CHAPTER 3

THEORETICAL FORMULATION FOR
HOOP WRAPPED CYLINDER

3.1 INTRODUCTION

The hoop wrapped construction consists of only hoop layers on the cylindrical portion and this makes the vessel construction as specially-orthotropic with the fiber orientation coinciding with one of the principal loading directions. Considering the vessel construction, a separate analytical formulation has been developed for the prediction of stresses and strains of hoop wrapped cylinder.

The basic idea of making hoop wrapped cylinder is to achieve a uniform strength construction. In a typical full-metallic cylinder, the longitudinal stress reaches only half the value of hoop stress at the operating pressure. In order to equalize the stresses in hoop and longitudinal directions, the metal thickness is reduced in the cylindrical portion and a composite overwrap is added such that equal stresses are developed in both the principal directions when the cylinder is loaded.

For hoop wrapped cylinders, since there is no composite wound on the dome, the liner thickness over the dome is designed to carry the full pressure load as per full-metallic cylinder design practice.
3.2 ANALYTICAL FORMULATION

The analytical model for the hoop wrapped cylinder is developed using the fundamental principles of force equilibrium between the applied pressure force and the material resisting force. The analytical model assumes the hoop wrapped cylindrical body of the vessel as a thin walled cylinder consisting of an inner metallic layer and an outer composite hoop layer. The composite thickness consists of a number of individual plies wound with the same angle continuously and hence considered as a single layer with orthotropic material properties. A segment is isolated by passing two planes perpendicular to the axis of the cylinder as shown in Fig. 3.1.

For the vessel subjected to internal pressure, the conditions of pressure vessel symmetry exclude the presence of any shear stresses in the planes of sections during normal operation. The shear stresses developed
between the liner and composite due to thermal effects in the initial curing process also gets released at the end of curing as analyzed in Chapter 2. The autofrettaging process results in a permanent compaction of composite overwrap on the liner surface, interface continuity is established for load transfer during normal loading cycles. Therefore, the stresses that exist on the sections of the cylinder are normal stresses in hoop and longitudinal directions only. A segment is isolated by passing two planes perpendicular to the axis of the cylinder as shown in Fig.3.1.

3.2.1 Elastic Loading

The stress pressure relationship for the hoop wrapped cylindrical section is obtained by equating the pressure force acting on the top and bottom of the wall section and the resisting force offered by the wall section. Referring to Fig.3.1, the force on the infinitesimal area of the cylinder caused by the internal pressure is given as:

\[
\text{Force on the infinitesimal area of the cylinder} = PLR \, d\Phi \quad (3.1)
\]

\[
\text{Horizontal component of this force} = PLR \, d\Phi \cos\Phi \quad (3.2)
\]

\[
\text{Total horizontal force for full cylinder} = \int_0^{\pi/2} PRL \, d\Phi \cos\Phi
\]
\[= 2 \, PRL \quad (3.3)\]

The resisting force offered by the wall section is calculated for the segment considering the top and bottom wall sections.

\[
\text{Resisting force offered by the wall section} = 2\sigma_{yl} \, t_1 \, L + 2 \, \sigma_{ye} \, t_c \, L \quad (3.4)
\]
Equating the applied pressure load to the resisting force,

\[ 2PRL = 2\sigma_{yl} t_I L + 2\sigma_{yc} t_c L \]

\[ PR = \sigma_{yl} t_I + \sigma_{yc} t_c \]  

Equation (3.5)

Where,  
- \( P \) - Internal Pressure  
- \( R \) - Inner radius of the cylinder  
- \( \sigma_{yl} \) - Hoop stress in the liner  
- \( \sigma_{yc} \) - Hoop stress in the composite

Equation (3.5) relates the hoop stresses in the liner and the composite overwrap. Applying the condition of hoop strain compatibility between liner and composite, the following equation is obtained.

\[ \frac{\sigma_{yl}}{E} - \frac{\nu \sigma_{xl}}{E} = \frac{\sigma_{yc}}{E_1} - \frac{\nu_{21} \sigma_{xc}}{E_2} \]  

Equation (3.6)

Where,  
- \( E \) - Modulus of elasticity of liner material  
- \( E_1 \) - Modulus of elasticity of composite in the fiber direction  
- \( E_2 \) - Modulus of elasticity of composite in the transverse direction  
- \( \nu_{12} \) - Major Poisson’s ratio  
- \( \nu_{21} \) - Minor Poisson’s ratio

For the isotropic liner, \( \sigma_{xl} = \frac{1}{2}\nu_{21}\sigma_{yl} \). Substituting this in equation (3.6) gives:

\[ \frac{\sigma_{yl}}{E} - \frac{\nu \sigma_{yl}}{2E} = \frac{\sigma_{yc}}{E_1} - \frac{\nu_{21} \sigma_{xc}}{E_2} \]
\[
\sigma_yl \left( \frac{1}{E} - \frac{\nu}{2E} \right) = \sigma_{yc} \frac{\nu_{21} \sigma_{xc}}{E_1} \]

\[
\sigma_yl \left( \frac{2 - \nu}{2E} \right) = \sigma_{yc} \frac{\nu_{21} \sigma_{xc}}{E_1} \quad (3.8)
\]

In equation (3.7), \( \nu_{21} \) is the Poisson's ratio of the composite that characterizes the strain in the hoop direction due to strain acting in the axial direction. \( \nu_{21} \) being a very small quantity for hoop layers, the axial strain contribution is neglected and hence equation (3.7) is rewritten as:

\[
\sigma_yl \left( \frac{2 - \nu}{2E} \right) = \sigma_{yc} \frac{\nu_{21} \sigma_{xc}}{E_1} \quad (3.8)
\]

Equations (3.5) and (3.8) are solved to get the hoop stress components in closed form in the liner and in the composite overwrap.

From equation (3.8):

\[
\sigma_y = \left[ \sigma_{yc} \frac{2E}{E_1(2 - \nu)} \right] \quad (3.9)
\]

Substituting equation (3.9) in (3.5) gives:

\[
\sigma_{yc} \cdot t_l \left( \frac{2E}{E_1(2 - \nu)} \right) + \sigma_{yc} \cdot t_c = PR
\]
Substituting equation (3.10) in (3.9) and simplification gives:

\[
\sigma_{yl} = \frac{2 \frac{PR}{E}}{2E \frac{t_l}{t_l + \left(\frac{E_l}{E}\right)(2 - v) t_c}}
\]

Equations (3.10) and (3.11) are used to predict the hoop stresses in the liner and the composite overwrap in terms of the applied pressure upto the point of liner yielding. The longitudinal stress in the liner, neglecting the transverse load carrying capacity of the hoop layered composite overwrap, is given as:

\[
P_R^2 = \frac{\sigma_{xl} (2\pi R)}{t_l}
\]

\[
\sigma_{xl} = \frac{PR}{2t_l}
\]
3.2.2 Prediction of Liner Yielding

By putting the yield strength value of the liner material in equation (3.13), the pressure corresponding to the liner yielding is computed. Liner yielding will occur when the induced hoop stress in the liner ($\sigma_{yi}'$) reaches the material yield strength ($S_{yi}$).

$$\sigma_{yi}' = S_{yi} \quad (3.14)$$

The liner yielding pressure ($P_{yi}$) is obtained as:

$$P_{yi} = \frac{S_{yi} \{ 2 t_i + (E_i/E) (2 - v) t_c \}}{2R} \quad (3.15)$$

The composite hoop stress at the point of liner yielding is obtained from equation (3.9) as:

$$\sigma_{yc}' = \frac{\sigma_{yi}'}{2E / (E_i(2 - v))}$$

$$\sigma_{yc}' = \frac{S_{yi}}{2E / (E_i(2 - v))} \quad (3.16)$$

The longitudinal liner stress at the point of liner yielding is given as:

$$\sigma_{xl}' = \frac{P_{yi}R}{2t_i} \quad (3.17)$$
3.2.3 Stresses at Autofrettage Pressure

Liner is assumed to yield at a single pressure under the assumptions of thin walled cylinder and hence upon increasing the pressure, the stress-strain relationship is governed by the same rules of multilayer construction; however the liner will be characterized by the plastic modulus and plastic Poisson’s ratio values in the elasto-plastic region.

For the purpose of analysis, the pressure above the liner yielding pressure is considered as the plastic pressure as done for fully wrapped cylinder analysis.

Plastic pressure component, \( P_{pl} = P_s - P_{yl} \)

Where,  
\( P_s \) - Autofrettage pressure  
\( P_{yl} \) - Liner yielding pressure

The stress components at autofrettageing pressure are given as:

\[
\sigma_{yl}^a = \sigma_{yl}^y + \sigma_{yl}^{pl} 
\]

\[
\sigma_{yl}^y = \frac{2 \ P_{yl}R}{2t_i + (E_i/E_p)(2 - v_p) \ t_c} 
\]

\[
\sigma_{yl}^{pl} = \frac{2 \ P_{pl}R}{2t_i + (E_i/E_p)(2 - v_{pl}) \ t_c} 
\]
\[ \sigma_{yl}^a = \frac{2 \, P_{yl}R}{2 \, t_i + (E_i/E) \,(2 - v) \, t_c} + \frac{2 \, P_{pl}R}{2 \, t_i + (E_i/E_{pl}) \,(2 - v_{pl}) \, t_c} \]

(3.21)

\[ \sigma_{yc}^a = \frac{\sigma_{yl}^y}{2E/(E_i(2 - v))} + \frac{\sigma_{yl}^{pl}}{2E_{pl}/(E_i(2 - v_{pl}))} \]

(3.22)

\[ \sigma_{sl}^a = \frac{P_{yl}R}{2t_i} + \frac{P_{pl}R}{2t_i} \]

(3.23)

Actually, there is no plastic component of stress in the composite overwrap. Since the applied autofrettage pressure is divided into two components based on liner elasto-plastic analysis, the plastic stress component of composite corresponds to the stress developed for the pressure component, \( P_{pl} \).

The stresses and strains in the liner and in the composite of hoop wrapped construction are predicted upto autofrettage pressure using the expressions developed in this section. The methodology developed for the fully wrapped cylinder is used for unloading analysis and residual stress prediction.
3.3 RESULTS AND DISCUSSION

3.3.1 Stress Output for Hoop Wrapped cylinder

The hoop wrapped cylinder is also designed with the same input specifications used for the fully wrapped cylinder. The final stress output is obtained after a few iterations and is given in Table 3.1. The stress-pressure plot is shown in Fig. 3.2. The stresses in the liner and the composite are optimized so as to reach closely their ultimate strength values at burst pressure. The optimized liner thickness is 4.3 mm and the composite thickness is 6.0 mm.

The important observations with the hoop wrapped cylinders are given below.

Table 3.1 Stress Output for Hoop Wrapped cylinder

<table>
<thead>
<tr>
<th>Pressure (MPa)</th>
<th>Liner stresses (MPa)</th>
<th>Fiber stresses (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hoop</td>
<td>Longitudinal</td>
</tr>
<tr>
<td>Zero Pressure</td>
<td>0</td>
<td>-508.6</td>
</tr>
<tr>
<td>Service Pressure</td>
<td>20</td>
<td>309.7</td>
</tr>
<tr>
<td>Test pressure</td>
<td>30</td>
<td>617.7</td>
</tr>
<tr>
<td>Autofrettage Pressure</td>
<td>31.5</td>
<td>665.0</td>
</tr>
<tr>
<td>Burst Pressure</td>
<td>60</td>
<td>831.4</td>
</tr>
<tr>
<td>Liner thickness: 4.3 mm</td>
<td>Hoop composite thickness: 6.0 mm</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3.2 Stress-Pressure Graph of Hoop Wrapped Cylinder

The compressive stress developed in the liner in hoop direction is 508 MPa while in the longitudinal direction a negligible value of 8 MPa is developed. This clearly demonstrates the absence of the longitudinal resistance by the hoop layers.

In hoop-layered construction, due to the absence of helical layers, there is no helical-to-hoop interaction. This provides the opportunity to design the hoop layers more precisely to meet the burst strength requirements.
3.3.2 Minimum Thickness of Liner

The basic idea of hoop wrapped construction is to equalize the stresses in the hoop and longitudinal directions in the cylinder. In order to achieve this, the metal thickness is reduced in the cylindrical portion and a composite overwrap is added in such a way that when the cylinder is loaded equal stresses are developed in both the principal directions. From this consideration, the thickness of the liner is decided based on the longitudinal stress due to the internal pressure load. In the hoop direction, the composite hoop layers will provide additional load carrying capacity so that the combined liner and the composite overwrap strength meets the hoop pressure load requirement.

Applying this condition, the required minimum thickness of the liner is half the value of a full metallic cylinder thickness. The liner thickness obtained through the present stress analysis (4.3 mm) very well matches with this condition. Hence, it is concluded that the minimum liner thickness is predominantly decided by the longitudinal load carrying capacity of the liner.

3.3.3 Thickness Ratio Between Liner and Composite

In the previous section, it is seen that the minimum liner thickness is predominantly decided by the longitudinal load carrying capacity of the liner. However, the hoop load carrying capacity of the liner is decided based on the amount of residual compressive stress developed in the liner. Interestingly, by increasing the liner residual stress, the hoop composite thickness gets reduced. Hence, it is possible to vary the thickness ratio between the liner and composite to achieve an optimum hoop stress distribution. Fig. 3.3 shows the residual
compressive stress developed in the liner for a thickness variation between 4mm to 5 mm. The composite thickness required for different liner thicknesses is shown in Fig.3.4.

![Residual Stress for Different Liner Thicknesses](image)

**Fig.3.3 Residual Stress for Different Liner Thicknesses**

The liner compressive stress is found to increase by 21% for the increase of liner thickness from 4mm to 5mm. The amount of residual stress increment in the above thickness range is not in proportion to the original rate.

From the above results, it is understood that increasing the liner thickness above the minimum required thickness (thickness that meets longitudinal load carrying capacity) does not result in substantial reduction of composite thickness. Since the steel liner density is more, any increment in liner thickness above the minimum value would not yield optimum weight.
3.4 CONCLUSIONS

The important conclusions derived out of the theoretical studies on hoop wrapped cylinders are given below.

1. The liner compressive stress in the longitudinal direction is negligible (1.6% of hoop compressive stress) compared to the hoop compressive stress. This clearly demonstrates the absence of the longitudinal resistance by the hoop layers.

2. In hoop layered construction, due to the absence of helical layers, there is no helical-to-hoop interaction. This provides the opportunity to design the hoop layers more precisely to meet the burst strength requirements.

Fig. 3.4 Composite Thickness for Different Liner Thicknesses
3. In the absence longitudinal residual compressive stress in the liner, the minimum liner thickness is predominantly decided by the longitudinal load carrying capacity of the liner for achieving a uniform strength construction.

4. Increasing the liner thickness above the minimum required value does not result in substantial reduction of composite thickness. Since the steel liner density is 4 times that of glassfiber-epoxy composite, any increment in liner thickness above minimum required thickness would not yield optimum weight.