\} \text{for } 1 \leq j \leq q \]

\[\cup \left\{ f(u_{2i-1}) \right\} \text{for } 1 \leq i \leq \delta, \text{ where } \delta = \left( \frac{n-1}{4} \right), \text{ when } n \equiv 1(\mod 4)\]

or \[\delta = \left( \frac{n+1}{4} \right), \text{ when } n \equiv 3(\mod 4)\]

\[\cup \left\{ f(u_{n-2i}) \right\} \text{for } 1 \leq i \leq \delta, \text{ where } \delta = \left( \frac{n-3}{4} \right), \text{ when } n \equiv 3(\mod 4)\]

(3.2)

and \(f(v_1) \neq f(x)\) for any vertex of \(x\) of \(G\) other than \(v_1\). Therefore the labels of all the vertices of \(G\) are distinct.

Let \(A\) denote the set of cycle edges \(\{u_1u_2, u_3u_4, u_5u_6, \ldots, u_{2\gamma-1}u_{2\gamma}\}\) of \(H_n\), where \(\gamma = \left( \frac{n+1}{4} \right)\) when \(n \equiv 1(\mod 4)\) or \(\gamma = \left( \frac{n-3}{4} \right)\) when \(n \equiv 3(\mod 4)\),

\(B\) denote the set of cycle edges \(\{u_2u_3, u_4u_5, u_6u_7, \ldots, u_{2\gamma+1}\}\) of \(H_n\), where \(\gamma = \left( \frac{n-3}{4} \right)\) when \(n \equiv 1(\mod 4)\) or \(\gamma = \left( \frac{n-3}{4} \right)\) when \(n \equiv 3(\mod 4)\),

\(C\) denote the set of cycle edges \(\{u_{n-1}u_{n-2}, u_{n-3}u_{n-4}, u_{n-5}u_{n-6}, \ldots, u_{n-(2\gamma-1)}u_{n-(2\gamma)}\}\) of \(H_n\), where \(\gamma = \left( \frac{n-1}{4} \right)\) when \(n \equiv 1(\mod 4)\) or \(\gamma = \left( \frac{n-3}{4} \right)\) when \(n \equiv 3(\mod 4)\),

\(D\) denote the set of cycle edges \(\{u_{n-2}u_{n-3}, u_{n-4}u_{n-5}, u_{n-6}u_{n-7}, \ldots, u_{n-(2\gamma)}u_{n-(2\gamma+1)}\}\) of \(H_n\), where \(\gamma = \left( \frac{n-5}{4} \right)\) when \(n \equiv 1(\mod 4)\) or \(\gamma = \left( \frac{n-3}{4} \right)\) when \(n \equiv 3(\mod 4)\),

\(E\) denote the set of cycle (chords) edges \(\{u_1u_{n-1}, u_2u_{n-2}, u_3u_{n-3}, \ldots, u_nu_{n-\gamma}\}\)

of \(H_n\), where \(\gamma = \frac{n-3}{2}\), \(F\) denote the edges \(\{u_0u_1, u_0u_{n-1} \text{ and } u_{n-1}u_1\}\)

and \(G\) denote the set of edges \(\{v_1w_1, v_1w_2, \ldots, v_{2p}w_1, v_{2p}w_2, \ldots, v_{2p}w_q\}\) of \(K_{p,q}\).

We give below the edge value of edges in the sets \(A, B, C, D, E, F\) and \(G\) and we denote these sets respectively by \(A', B', C', D', E', F'\) and \(G'\).
\[ A' = \{ M - 3, M - 9, M - 15, \ldots, M - (6\gamma - 3) \}, \text{ where } \gamma = \left( \frac{n-1}{4} \right) \text{ when } \\
n \equiv 1(\text{mod } 4) \text{ or } \gamma = \left( \frac{n-3}{4} \right) \text{ when } n \equiv 3(\text{mod } 4), \]
\[ B' = \{ M - 5, M - 11, M - 17, \ldots, M - (6\gamma - 1) \}, \text{ where } \gamma = \left( \frac{n-1}{4} \right) \text{ when } \\
n \equiv 1(\text{mod } 4) \text{ or } \gamma = \left( \frac{n-3}{4} \right) \text{ when } n \equiv 3(\text{mod } 4), \]
\[ C' = \{ M - 2, M - 8, M - 14, \ldots, M - (6\gamma - 4) \}, \text{ where } \gamma = \left( \frac{n+1}{4} \right) \text{ when } \\
n \equiv 1(\text{mod } 4) \text{ or } \gamma = \left( \frac{n-1}{4} \right) \text{ when } n \equiv 3(\text{mod } 4) \]
\[ D' = \{ M - 6, M - 12, M - 18, \ldots, M - (6\gamma) \}, \text{ where } \gamma = \left( \frac{n-1}{4} \right) \text{ when } \\
n \equiv 1(\text{mod } 4) \text{ or } \gamma = \left( \frac{n-3}{4} \right) \text{ when } n \equiv 3(\text{mod } 4) \]
\[ E' = \{ M - 1, M - 4, M - 7, \ldots, M - (3\gamma - 2) \}, \text{ where } \gamma = \left( \frac{n-3}{4} \right) \text{ when } \\
n \equiv 1 \text{ or } 3(\text{mod } 4), \]
\[ F' = \{ M, 1, M - \left( \frac{3n-7}{2} \right) \} \]
and \[ G' = \{ 2, 3, 4, \ldots, pg + 1 \}. \quad (3.3) \]

We observe that the values in the sets \( A', B', C', D', E', F' \) and \( G' \) are all distinct and \( A' \cup B' \cup C' \cup D' \cup E' \cup F' \cup G' = \{ 1, 2, 3, \ldots, M \} \).

Hence the graph \( G \) is graceful. □
Illustrative examples of the labeling given in the Proof of Theorem 3.2.1 are provided in Figures 3.1 and 3.2.

Figure 3.1. Graceful labeled $H_{13} \cup K_{4,6}$. 
Figure 3.2. Graceful labeled $H_{15} \cup K_{5,8}$. 
3.3 GRACEFULNESS OF UNION OF $H_n$ WITH PATHS

In this section we obtain the following result on the gracefulness of $H_n \cup P_m$.

**Theorem 3.3.1.** For $n \geq 6$, $H_n \cup P_m$ is graceful,

$$m = \begin{cases} 
  n & \text{if } n \equiv 1 \pmod{4} \\
  n-2 & \text{if } n \equiv 3 \pmod{4} \\
  n-1 & \text{if } n \equiv 0 \pmod{4} \\
  n-3 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}$$

**Proof.** Let $H_n$ be a cycle $C_n$ with parallel chords. Let $u_0, u_1, u_2, \ldots, u_{n-1}$ be the vertices of $H_n$ and let $v_1, v_2, \ldots, v_m$ be the vertices of $P_m$. Let $G = H_n \cup P_m$.

We define the vertex labeling of $G$ in the following four cases, $G = H_n \cup P_m$,

$$m = \begin{cases} 
  n & \text{if } n \equiv 1 \pmod{4} \\
  n-2 & \text{if } n \equiv 3 \pmod{4} \\
  n-1 & \text{if } n \equiv 0 \pmod{4} \\
  n-3 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}$$

Observe that $G$ has $n+m$ vertices and $M = \frac{3(n-p)+2(m-1)}{2}$ edges, where $p = 3$ if $n$ is odd, $p = 2$ if $n$ is even.

**Case 1:** $n \equiv 1 \pmod{4}$.

Then by assumption $m = n$.

Define $f(u_0) = 0$

$$f(u_{2i-1}) = M - 2(i - 1), \quad \text{for } 1 \leq i \leq \left( \frac{n-1}{4} \right)$$

$$f(u_{2i}) = 4i - 1, \quad \text{for } 1 \leq i \leq \left( \frac{n-1}{4} \right)$$

$$f(u_{n-(2i-1)}) = 4i - 3, \quad \text{for } 1 \leq i \leq \left( \frac{n-1}{4} \right)$$
\[ f(u_{n-2i}) = M - 2i + 1, \quad \text{for } 1 \leq i \leq \left( \frac{n-1}{4} \right) \]
\[ f(v_{2i-1}) = \frac{n + 4i - 5}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m+1}{2} \right) \]
\[ f(v_{2i}) = \frac{3n - 4i + 3}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m-1}{4} \right) \]
\[ f(v_{m-(2i-1)}) = \frac{3n + 4i - 3}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m-1}{4} \right) \]

From the above vertex labeling the sets \( \{f(u_{2i})\} \) \( 1 \leq i \leq \left( \frac{n-1}{4} \right) \), \( \{f(u_{m-(2i-1)})\} \) \( 1 \leq i \leq \left( \frac{m+1}{2} \right) \) and \( \{f(v_{2i-1})\} \) \( 1 \leq i \leq \left( \frac{m-1}{4} \right) \) form a monotonically increasing sequence and the sets \( \{f(u_{2i})\} \) \( 1 \leq i \leq \left( \frac{n-1}{4} \right) \), \( \{f(u_{m-2i})\} \) \( 1 \leq i \leq \left( \frac{n-1}{4} \right) \), \( \{f(v_{2i})\} \) \( 1 \leq i \leq \left( \frac{m-1}{4} \right) \) is form a monotonically decreasing sequence.

Also, observe that
\[
\max\left\{ \{f(u_{2i})\} \mid 1 \leq i \leq \left( \frac{n-1}{4} \right) \right\} \cup \{f(u_{m-(2i-1)})\} \mid 1 \leq i \leq \left( \frac{m+1}{2} \right) \}
\cup \{f(v_{2i-1})\} \mid 1 \leq i \leq \left( \frac{m-1}{4} \right) \}
\leq \min\left\{ \{f(u_{2i})\} \mid 1 \leq i \leq \left( \frac{n-1}{4} \right) \right\} \cup \{f(u_{m-2i})\} \mid 1 \leq i \leq \left( \frac{n-1}{4} \right) \}
\cup \{f(v_{2i})\} \mid 1 \leq i \leq \left( \frac{m-1}{4} \right) \}
\] (3.5)

Therefore the labels of all vertices of \( G \) are distinct.

Let A denote the set of cycle edges \( \{u_1u_2, u_3u_4, u_5u_6, \ldots, u_{m-2}u_{m-1}\} \) of \( H_n \),
B denote the set of cycle edges \( \{u_2u_3, u_4u_5, u_6u_7, \ldots, u_{m-1}u_m\} \) of \( H_n \),
C denote the set of cycle edges \( \{u_{m-1}u_1, u_{m-3}u_{m-4}, u_{m-5}u_{m-6}, \ldots, u_{m-2}u_{m-3}\} \) of \( H_n \),
D denote the set of cycle edges \( \{u_{n-2}u_{n-3}, u_{n-4}u_{n-5}, u_{n-6}u_{n-7}, \ldots, u_{n-2}u_{n-3}\} \) of \( H_n \),
E denote the set of cycle (chords) edges \( \{u_1u_{n-1}, u_2u_{n-2}, u_3u_{n-3}, \ldots, u_{\frac{m-1}{2}}u_{\frac{m+3}{2}}\} \)
of \( H_n \),

F denote the edges \( \{u_0u_1, u_0u_{n-1}, u_{\frac{m-1}{2}}u_{\frac{m+1}{2}}\} \),

G denote the set of edges \( \{u_1u_2, v_2v_3, \ldots, v_{\frac{m-1}{2}}v_{\frac{m+1}{2}}\} \) of a path \( P_m \).

and \( H \) denote the set of edges \( \{v_1v_{m-1}, v_{m-2}v_{m-2}, \ldots, v_{\frac{m-1}{2}}v_{\frac{m+1}{2}}\} \) of a path \( P_m \).

We give below the edge value of edges in the sets \( A, B, C, D, E, F, G \) and \( H \), we denote these sets respectively by \( A', B', C', D', E', F', G' \) and \( H' \).

\[
\begin{align*}
A' &= \{M - 3, M - 9, M - 15, \ldots, M - (6\gamma - 3)\}, \text{ where } \gamma = \left(\frac{n-1}{4}\right), \\
B' &= \{M - 5, M - 11, M - 17, \ldots, M - (6\gamma - 1)\}, \text{ where } \gamma = \left(\frac{n-3}{4}\right), \\
C' &= \{M - 2, M - 8, M - 14, \ldots, M - (6\gamma - 4)\}, \text{ where } \gamma = \left(\frac{n-5}{4}\right), \\
D' &= \{M - 6, M - 12, M - 18, \ldots, M - (6\gamma - 2)\}, \text{ where } \gamma = \left(\frac{n-7}{4}\right), \\
E' &= \{M - 1, M - 5, M - 9, \ldots, M - (3\gamma - 2)\}, \text{ where } \gamma = \left(\frac{n-3}{2}\right), \\
F' &= \{M, 1, n + 1\} \quad (3.6) \\
G' &= \{m, m - 2, m - 4, \ldots, 5, 3\} \\
\text{and } H' &= \{2, 4, 6, \ldots, m - 1\}.
\end{align*}
\]

Observe that the values in the sets \( A', B', C', D', E', F', G' \) and \( H' \) are all distinct and \( A' \cup B' \cup C' \cup D' \cup E' \cup F' \cup G' \cup H' = \{1, 2, 3, \ldots, M\} \).

Hence the graph \( G \) is graceful.

**Case 2:** \( n \equiv 3(\text{mod} \ 4) \).

Then by assumption \( m = n - 2 \).

Define \( f(u_0) = 0 \)

\[
\begin{align*}
f(u_{2i-1}) &= M - 2(i - 1), \quad \text{for } 1 \leq i \leq \left(\frac{n+1}{4}\right), \\
f(u_{2i}) &= 4i - 1, \quad \text{for } 1 \leq i \leq \left(\frac{n-3}{4}\right), \\
f(u_{n-(2i-1)}) &= 4i - 3, \quad \text{for } 1 \leq i \leq \left(\frac{n+1}{4}\right)
\end{align*}
\]
As in the Case 1, in this case also it follows that the labels of all vertices of \( G \) are distinct and edge values of the edges range from 1 to \( M \). Hence graph \( G \) is graceful.

**Case 3:** \( n \equiv 0 \pmod{4} \).

Then by assumption \( m = n - 1 \).

Define \( f(u_0) = 0 \)

\[
\begin{align*}
  f(u_{2i-1}) &= M - 2(i - 1), \quad \text{for } 1 \leq i \leq \left( \frac{n}{4} \right) \\
  f(u_{2i}) &= 4i - 1, \quad \text{for } 1 \leq i \leq \left( \frac{n}{4} \right) \\
  f(u_{n-(2i-1)}) &= 4i - 3, \quad \text{for } 1 \leq i \leq \left( \frac{n}{4} \right) \\
  f(v_{2i-1}) &= \frac{n + 4i - 4}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m + 1}{4} \right) \\
  f(v_{2i}) &= \frac{3n - 4i + 6}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m + 1}{4} \right) \\
  f(v_{m-(2i-1)}) &= \frac{3m + 4i}{2}, \quad \text{for } 1 \leq i \leq \left( \frac{m - 3}{4} \right)
\end{align*}
\]
As in the Case 1, in this case also it follows that the labels of all vertices of $G$ are distinct and edge values of the edges range from 1 to $M$. Hence graph $G$ is graceful.

Case 4: $n \equiv 2 \pmod{4}$.

Then by assumption $m = n - 3$.

Define $f(u_0) = 0$

$$f(u_{2i-1}) = M - 2(i - 1), \quad \text{for } 1 \leq i \leq \left(\frac{n - 2}{4}\right)$$

$$f(u_{2i}) = 4i - 1, \quad \text{for } 1 \leq i \leq \left(\frac{n - 2}{4}\right)$$

$$f(u_{n-(2i-1)}) = 4i - 3, \quad \text{for } 1 \leq i \leq \left(\frac{n + 2}{4}\right)$$

$$f(u_{n-(2i)}) = M - 2i + 1, \quad \text{for } 1 \leq i \leq \left(\frac{n + 1}{2}\right)$$

$$f(v_{2i-1}) = \frac{n + 4i - 6}{2}, \quad \text{for } 1 \leq i \leq \left(\frac{m + 1}{2}\right)$$

$$f(v_{2i}) = \frac{3n - 4i}{2}, \quad \text{for } 1 \leq i \leq \left(\frac{m + 1}{4}\right)$$

$$f(v_{m-(2i-1)}) = \frac{3n + 4i - 6}{2}, \quad \text{for } 1 \leq i \leq \left(\frac{m - 3}{4}\right)$$

As in the Case 1, in this case also it follows that the labels of all vertices of $G$ are distinct and edge values of the edges range from 1 to $M$. Hence the graph $G$ is graceful. □
Illustrative examples of labeling given in the Proof of Theorem 3.3.1 are provided in Figures 3.3 and 3.4.

Figure 3.3. Graceful labeled $H_n \cup P_{13}$. 
Figure 3.4. Graceful labeled $H_{16} \cup P_{15}$. 
3.4 DISCUSSION

Here in this chapter, we have shown that disjoint union of cycle with parallel chords $H_n$ and complete bipartite graph $K_{p,q}$ is graceful for all odd $n \geq 7$ and for all $p, q \geq 1$. It appears that proving the gracefulness of disjoint union of cycle with parallel chords $H_n$ and complete bipartite graph $K_{p,q}$ for all even $n \geq 6$ and for all $p, q \geq 1$ is difficult. However, proving the above results negatively also appears equally hard. So we conclude this chapter with the following question.

Whether for every $n \geq 6$ and for all $p, q \geq 1$ disjoint union of cycle with parallel chords $H_n$ and complete bipartite graph $K_{p,q}$ is graceful?