CHAPTER 1

INTRODUCTION

1.1 PERTURBATION EXPANSIONS: BASIC CONCEPTS

A theoretician translates a physical problem into a mathematical model and attempts an analysis on it. But quite often, it is found difficult to obtain exact analytical solutions for these problems. These difficulties arise due to a variety of features exhibited by these models, such as nonlinearities in the governing equations (endangered more due to the presence of variable coefficients), complex boundary shapes, nonlinear boundary conditions and so on.

The fundamental equations that govern fluid flows are the Navier-Stokes equations. The constant property Navier-Stokes equations (adopted in this thesis) are

\[
\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q},
\]

\[\nabla \cdot \mathbf{q} = 0,\]  \hspace{1cm} (1.1)

where \(\mathbf{q}(x,t)\), a function of space \(x\) and time \(t\) is the fluid velocity vector, \(p(x,t)\) is the pressure. The density \(\rho\) and the kinematic viscosity \(\nu\) are constants. Although \(\rho\) can be eliminated by taking the curl of Equation (1.1), the constant \(\nu\) (or, in non-dimensional form, the Reynolds number) is a basic parameter. An exact solution is defined as a solution of Equations (1.1) and (1.2) that is valid for all \(x,t\) and for all values of \(\nu\). Obviously, all closed form solutions, including similarity solutions of Equations (1.1) and (1.2) are exact solutions. Solutions of Navier-Stokes equations have come to be accepted as constituting a primary
standard for data on laminar flows. A comprehensive review of these exact solutions is given by Berker (1936, 1963), Dryden et al (1932), Whitham (1963) and Schlichting (1979). Of more recent origin are the review articles by Wang (1989, 1991) for unsteady and steady exact solutions, highlighting the current status. However all these solutions are restricted to a limited class of problems involving simple geometries and simple conditions. Even for these problems the analytical techniques adopted are of varied nature, requiring ingenious methods. This is because of the insurmountable mathematical difficulties due to the number of coupled nonlinear partial differential equations to be simultaneously satisfied. Also, because of the non-linearities, the superposition principle is not applicable and complex flows may not be compounded from simple flows. Hence no unified approach is possible and no standard analysis exists to obtain an exact solution in general.

Therefore, we have to resort to approximate analytical methods or fully numerical procedures or a suitable combination of both, in order to obtain relevant information about solutions of relatively more complex problems. During the past three decades, there had been an explosive growth of numerical computations as a tool for fluid mechanical and heat transfer analysis, dominated by two powerful procedures involving finite differences and finite elements. Inspite of this overwhelming trend, approximate analytical methods have continued to develop and provide useful solutions to a variety of problems. These solutions serve as an yardstick to strengthen the validity of numerical computations as well, hence they retain their usefulness all the more.

The method of perturbation (asymptotic) expansion stands foremost among the approximation methods. This method is implicitly based on the assumption that an exact solution of a physical problem has an asymptotic expansion of certain form. To identify this form explicitly, first the model is recasted into a dimensionless form, so that the parameters and or variables governing the system are established. Assessing their order of magnitude, a parameter or a variable that is small compared to others is identified. This quantity, \( \epsilon \), is designated as the
perturbation quantity. A relevant expansion describing the analytic dependence of the exact solution $f$ on this parameter $\varepsilon^*$, is the desired asymptotic expansion for $f$ as $\varepsilon^* \to 0$. Knowing the limit behaviour of $f$ as $\varepsilon^*$ approaches zero, more precisely the rate at which this limit is approached, an asymptotic sequence of appropriate gauge functions $\{g_n(\varepsilon^*)\}$, characterised by the property

$$\lim_{\varepsilon^* \to 0} \frac{g_n(\varepsilon^*)}{g_{n-1}(\varepsilon^*)} = 0,$$

is constructed. The exact form of the asymptotic expansion of $f(x;\varepsilon^*)$, where the independent variable $x$ may be a scalar or a vector is now developed in a straightforward expansion of the form

$$f(x;\varepsilon^*) = \sum_{n=0}^{\infty} f_n(x) g_n(\varepsilon^*). \quad (1.3)$$

A special form of Equation (1.3), used most commonly in conventional perturbation theory, is a power series expansion in terms of the parameter $\varepsilon^*$, viz.,

$$f(x;\varepsilon^*) = \sum_{n=0}^{\infty} (\varepsilon^*)^n f_n(x). \quad (1.4)$$

Substituting the expansion given by Equation (1.3) or (1.4) in the governing equations of the problem and equating like terms on either side, we are led to a sequence of problems which can be solved in succession to obtain the coefficients $f_n(x)$. This helps us to learn qualitatively about the solution, with a comparatively lesser effort than otherwise from the exact solution. With this evaluation of the expansion given by Equation (1.3) to a desired level, one important question that can be asked is the following. How far is the asymptotic solution valid in the domain of the independent variable?
The expansion may be uniformly valid in the entire domain. By this we mean that the expansion is a valid representation of the solution $f$ throughout the domain under consideration. Mathematically this means that if we write $f$ in the form

$$f(x; \epsilon^*) = \sum_{n=0}^{N-1} f_n(x) g_n(\epsilon^*) + R_N(x; \epsilon^*), \quad (1.5)$$

the truncation error $R_N$ should be such that

$$R_N(x; \epsilon^*) = O(g_N(\epsilon^*)), \quad (1.6)$$

i.e., $\lim_{\epsilon^* \to 0} \left| \frac{R_N}{g_N} \right| < \infty$ uniformly for all $x$.

In simple language this only means that each term must be a small correction to the preceding term irrespective of the value of $x$. Such an uniformly valid expansion is termed as regular perturbation expansion. Instead, the expansion $(1.3)$ may violate the condition $(1.6)$, with the result it lacks the feature of uniform validity. Such an expansion is called singular perturbation expansion. This nonuniformity manifests in several forms: (i) the solution becomes infinite at some values of the independent variable (ii) the solution becomes discontinuous within the domain (iii) the solution violates the boundary conditions (iv) the solution contains an essential singularity. Some of the sources of these nonuniformities are (i) infinite domain (ii) small parameter multiplying the highest derivative (iii) character/type change of the partial differential equation and (iv) presence of singularities. A good description of these nonuniformities and detailed account of special techniques employed for achieving uniform validity from these singular expansions can be found in the books by Cole (1968), Nayfeh (1973, 1981), van Dyke (1975) and Aziz and Na (1984).
The present thesis employs both regular and singular perturbation techniques, to solve some problems in viscous flows including convective heat transfer. The study undertaken in this work is the result of motivation derived from the applications of fluidmechanical problems to viscometry, suspension rheology, human circulatory system, industrial design and manufacture of products involving improved transfer of heat and mass such as power generators, chemical reactors, superheaters/coolers and so on. The thesis consists of six chapters. The present chapter, giving a general introduction, is listed as Chapter 1.

1.2 HYDRODYNAMIC INTERACTION

Transport phenomena in a two phase (heterogeneous) media involving the motion of aggregate of small particles, moving relative to fluids in which they are immersed, occurs in a variety of fields. Particles may move together in bulk through a fluid as in sedimentation, or, remain more or less stationary as in a packed bed. When two particles in a suspension approach each other, the velocity field generated by the motion of one particle is transmitted through the fluid medium and felt by the other particle, even in the absence of inter-particle interactions such as van der Waal's and electrostatic forces. Hence the trajectory of each particle is influenced by the presence of the other. This effect is called hydrodynamic interaction. The magnitude of this interaction among the particles is, in general, governed by the following factors: (i) their shapes and sizes (ii) the distances between them (iii) their orientation with respect to each other (iv) their individual orientations relative to external force field (v) their velocities and spins relative to the fluid at infinity. For specified particle motions (prescribed by translational and angular particle velocities) in the ambient fluid, the macroscopic parameters of primary physical interest are the hydrodynamic forces, torques and stresslets exerted by the fluid on the particles. The problem of determining such forces is defined as the resistance problem, Brenner and O'Neill (1972). Once the particle forces and torques are prescribed for a given array in the ambient fluid, we can solve the inverse problem of determining the particle motion and stresslets. Such a problem is defined as mobility problem, Batchelor (1976).
These problems jointly characterise the phenomena of hydrodynamic interaction. Information concerning such interaction is needed in the theoretical investigations of the behaviour of suspensions of small (submicron) particles as shown in the review articles by Batchelor (1974), Jeffrey and Acrivos (1976). Specific applications are found in microhydrodynamics and colloid science, which includes particle pair interactions in polydisperse suspensions (aggregate growth from monomers, seeded growth), interaction between non-spherical particles (clay suspensions and fibrous suspensions) and N-body interactions (direct simulation of suspension behaviour). In problems of this kind, in general, the particle based Reynolds number satisfies $\rho V a / \mu << 1$, where $a$ is the sphere radius, $V$ is the characteristic velocity of the particle (relative to the ambient fluid motion) and $\rho$ and $\mu$ are the density and viscosity of the fluid medium. For such cases, the fluid motion is adequately described by the Stokes equation

$$\nabla p + \mu \nabla^2 q = 0,$$

(1.7)

$$\nabla \cdot q = 0.$$

(1.8)

The velocity field $q$ and the hydrodynamic interactions between the particles can be determined by solving the Equations (1.7) and (1.8) with appropriate boundary conditions on the particles. For rigid particles these are the no slip conditions, while for viscous drops we assume kinematic conditions for the velocity component normal to the fluid-fluid interface and continuity of tangential velocities and stresses. Because of the linearity of the governing equations of motion and boundary conditions, the two general motions which the rigid particles may undergo, namely translation and rotation, may be independently investigated and the results superposed. Further, the principle of superposition helps in developing a suitable geometrical frame work for the description of the geometry of the problem and for satisfying the boundary conditions on different bodies of the fluid. Various methods are developed for studying these problems. They can be classified as follows: (i) use of a suitable (bispherical) coordinate system, Stimson and Jeffrey (1926) (ii) method of reflections or

In the present thesis, the problem of hydrodynamic interaction between two rigid (identical) spheres in low Reynolds number-flow is studied. The flow is caused due to slow and steady rotation of the bodies about their line of centres. The flow pattern is no longer symmetric with regard to the equatorial planes of the spheres. Consequently, the spheres must experience drag along the axis in addition to the torque. The calculation of these resistance functions forms the subject matter of Chapter 2. The Navier-Stokes equations are linearized following an asymptotic expansion for the flow field in terms of the parameter \( R_0 \) (< 1, Reynolds number due to rotation). Method of reflections is used to solve these equations, but in a slightly different way. Since the equations are linear and are also form-invariant under a translation of the origin, we can write down the solutions independently with respect to the centres of two spheres as working origins and superpose them. Use of the identities on spherical harmonics (these identities are given in Appendix 1), Whittaker and Watson (1962), enables us to transfer the solutions to either origin and consequently satisfy the boundary conditions simultaneously on both the spheres. As a result we obtain locally valid solutions, which are sufficient to calculate the resistance functions. Interestingly, it is seen that the present result predicts a reduction in torque with increasing speed thereby showing the influence of second body in the medium. The drag on either body is also found to be considerably reduced in comparison with the results of Arunachalam and Majhi (1987).
Chapter 3 deals with a similar two-body problem, with the spheres replaced by two nearly spherical solids of revolution (not necessarily identical), \( r = 1 + \eta f_1(\theta) \) and \( r' = 1 + \eta f_2(\theta') \). Assuming that \( R_0 \) and \( \eta \) (roughness parameter) are of same order of smallness, general expressions for torque and drag are obtained up to \( O(R_0^m \eta^n)_{m+n \leq 2} \) for arbitrary (complex) shapes of the bodies. Corresponding particular computations for typical solids characterised by \( f(\theta) = \cos \theta, P_3(\cos \theta) \) and \( \sin 2\theta \) are derived. The dependence of the shape of the bodies, distance between them etc. on the drag are discussed through graphs.

1.3 INTERNAL FLUID FLOW

Study of internal fluid flow is of fundamental importance in engineering and provides information concerning design data for friction and pressure losses in pipes. If a fluid, flowing along an initially straight pipe of constant cross-section, encounters a change in direction, the fluid near the axis of duct, having the highest velocity, is subjected to a larger centrifugal force than the fluid moving slowly in the neighbourhood of the duct walls. This gives rise to a flow transverse to the primary axial flow. This transverse flow is known as secondary flow. Due to the interaction of primary and secondary flows, fluid in the central region of the pipe moves away from the centre of curvature and the fluid near the duct walls flows towards the centre of curvature. Hence for ducts of symmetrical cross-section with respect to the plane of curvature, the secondary flow consists of a pair of helical vertices. Subsequent development of the primary and secondary flows depends critically on the geometry of the bend giving rise to the change in flow direction, Ward-Smith (1980). Dean (1927) studied fully developed flow in a curved pipe and demonstrated theoretically the screw-like motion of the fluid. The dye injection experiments of Eustice (1910, 1911) also show the existence of such secondary flows. But the volume flow rate remained independent of the curvature \( \epsilon \) of the pipe and was still given by Poiseuille law. Using an approximate analysis, Dean (1928) extended his earlier work and obtained an expression for flow rate up to \( O(\epsilon^4) \). This expression has recently been confirmed by van Dyke (1978). The exact analytical solutions for the flow field, up to \( O(\epsilon^2) \), was later obtained.
by Topakoglu (1967). His expression for flow rate, which incidentally is a correction to Dean's result, is valid for moderately large Dean number and brings out the full effect of curvature of the coiled pipe on the volume transport. This expression also agrees with Larrain and Bonilla (1970). The flow at high Dean numbers was analysed by Barua (1963), Mori and Nakayama (1965) and Ito (1969). A full description of the numerical determination of flow in curved pipe is given by McConalogue and Srivastava (1968), Austin and Seader (1973), Collins and Dennis (1975) and Smith (1976). Patankar, Pratap and Spalding (1974) have obtained numerical solutions for developing flows. These analytical and numerical results are very well correlated to the empirical equation for the friction factor given by White (1929), based on his experimental results. Murata et al (1976) studied the problem of varying curvature. However, limited results, Truesdell and Adler (1970), Murata et al (1981), are available in the literature concerning flows in a helical pipe. Helical pipes involve both a curvature and a torsion or twist. Wang (1981) is the first person to examine the effect of both curvature and torsion on the flow in a helical pipe. The volumetric flow rate obtained by Wang is identical to that of Topakoglu (1967) and hence the effect of torsion is not seen. Chapter 4 is born out of motivation to realise this effect in the helical pipe flow. The curvature $\epsilon$ and torsion $\sigma$ of the central generic curve are assumed to be small such that $\delta = \tan \xi = \epsilon/\alpha < 1$, where $\xi$ is the helical angle. Pursuing a regular perturbation analysis in terms of $\delta$, the expression for flow rate is obtained up to $O(\delta^m \sigma^n)_{m+n\leq 6}$. This expression is numerically computed and compared with the predicted results of Wang.

1.4 CONVECTIVE HEAT TRANSFER

Heat transfer considerations are often of crucial importance in modern engineering design. A detailed analysis is needed to finalise the design or size of the equipment necessary to transfer a specified amount of heat in a given time. More precisely, we need to work out the details concerning the rate of heat transfer under the specified conditions. The design of boilers, airconditioning and refrigerator systems or other types of heat exchangers, the storage, transfer and use of cryogenic substances and a large number of similar applications involve
consideration of heat transfer rates. The mechanism by which heat is transferred in a heat exchanger or an energy conversion system is classified under three distinct modes. These are (i) conduction (ii) convection and (iii) radiation. Transfer of heat energy through thermal motion of the microscopic particles of which the material is composed, supplemented in some cases by the flow of free electrons, thereby diffusing the energy through the material, from a region of higher temperature to a region of lower temperature, is called conduction. Radiation is the transport of energy from the material into the surrounding space by electromagnetic waves. This radiant emission is similar to heat conduction, with a difference, the energy is now transmitted electromagnetically. Also, radiation can occur even in a vacuum, whereas conduction requires a material medium. The process of energy transport due to the combined effect of conduction (and radiation) in a fluid medium is basically altered by the relative motion in the fluid. Thermal energy is carried in the flow region by molecular motion. This mechanism of heat transfer is called convection. This is important not only between layers of a fluid but also between a fluid and solid surface when they are in contact.

The convective mode of heat transfer is generally divided into two basic categories, motivated by any of the following two processes: (i) the flow field is imposed, as in placing an object in a stream, or induced by external agencies such as a pump, blower or fan or the motion of the heated object itself. Such a process is called 'forced convection'. (ii) The fluid velocities arise naturally as a result of density gradients due to temperature or concentration difference in a body force field such as gravity. Heat exchange in such a situation is called 'free or natural convection'. A detailed account of these processes can be found in the works of Gebhart (1961), Ozisik (1978, 1980), Kakac and Yener (1980), Schlichting (1979), Bejan (1984). The latest design and research information in the area of single-phase convective heat transfer is given in the hand book by Kakac, Shah and Win Aung (1987).
In the problem of external flow forced convection, the primary interest is the computation of heat transferred from the body to the ambient medium or vice-versa. It is customary to express this characteristic in terms of a surface-coefficient $h$ (also called heat transfer coefficient or film coefficient). This is defined as (Newton's law of cooling)

$$q_w = h (T_w - T_\infty) ,$$  \hspace{1cm} (1.9)

where $q_w$ is the heat flux at the wall, $T_w$ is the wall temperature and $T_\infty$ is the free stream/reference temperature. Also, from the Fourier's law (the basic law governing heat conduction based on continuum concept)

$$q_w = -k \left( \frac{dT}{dn} \right)_w ,$$  \hspace{1cm} (1.10)

where $k$ is the thermal conductivity and $T$ is the temperature of the fluid. The subscript $n$ denotes the normal direction to the surface and the subscript $w$ means the evaluation of the said quantities on the surface. From Equations (1.9) and (1.10), we get

$$h = \frac{-k \left( \frac{dT}{dn} \right)_w}{T_w - T_\infty} .$$  \hspace{1cm} (1.11)

Multiplying the Equation (1.11) by $L / k$, where $L$ is a characteristic length, a dimensionless combination is formed on the left-hand side, which is called the Nusselt number. It is given by the relation

$$N = \frac{hL}{k} = \frac{L \left( \frac{dT}{dn} \right)_w}{T_w - T_\infty} = \frac{L q_w}{k(T_w - T_\infty)} .$$  \hspace{1cm} (1.12)

The last form suggests that Nusselt number measures the ratio of the actual wall heat flux to that which would occur by conduction alone, across a layer of thickness $L$. This parameter $N$ thus characterises the physical process of
convective heat transport. Chapters 5 and 6 are case studies of forced convection.

Numerous theoretical investigations have been reported in recent years, Acrivos (1962), Rimmer (1968, 1969) and Dennis et al (1973) centering around the classical problem of heat and mass transfer from a solid sphere in a low Reynolds number flow. The primary problem in this context is the flow past a sphere, which has an extensive literature. An analysis to obtain higher order approximations beyond the Stokes' (1851) solution, is complicated and leads to the well-known Whitehead's paradox. The complication is due to the fact that the perturbation expansion for the flow field in terms of the Reynolds number, for the flow in the vicinity of the sphere, is not valid at large distances from the sphere. It has therefore to be matched with another expansion valid far from the sphere. Hence the problem leads to singular perturbation expansions. A good historical survey of the problem of slow viscous flow past a sphere and a detailed description of the application of matched asymptotic expansions to this problem is given by Proudman and Pearson (1957). This work is a case study based on the general principles of singular expansions proposed by Lagerstrom and Cole (1955). The solutions for heat transfer in a viscous liquid confined between a pair of concentric rotating spheres is given recently by Bhatnagar and Vayo (1989), using regular perturbation technique.

The point of natural interest that can now arise is to examine how heat transfer characteristics get affected when a sphere is caused to spin in a streaming flow. The corresponding flow problem was investigated by Rubinow and Keller (1961) and Takagi (1974). Ranger (1971) presented a similar problem of axially symmetric flow past a rotating sphere due to a uniform stream at infinity. Based on creeping flow, he observed an interesting phenomena of reverse flow in the aft region of the sphere. However, the drag force was still given by Stokes law. This was later improved by Vasudevaiah and Majhi (1982) to account for the effect of spin. This velocity field is adopted to study the convective heat transfer. This forms the subject matter of Chapter 5. The physical parameters of interest here,
namely, \( \text{Re} \) (Reynolds number due to uniform streaming), \( R_0 \) (Reynolds number based on axial spin), \( \text{Pr} \) (Prandtl number) are such that \( \text{Re} < 1, \ R_0 < 1, \ R_0^2 / \text{Re} = O(1) \) and \( \text{Pr} = O(1) \). Using the method of matched asymptotic expansions, locally valid expansions for the temperature field close to and far away from the sphere are obtained upto \( O(\text{Re}^2 \text{Pr}^2) \). The expression for the mean Nusselt number, \( \text{Nu} \), describing the rate of heat transfer from the body to the fluid medium is obtained. It is found that while the spin affects the drag force by a term of \( O(R_0^2) \), its contribution to \( \text{Nu} \) is of \( O(R_0^4) \). When \( \text{Re}.\text{Pr} < 1 \), the spin tends to increase the rate of heat transfer, while the effect gets reversed when \( \text{Re}.\text{Pr} > 1 \).

Chapter 6 deals with heat transfer in parallel flow between corrugated plates. The axial corrugations are designed as sinusoidal waves of small amplitude \( b \ (<< a, \ \text{the mean gap width between the plates}) \) and are characterised by a phase difference \( \beta \). From thermal considerations, it is important to examine how and when, such a design can maximise/minimise heat transfer in the channel. This chapter consists of three sections. In the first section, the flow field imposed is the classical Couette flow. It is found that heat transfer from the warmer (moving) plate increases with \( \beta \) and reaches a maximum when \( \beta = \pi \). This feature is physically consistent with the drag behaviour, Wang (1976 a). In the second section, a similar analysis is pursued when both the plates are at equal temperature. In the third section, heat transfer in Poiseuille flow between fixed corrugated plates, Wang (1976 b) is considered. It is found that, by designing the warmer plate with a suitable phase difference, maximum heating/cooling can be achieved depending on the values of the parameter \( \text{Ec}.\text{Pr} \), where \( \text{Ec} \) is the Eckert number and \( \text{Pr} \) is the Prandtl number.