3. TIME MOMENT METHOD

3.1 INTRODUCTION

This chapter is primarily devoted to presenting a method for effecting model approximation when $T_m(s)$ and $T_a(s)$ are scalar transfer functions. This method seeks to match or approximate time moments, subject to the limitations imposed by the use of state variable feedback. To ensure stability, a check is incorporated into the algorithm which forces some degree of mismatch in some time moments, if the needs of stability so demand. The time moment method, as this method shall be called, can be applied regardless of whether the number of poles of $T_m(s)$ is greater than, equal to or less than the number of poles of $T_a(s)$. The primary assumption made is that $T_m(s)$ is asymptotically stable. It is shown that the proposed method will result in exact model matching in case such matching is possible. The procedure developed is illustrated by means of examples.

3.2 DOMINANT POLE PROCEDURE

Some reduced order modelling methods seek to formulate a lower order system which retains the dominant poles of a higher order system and also matches the time moments of the higher order system. In the case when only state feedback is permissible there are major restrictions in the
freedom available for effecting approximation. The primary purpose of this section is to illustrate the effect of such restriction in the freedom available. It is also intended to show that matching of additional time moments does not necessarily effect improvement in closeness of impulse responses. The purpose is to highlight the mutually contradictory nature of some of the approximation criteria discussed in chapter 2.

Let $T_a(s)$ and $T_m(s)$ be strictly proper scalar transfer functions of the actual and model systems with $T_m(s)$ being asymptotically stable. Let it be further assumed that there are no pole zero cancellations in $T_m(s)$ and that it has $m$ poles while $T_a(s)$ has $n$ poles, with $m \geq n$. With these assumptions one has,

Proposition 3.1

By the use of state variable feedback, $T_a(s)$ can be made to retain $n$ of the $m$ poles of $T_m(s)$ and thereafter there will be freedom left for arbitrary assignment of only one time moment of $T_a(s)$.

Proof: After state feedback the general form of $T_a(s)$ is given by:

$$T_{aG}(s) = \frac{a_0 a_1 s + a_2 s + \cdots + a_n \cdot s^{n-1}}{c_0 + c_1 s + c_2 s^2 + \cdots + c_n \cdot s^{n-1} + s^n}$$
where \( g \neq 0 \) and \( c_i, i = 0,1,2,\ldots,n-1 \) are arbitrary. If \( -\alpha_1,\alpha_2,\ldots,\alpha_n \) are the \( n \) poles of \( T_m(s) \) sought to be retained in \( T_{kg}(s) \) then one must have,

\[
c_0 + c_1 s + \ldots + c_{n-1} s^{n-1} + s^n = (s + \alpha_1) (s + \alpha_2) \ldots (s + \alpha_n)
\]

This relation uniquely fixes \( c_0, c_1, \ldots, c_{n-1} \) because the L.H.S. is a \( n \)th degree polynomial in \( s \) and as such must have precisely \( n \) roots and these are specified in the R.H.S.

Hence if state feedback is used to fix \( n \) poles of \( T_{kg}(s) \) then \( c_0, c_1, \ldots, c_{n-1} \) will get fixed leaving only \( g \) as the parameter available for further manipulation. It follows that after fixing \( n \) poles there is freedom to fix only one time moment arbitrarily. This completes the proof.

An example can now be considered which will illustrate how state feedback can be used to retain dominant poles of \( T_m(s) \) in \( T_a(s) \) and to match the initial time moments of the two systems.

**Example 3.1**

\[
T_m(s) = \frac{10(s+4)}{(s+1)(s+2)(s+3)}
\]

\[
T_a(s) = \frac{1}{(s+7)(s+5)}
\]

After state feedback \( T_a(s) \) transforms to \( T_{kg}(s) \) where,

\[
T_{kg}(s) = \frac{g}{s^2 + c_1 s + c_0}
\]
To retain the dominant poles of $T_m(s)$ in $T_{kg}(s)$ one must have,

$$s^2 + c_1s + c_0 = (s+1)(s+2)$$

This yields,

$c_0 = 2$ and $c_1 = 3$

At the end of this stage one has,

$$T_{kg}(s) = \frac{g}{s^2 + 3s + 2}$$

Now it can be seen that,

$$T_m(s) = \frac{20}{3} - \frac{95}{9} s + \ldots$$

Hence $k_{o,m} = \frac{20}{3}$ and $k_{1,m} = -\frac{95}{9}$ (Notations are same as in chapter 2)

Further $T_{kg}(s) = \frac{g}{s^2 + 3s + 2} = \frac{g}{2} - \frac{3g}{4} s + \ldots$

Hence $k_{o,a} = \frac{g}{2}$ and $k_{1,a} = -\frac{3g}{4}$

$k_{o,a} = k_{o,m}$ implies $\frac{g}{2} = \frac{20}{3}$ or $g = \frac{40}{3}$

There is no freedom left for setting $k_{1,a} = k_{1,m}$

Hence the final form of $T_{kg}(s)$ is,

$$T_{kg}(s) = \frac{40}{3} \frac{1}{s^2 + 3s + 2}$$
The case will now be considered where there is freedom to set the numerator polynomial also. This will correspond to some of the standard reduced order modelling problems. The first step will be to set the denominator to \( s^2 + 3s + 2 \) (in order to preserve the dominant modes of \( T_m(s) \)). The transfer function of the lower order system after dominant pole retention has been effected will be \( T_r(s) \) where,

\[
T_r(s) = \frac{g(s+a)}{s^2 + 3s + 2}
\]

Expanding about \( s = 0 \) one has,

\[
T_r(s) = \frac{g a}{2} + g \frac{2-3a}{4} s + \ldots
\]

Matching of the first time moment yields,

\[
g \frac{a}{2} = \frac{20}{3} \quad \text{or} \quad g \frac{a}{3} = \frac{40}{3}
\]

Matching of the second time moment yields,

\[
g \frac{2-3a}{4} = \frac{-95}{9} \quad \text{or} \quad g \frac{a}{3} = \frac{2}{3} g + \frac{380}{27}
\]

Hence \( g = -\frac{10}{9} \) and \( a = -12 \).

So,

\[
T_r(s) = \frac{-10}{9} \frac{(s-12)}{s^2 + 3s + 2}
\]
It is thus seen that if there is freedom to set the numerator polynomial also then two time moments can be matched unlike in the state variable feedback case when only one time moment could be matched. This serves as an illustration to highlight the restrictions imposed by the employment of only state variable feedback for approximation.

$T_{kg}(s)$ and $T_r(s)$ both retain two dominant poles of $T_m(s)$. However, while $T_{kg}(s)$ has its first time moment matching that of $T_m(s)$, the first two time moments of $T_r(s)$ match those of $T_m(s)$. So, at the first sight, it may appear that the impulse response of $T_r(s)$ will approximate the impulse response of $T_m(s)$ better than the impulse response of $T_{kg}(s)$ will approximate that of $T_m(s)$.

Let $y_m$, $y_a$, and $y_r$ denote the impulse responses of $T_m(s)$, $T_{kg}(s)$, and $T_r(s)$ respectively.

$$y_m(t) = L^{-1} T_m(s) = 15 e^{-t} - 20 e^{-2t} + 5 e^{-3t}$$

$$y_a(t) = L^{-1} T_{kg}(s) = \frac{40}{3} e^{-t} - \frac{40}{3} e^{-2t}$$

$$y_r(t) = L^{-1} T_r(s) = \frac{130}{9} e^{-t} - \frac{140}{9} e^{-2t}$$

Let $e_1(t) = y_m(t) - y_a(t)$ and $e_2(t) = y_m(t) - y_r(t)$
Then

\[ e_1(t) = \frac{5}{3} e^{-t} - \frac{20}{3} e^{-2t} + 5 e^{-3t} \]

and \[ e_2(t) = \frac{5}{9} e^{-t} - \frac{40}{9} e^{-2t} + 5 e^{-3t} \]

Define \[ J_1 = \int_0^\infty e_1^2(t) \, dt \] and \[ J_2 = \int_0^\infty e_2^2(t) \, dt \].

\[ J_1 \] and \[ J_2 \] will serve as measures of closeness of the impulse responses.

In the present case

\[ J_1 = 0.092592 \] and \[ J_2 = 0.113168 \]

It is obvious that \[ J_1 < J_2 \] which means that the impulse response of \( T_m(s) \) is better approximated by \( T_{kg}(s) \) than by \( T_k(s) \) (in the \( L_2 \) norm sense). This example shows that matching more time moments does not, in all cases, guarantee better closeness between the impulse responses of the actual and model systems. In the next section the restriction of freedom caused by retention of dominant poles will be removed. Thus what is developed in the sequel will be applicable even when the number of poles of \( T_m(s) \) is less than that of \( T_a(s) \).

3.3 PRELIMINARY RELATIONSHIPS

The time moment method seeks to use state feedback to achieve matching of successive time moments of the actual
and model systems. The advantages of such matching have already been considered in chapter 2. For performing time moment matching it is necessary to arrive at a set of relationships to compute time moments and we now proceed to derive it. Let \( G(s) \) be a strictly proper rational transfer function with no pole at the origin.

Let 
\[
G(s) = \frac{x_0 + x_1 s + x_2 s^2 + \ldots + x_{q-1} s^{q-1}}{y_0 + y_1 s + y_2 s^2 + \ldots + y_{q-1} s^{q-1} + y_q s^q}
\]

\[
= k_0 + k_1 s + k_2 s^2 + k_3 s^3 + \ldots
\]

Hence

\[
(x_0 + x_1 s + x_2 s^2 + \ldots + x_{q-1} s^{q-1}) = (k_0 + k_1 s + k_2 s^2 + \ldots) \left( y_0 + y_1 s + y_2 s^2 + \ldots + y_{q-1} s^{q-1} + y_q s^q \right)
\]

Equating coefficients of like powers of \( s \) on both sides yields,

\[
x_0 = k_0 y_0
\]
\[
x_1 = k_0 y_1 + k_1 y_0
\]
\[
x_2 = k_0 y_2 + k_1 y_1 + k_2 y_0
\]
\[
\vdots
\]
\[
x_{q-1} = k_0 y_{q-1} + k_1 y_{q-2} + \ldots + k_{q-1} y_0
\]
\[
0 = k_1 y_{q-1} + k_1 y_{q-2} + \ldots + k_{q-1} y_0, \quad i = 0, 1, 2, \ldots
\]

\[\text{... (3.1)}\]

\[
V_l = k_0 y_{q-l} + k_1 y_{q-2} + \ldots + k_{q-1} y_0
\]

\[
= k_0 y + k_1 y + k_2 y + \ldots + k_{q-1} y, \quad i = 0, 1, 2, \ldots
\]
These constitute the set of recursive relationships that can be employed to compute \( k_0,k_1,k_2, \ldots \).

Let the model system have a transfer function \( T_m(s) \) where

\[
T_m(s) = \frac{d_1 s + \ldots + d_{m-1} s^{m-1} + d_m s^m}{e_0 s + \ldots + e_{m-1} s^{m-1} + e_m s^m} \quad \ldots (3.2)
\]

Here it will be assumed that \( e_0 \neq 0 \), \( d_0 \neq 0 \) and that \( e_0 + e_1 s + \ldots + e_m s^m \) is a strictly Hurwitz polynomial.

Let the actual system have a transfer function \( T_a(s) \) where

\[
T_a(s) = \frac{a_0 s + \ldots + a_{n-1} s^{n-1} + a_n s^n}{b_0 s + \ldots + b_{n-1} s^{n-1} + b_n s^n} \quad \ldots (3.3)
\]

It will be assumed that \( a_0 \neq 0 \).

After state feedback \( T_a(s) \) becomes \( T_{kg}(s) \) where

\[
T_{kg}(s) = \frac{a_0 s + \ldots + a_{n-1} s^{n-1} + a_n s^n}{f_0 s + \ldots + f_{n-1} s^{n-1} + f_n s^n}, \quad f_n \neq 0 \quad \ldots (3.4)
\]

It will be assumed \( f_0 \neq 0 \).

Let \( T_m(s) = k_0 s + k_1 s^2 + k_2 s^3 + \ldots \quad \ldots (3.5) \)

and \( T(s) = k_0 s + k_1 s^2 + k_2 s^3 + \ldots \quad \ldots (3.6) \)
In matching time moments, the first step is to use equation 3.1 to get at the values of \( k_{i,m} \), \( i = 0,1,2,... \) for the model system. The next step is to use equation 3.1 to formulate relations for the actual system after feedback. Here \( f_0, f_1, ..., f_n \) as also \( k_{i,a}, i = 0,1,2,... \) will be unknowns. The third step would consist in replacing \( k_{i,a} \) by \( k_{i,m} \) for \( i = 0,1,2,... \) and solving at each stage for the corresponding \( f_i \). The following example illustrates the steps involved.

**Example 3.2**

\[
T_m(s) = \frac{6s^2 + 26s + 24}{s^3 + 6s^2 + 11s + 6}
\]

\[
T_a(s) = \frac{6s + 8}{s^2 + 2s + 1}
\]

Hence \( T_{ag}(s) = \frac{6s + 8}{f_2s^2 + f_1s + f_0} \), \( f_2 \neq 0 \)

Solution of equation 3.1 yields

\[
k_{0,m} = 4, \quad k_{1,m} = -3 \quad \text{and} \quad k_{2,m} = \frac{5}{2}
\]

Using equation 3.1 for the actual system after feedback and substituting the already known values of \( k_{0,m}, k_{1,m} \) and \( k_{2,m} \) in place of \( k_{0,a}, k_{1,a} \) and \( k_{2,a} \), we obtain \( f_0 = 2, f_1 = 3 \) and \( f_2 = 1 \). Hence,
It may be noted that $T_{kg}(s)$ is such that its first three time moments match those of $T_m(s)$. Fig. 3.1 shows the step responses of $T_m(s)$ and $T_{kg}(s)$.

The plots were obtained by means of a BAI TR 20 analog computer system. Since only a rough visual comparison is aimed at, the time and output scales are not indicated in Fig 3.1.

3.4 STABILITY CONSIDERATIONS

If $f_1$ are directly fixed so as to effect matching of $k_i,m$ and $k_i,a$ there is a possibility of $T_{kg}(s)$ becoming unstable. This aspect has been illustrated in chapter 2. It is imperative to overcome this instability problem if a useful $T_{kg}(s)$ is to be arrived at. The purpose of this section is to discuss some means of averting the instability problem by imposing some restrictions on $f_1$.

Given a set of arbitrary numbers $k_i,m$ (in actual practice they may be the Taylor series expansion
i = 0,1,2,...n and which at the same time remain
tically stable. It may turn out that for a cert
ification of \{k_i, q\} no such n-th order system T_{kg}(s)
If that be the case it will be better to know the
tion apriori. Theorem 3.1 and proposition 3.4 w
propose to solve this problem in a satisfactory
Briefly the approach is as follows:

Theorem 3.1 establishes a Lyapunov type matrix e
using which one may solve for a lower triangular
If the solution exists then, according to the th
T_{kg}(s) is stable for the given set of parameters
However the matrix equation defined by theorem 3
besides known parameters \{k_i, q\} and \{a_i\} an unk
f_n. Proposition 3.4 expresses f_n in terms of k_i
and hence it is possible to apriori determine th
of obtaining a stable T_{kg}(s).

The proof of theorem 3.1 and the accompanying pr
follow.

Theorem 3.1
$T_{kg}(s)$ will be asymptotically stable if there exists a non-singular lower triangular matrix $L$ such that

$$E^T L^T D + D^T L^T E = -I$$

where

$$E = \begin{bmatrix} 0 & k_{0a} & k_{1a} & \cdots & k_{n-2,a} \\
0 & 0 & k_{0a} & \cdots & k_{n-3,a} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-f_{n-2} & -f_{n-1} & -f_{n} & \cdots & -f_{n-1} \\
f_{n} & f_{n} & f_{n} & \cdots & f_{n} \end{bmatrix}$$

$$D = \begin{bmatrix} k_{0a} & k_{1a} & k_{2a} & \cdots & k_{n-1,a} \\
0 & k_{0a} & k_{1a} & \cdots & k_{n-2,a} \\
0 & 0 & k_{0a} & \cdots & k_{n-3,a} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_{0a} \end{bmatrix}$$

and $\{k_{ia}\}$, $\{f_{i}\}$ and $\{e_{i}\}$ are the parameters associated with $T_{kg}(s)$ (as defined earlier). Further $k_{0a} \neq 0$. 


Proof:

\( T_{kg}(s) \) will be asymptotically stable if and only if
\[
f_0 + f_1 s + f_2 s^2 + \cdots + f_n s^n = 0
\]
has all its roots with negative real parts.

Consider

\[
H = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
-f_0 & -f_1 & -f_2 & -f_3 & \cdots & -f_{n-2} & -f_{n-1} \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & f_{n-1} \\
f_2 & f_3 & f_4 & f_5 & \cdots & f_{n-1} & f_n \\
f_3 & f_4 & f_5 & f_6 & \cdots & f_n & f_{n+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & f_{n+1} \\
f_{n-1} & f_n & f_{n+1} & f_{n+2} & \cdots & f_{n+1} & f_{n+2} \\
f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_{n+2} & f_{n+3} \\
f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots & f_{n+3} & f_{n+4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & f_{n+1} \\
f_{n-3} & f_{n-2} & f_{n-1} & f_n & \cdots & f_{n-1} & f_n \\
f_{n-4} & f_{n-3} & f_{n-2} & f_{n-1} & \cdots & f_{n-2} & f_{n-1} \\
f_{n-5} & f_{n-4} & f_{n-3} & f_{n-2} & \cdots & f_{n-3} & f_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_2 & f_3 & f_4 & f_5 & \cdots & f_{n-1} & -f_1 \\
f_3 & f_4 & f_5 & f_6 & \cdots & f_n & -f_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & -f_{n-1} \\
f_{n-1} & f_n & f_{n+1} & f_{n+2} & \cdots & f_n & -f_{n-1} \\
f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_n & -f_{n-1} \\
f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots & f_n & -f_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_2 & f_3 & f_4 & f_5 & \cdots & f_{n-1} & -f_1 \\
f_3 & f_4 & f_5 & f_6 & \cdots & f_n & -f_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & -f_{n-1} \\
f_{n-1} & f_n & f_{n+1} & f_{n+2} & \cdots & f_n & -f_{n-1} \\
f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_n & -f_{n-1} \\
f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots & f_n & -f_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_2 & f_3 & f_4 & f_5 & \cdots & f_{n-1} & -f_1 \\
f_3 & f_4 & f_5 & f_6 & \cdots & f_n & -f_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & -f_{n-1} \\
f_{n-1} & f_n & f_{n+1} & f_{n+2} & \cdots & f_n & -f_{n-1} \\
f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_n & -f_{n-1} \\
f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots & f_n & -f_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-2} & -f_0 \\
f_2 & f_3 & f_4 & f_5 & \cdots & f_{n-1} & -f_1 \\
f_3 & f_4 & f_5 & f_6 & \cdots & f_n & -f_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n-2} & f_{n-1} & f_n & f_{n+1} & \cdots & f_n & -f_{n-1} \\
f_{n-1} & f_n & f_{n+1} & f_{n+2} & \cdots & f_n & -f_{n-1} \\
f_n & f_{n+1} & f_{n+2} & f_{n+3} & \cdots & f_n & -f_{n-1} \\
f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots & f_n & -f_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{bmatrix}
\]

The characteristic equation of \( H \) is given by \( \det [aI - H] = 0 \).
That is
\[
\alpha^n + \frac{f_{n-1}}{f_1} \alpha^{n-1} + \frac{f_{n-2}}{f_2} \alpha^{n-2} + \cdots + \frac{f_1}{f_n} \alpha + \frac{f_0}{f_n} = 0
\]

Since \( f_n \neq 0 \),
\[
f_{n-1} \alpha^{n-1} + f_{n-2} \alpha^{n-2} + \cdots + f_1 \alpha + f_0 = 0
\]

Thus it is seen that \( T_{kg}(s) \) will be asymptotically stable if and only if all the eigen values of \( H \) have negative real parts. Consider \( \dot{X} = HX \) where \( X \) is a vector of dimension \( n \).
This system will be asymptotically stable if and only if the eigen values of $H$ have negative real parts.

Define a function,

$$ V = X^T P X $$

where $P$ is a symmetric positive definite matrix.

Then $\dot{V} = X^T P \dot{X} + (X^T) F X = X^T P \dot{X} + X^T H^T F X = X^T (H^T P + PH) X$

$V$ is positive for $X \neq 0$ and zero for $X = 0$ (since $P$ is positive definite). According to the method of Lyapunov

if $\dot{V}$ is negative for $X \neq 0$ and zero for $X = 0$ then the system under consideration will be asymptotically stable. For this condition to be satisfied

$$ H^T P + PH = -Q \quad \ldots \quad (3.4) $$

where $Q$ is positive definite.

In the light of the above, $T_k g(s)$ will be asymptotically stable if there exists a symmetric positive definite matrix $P$, such that for any positive definite $Q$, equation 3.4 is satisfied.

Let $Q = (D^{-1})^T D^{-1} = (D^{-1})^{-1} D^{-1}$

Such a $Q$ will be positive definite since it is expressible as the product of a non singular matrix and its transpose. Then,

$$ H^T P + PH = -(D^{-1})^{-1} D^{-1} $$

Premultiplying by $D^T$ and post multiplying by $D$ gives:

$$ D^T H^T PD + D^T PHD = -I $$
That is, 
\[(HD)^T PD + D^T P(HD) = -I \quad \ldots (3.5)\]

It can be seen that

\[
\begin{pmatrix}
0 & k_{0a} & k_{1a} & \cdots & k_{n-2,a} \\
0 & 0 & k_{0a} & \cdots & k_{n-1,a} \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & k_{0a} \\
-s_0 & -s_1 & -s_2 & \cdots & -s_{n-1} \\
\end{pmatrix}
\]

The last row is got by noting (vide equation 3.1) that

\[
f_{0a} k_{0a} = a_0
\]

\[
f_{1a} k_{0a} + f_{0a} k_{1a} = a_1
\]

\[\vdots\]

\[
f_{n-1,a} k_{0a} + f_{n-2,a} k_{1a} + \cdots + f_{0a} k_{n-1,a} = a_{n-1}
\]

Thus \(HD = E\)

Substitution in equation 3.5 gives

\[(T P D + D^T P E) = -I \quad \ldots (3.6)\]

Thus if there exists a symmetric positive definite P satisfying eqn. 3.6 then \(T_{kG}(s)\) will be asymptotically stable. If such a \(P\) exists then it can be expressed as
\[ F = LL^T \], where \( L \) is a lower triangular matrix which is non-singular. (This aspect is discussed in greater depth in chapter 4).

Hence \( T_k(s) \) will be asymptotically stable if the following equation is satisfied:

\[ T_k^2D + D^2LT = -I \]

This completes the proof.

**Proposition 3.2**

\[ t_n = \frac{E_1}{k_{0a}} \]

where,

\[ E_1 = \text{det} \begin{bmatrix} k_{0a} & 0 & 0 & \cdots & 0 & a_0 \\ k_{1a} & k_{0a} & 0 & \cdots & 0 & a_1 \\ k_{2a} & k_{1a} & k_{0a} & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{na} & k_{n-1,a} & k_{n-2,a} & \cdots & k_{1a} & 0 \end{bmatrix} \]

Proof:

The relationships between \( k_{1a}, f_1 \) and \( a_1 \), as defined in equation 3.1 can be written in matrix form as,
On solving for $f_n$ from the above equation, the result of proposition follows.

In the light of these results we note that if $k_0, k_1, \ldots, k_n$ satisfy the condition laid down in Theorem 3.1 then $T_{kg}(s)$ is stable. Use of the theorem and proposition permit one to know a priori whether matching of $k_{ia}$ with $k_m$, $i = 0, 1, 2, \ldots$ is likely to result in a stable $T_{kg}(s)$.

Example 3.3

Let $k_{om} = 1$, $k_{im} = -2$ and $k_{2m} = 6$

Further let $a_0 = 1$, $a_1 = 2$ and $n = 2$

It will now be examined whether setting $k_{im} = k_{ia}$, $i = 0, 1, 2$ will result in a stable $T_{kg}(s)$.

We have,
\[ f_n = f_2 = \frac{\det \begin{vmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 6 & -2 & 0 \end{vmatrix}}{(1)^3} = 2 \]

So,
\[ E = \begin{bmatrix} 0 & 1 \\ -1/2 & -2/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \end{bmatrix} \]
\[ D = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \]
Let \( L = \begin{bmatrix} m_0 & 0 \\ m_1 & m_2 \end{bmatrix} \)
Then \( T_L = \begin{bmatrix} m_0 & 0 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} m_0 & m_1 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} m_0^2 + m_1^2 \\ 2m_0m_1 \end{bmatrix} \]
Hence \( T_L^T D + D^T T_L E = -I \), gives
\[
\begin{bmatrix} 0 & -1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_0 & 2m_1 \\ m_0m_1 & m_1^2 + m_2^2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}
\]
The equations are consistent and yield
\[ m_0^2 = 7/8, \quad m_0m_1 = 1 \quad \text{and} \quad m_1^2 + m_2^2 = 7/4 \]
Hence it follows that setting \( k_{ia} = k_{im} \), \( i = 0,1,2 \) will result in a stable \( T_{kg}(s) \).

The \( T_{kg}(s) \) that results in the present case is \( \frac{2s+1}{2s^2+4s+1} \).

This method can be directly used for checking whether setting \( k_{ia} = k_{im} \) will yield a stable system. It can also be used to impose conditions on \( k_{ia} \) to ensure stability. To alter \( \{k_{ia}\} \) to ensure stability, some of the elements can be treated as parameters. Such a procedure, however, would be somewhat complicated to use. In the specific case, where top priority is given to matching the first time moment, next priority to matching the second time moment and so on, a simpler method can be devised, where \( f_0, f_1 \) etc. will be generated in that order. It will be necessary to check, at each stage, whether a new \( f_i \) will induce instability. For this purpose the Routh array may be written down but that requires the generation of coefficients in the order \( f_n, f_{n-1}, f_{n-2} \) etc. This is circumvented with the help of the following proposition.

**Proposition 3.3**

\[ f_n s^n + f_{n-1} s^{n-1} + \ldots + f_1 s + f_0 = 0 \]  
will have all its roots with negative real parts if and only if \( f_n s^n + f_{n-1} s^{n-2} + \ldots + f_1 s + f_0 = 0 \) has all its roots with negative real parts.
Proof:

Let $Q_1(s) = f_n s^n + f_{n-1} s^{n-1} + \cdots + f_1 s + f_0$

Let $s_1 = 1/s$

Then $Q_2(s_1) = Q_1(1/s_1) = \frac{f_n}{s_1} + \frac{f_{n-1}}{s_1} + \frac{f_1}{s} + f_0$

It follows that if $s = -\alpha$ satisfies $Q(s) = 0$, then

$s_1 = -\frac{1}{\alpha}$ will satisfy $Q_2(s_1) = 0$

Regardless of whether $-\alpha$ is real or complex, if it has a negative real part then $-\frac{1}{\alpha}$ will also have a negative real part. Hence $Q_1(s) = 0$ is equivalent, as far as stability is concerned, to, $Q_2(s_1) = 0$. For $s \neq 0$, $Q_2(s_1) = 0$ implies $s_1 Q_2(s_1) = 0$. This completes the proof.

The procedure suggested is to generate $f_1$ in pairs from $f_0$ onwards and build a Routh table. The advantage in using $f_1$ in pairs is that large portions of the previously constructed Routh array remain undisturbed, thus leading to saving in computation. The effect of adding a new pair of $f_1$ is to increase the number of elements per row of the Routh array by one and to increase the number of rows by two, with each of the added rows having only one entry. If at any stage the addition of an $f_1$ introduces instability then it can be treated as a parameter and limits on it set. Then that particular $f_1$ can be chosen to be as
Close as possible to the value desired, without at the same time violating the stability conditions.

The following figure (Fig. 3.2) illustrates the effect of an additional pair of $f_i$ on an existing table. $N$ denotes a new element, $X$ an old element left unaltered and $\overline{X}$ an old element altered due to the addition of a new pair of $f_i$.

![Diagram showing the effect of addition of a new pair of $f_i$]

Fig. 3.2 Effect of addition of a new pair of $f_i$
Treating an \( f_1 \) as a parameter need not lead to a need to deal with complex polynomials, because of the following result:

**Proposition 3.4**

If any entry of any row of the Routh array is negative then the original polynomial under consideration is not Hurwitz.

**Proof:**

This follows from the fact that a change of sign anywhere in the array ultimately leads to a change of sign in the first column of Routh array. The result follows.

Since a yes or no answer is sought for the stability of the system, in many cases the problem may be settled by solving linear or quadratic expressions in the unknown parameter. The case of dealing with a much higher degree polynomial will be more of an exception, rather than the rule. The next question is whether a stable solution exists at all for the chosen parameter. If there is no solution for stability, then we can take advantage of the following result.
Proposition 3.5

If \( f_0 + f_1 s + f_2 s^2 + \ldots + f_q s^q = 0 \) with \( f_0 > 0 \) has all its roots with negative real parts then there always exists a real \( f_{q+1} \) such that \( f_0 + f_1 s + f_2 s^2 + \ldots + f_q s^q + f_{q+1} s^{q+1} = 0 \) also has its roots with negative real parts.

Proof:

Let \( Q_1(s) = f_0 + f_1 s + f_2 s^2 + \ldots + f_q s^q \)

and \( Q_2(s) = f_0 + f_1 s + f_2 s^2 + \ldots + f_q s^q + f_{q+1} s^{q+1} \)

If \( f_{q+1} \) be chosen to be an infinitely small positive quantity then one notes that \( Q_2(s) \approx Q_1(s) \) in the neighbourhood of the roots of \( Q_1(s) = 0 \). Hence \( Q_2(s) = 0 \) will contain roots at virtually the same locations as those of \( Q_1(s) = 0 \).

There will, however, be an additional root. Since \( f_0, f_1, \ldots, f_{q+1} \) are all positive there can be no root for a real positive \( s \). At best there can only be complex roots with positive real parts. In the present case only one extra root is there. Hence it must be real and negative and will be situated near \( s = -\infty \). Thus \( Q_2(s) = 0 \) also has roots with negative real parts. It follows that there exists an \( f_{q+1} \) such that \( Q_2(s) \) is strictly Hurwitz if \( Q_1(s) \) is strictly Hurwitz. The actual bounds for \( f_{q+1} \) will
be set by $f_0, f_1, \ldots, f_q$. This completes the proof.

If for no value of the final $f_1$ (of the pair introduced)
$T_k(g)(s)$ is stable then set the last parameter $f_1$ in the given
pair to zero treating the other member of the pair as an
unknown parameter. This will give rise to a stable solu-
tion (because of proposition 3.5). Then proceed to assign
the second element of the pair. In this case also a stable
solution can be found, again due to proposition 3.5.

In the light of these results one can achieve time
moment matching by using the earlier described algorithm
(vide section 3.3) for matching a pair of $k_{ia}$ with $k_{im}$ and
generating the corresponding pair of $f_i$. By use of the stabili-
ity test it can be decided whether any alteration is
necessary. If not one proceeds to the next pair. Else
the alteration is made in such a way that the new value
of $f_i$ satisfies the stability constraint and is also as
close as possible to the desired value.

Example 3.4

\[
T_{12}(s) = \frac{1+2s^2-3s^3}{1+s+3s^2+2s^3+s^4}
\]

\[
T_{12}(s) = \frac{2+s+s^2}{6+11s+6s^2+s^3}
\]
Hence $T_{kg}(s) = \frac{g(2+s+s^2)}{c_0 + c_1 s + c_2 s^2 + s^3}$, $g \neq 0$

$= \frac{2+s+s^2}{f_0 + f_1 s + f_2 s^2 + f_3 s^3}$, $f_3 \neq 0$

Use of equation 3.1 for $T_m(s)$ yields,

$k_{0,m} = 1$, $k_{1,m} = -1$, $k_{2,m} = 0$ and $k_{3,m} = -2$

Matching of the first two time moments of $T_m(s)$ and $T_{kg}(s)$ gives $f_0 = 2$ and $f_1 = 3$.

At this stage Routh stability test is satisfied. Hence matching the next two time moments is attempted and this yields,

$f_2 = 4$ and $f_3 = 8$

Now the polynomial whose stability is under investigation is $u(s) = 2s^3 + 3s^2 + 4s + 8$ whose Routh array becomes,

\[
\begin{array}{ccc}
 s^3 & 2 & 4 \\
 s^2 & 3 & 8 \\
 s^1 & -4 \\
 s^0 & 8 \\
\end{array}
\]

(The fractions have been cleared)
Clearly the polynomial is unstable. Hence \( f_3 \) is treated as an unknown parameter \( x \) and the Routh array constructed in terms of \( x \). The modified Routh table now reads:

\[
\begin{array}{c|cccc}
0 & s^3 & 2 & 4 \\
1 & s^2 & 3 & x \\
2 & s^1 & 12-2x \\
3 & s^0 & x \\
\end{array}
\]

(Fractions have been cleared)

Stability considerations demand,

\[ x < 6 \text{ and } x > 0 \]

i.e. \( 0 < x < 6 \)

It is noted that for time moment matching \( f_3 \) should be 8. Hence the nearest one can approach this value, consistent with stability is by assigning \( f_3 \) some value slightly less than 6. A value of \( f_3 = 5.9 \) will be adequate for the present purpose. Clearly the denominator polynomial of \( T_{kg}(s) \), whose coefficients were generated in the reverse order, takes the form \( 5.9s^3 + 4s^2 + 3s + 2 \).

Hence

\[
T_{kg}(s) = \frac{s^2 + s + 2}{5.9s^3 + 4s^2 + 3s + 2}
\]
3.5 CONCLUSION

In this chapter a method of matching time moments, which also assures system stability, was presented. This method will lead to exact matching, if it is possible, since it is known that \( T_{kg}(s) \) matches \( T_m(s) \) if and only if the time moments are identical.