6.1 INTRODUCTION

Extensive research work has been published on an impulsively started vertical plate with different boundary conditions. Stokes (1851) first presented an exact solution to the Navier-Stokes equation, which was for the flow of viscous incompressible fluid past an impulsively started infinite horizontal plate in its own plane. It is often called Rayleigh problem in the literature. Stewartson (1951) presented analytic solution to the viscous flow past an impulsively started semi-infinite horizontal plate. Hall (1969) solved the problem of Stewartson (1951) by finite difference method of a mixed explicit-implicit type, which is convergent and stable and hence it is free from any restriction on the mesh sizes. Instead of horizontal plate, if an infinite isothermal vertical plate is given an impulsive motion in its own plane, how the free convection currents affect the flow, which exists due to the temperature difference between the plate temperature and that of fluid far away from the plate? This was first studied by Soundalgekar (1977) again presented an exact solution. The effects of heating and cooling on the plate were discussed. Soundalgekar (1979) studied the problem of the flow past an impulsively started isothermal vertical plate with mass transfer by using Laplace transform technique. Moutsoglou and
Chen (1980) considered the buoyancy effects in boundary layer on inclined, continuous moving sheets.

Free convection flow involving coupled heat and mass transfer occurs frequently in nature. It occurs not only due to temperature difference, but also due to concentration differences or a combination of these two, e.g., in atmospheric flows, there exists a difference in the H₂O concentration. These concentration differences also affect the flow and temperature near the body. In the body of water, the temperature, the concentration of dissolved material, suspended particulate matter also affect the density of water, and hence the flow of water is affected considerably. In engineering applications, the concentration differences are created by either injecting the foreign gases or by coating the body with evaporating material, which evaporates due to the heat of the body, by frictional effects. These mass transfer differences do affect the rate of heat transfer. In practice, H₂, O₂, H₂O, CO₂ etc. are the foreign gases, which are injected in the air flowing past bodies. Thus for flows past inclined bodies, there are buoyancy forces which arise due to temperature differences and concentration differences. Such a physical phenomenon in case of flow past an impulsively started semi-infinite inclined plate has not received the attention of any researchers in the literature. Hence in this chapter, it is now proposed to study the natural convection effects on impulsively started isothermal inclined plate with mass transfer.

Section 6.2 describes the Mathematical Formulation of the problem. Numerical Technique is discussed in section 6.3 and Section 6.4 deals with Results and Discussion.
\[0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} + g \beta \cos \phi (T' - T_{I}) + g \beta' \cos \phi (C' - C_{w})\]  \hspace{1cm} (6.3)

\[\frac{\partial T'}{\partial t'} + u \frac{\partial T'}{\partial x} + v \frac{\partial T'}{\partial y} = \alpha \frac{\partial^2 T'}{\partial y^2}\]  \hspace{1cm} (6.4)

\[\frac{\partial C'}{\partial t'} + u \frac{\partial C'}{\partial x} + v \frac{\partial C'}{\partial y} = D \frac{\partial^2 C'}{\partial y^2}\]  \hspace{1cm} (6.5)

The initial and boundary conditions are

\[t' \leq 0 : u = 0, \ v = 0, \ T' = T_{I}, \ C' = C_{w}\]
\[t' > 0 : u = u_0, \ v = 0, \ T' = T_{I}, \ C' = C_{w} \quad \text{at} \ y = 0\]
\[u = 0, \ T' = T_{I}, \ C' = C_{w} \quad \text{at} \ x = 0\]
\[u \rightarrow 0, \ T' \rightarrow T_{I}, \ C' \rightarrow C_{w} \quad \text{as} \ y \rightarrow \infty\]  \hspace{1cm} (6.6)

Here, u and v are the velocity components in x and y directions respectively, \(u_0\) - velocity of the plate, \(\rho\) - the density of the fluid, \(t'\) - the time, \(v\) - the kinematic viscosity, \(g\) - the acceleration due to gravity, \(\beta\) - the volumetric coefficient of thermal expansion, \(\beta'\) - the volumetric coefficient of expansion with concentration, \(C'\) - the species concentration in the boundary layer, \(C_{w}\) - the species concentration in fluid far away from the plate, \(T'\) - the temperature of the fluid in the boundary layer, \(T_{I}\) - the temperature of the fluid far away from the plate, \(\alpha\) - the thermal diffusivity and \(D\) - the species diffusion coefficient.
After eliminating $P$ from Equation (6.2) using Equation (6.3), one can get

$$\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta \cos \phi \frac{\partial}{\partial x} \int_{y}^{\infty} (T' - T'_w) \, dy$$

$$+ g\beta' \cos \phi \frac{\partial}{\partial x} \int_{y}^{\infty} (C' - C'_w) \, dy$$

$$+ g\beta \sin \phi (T' - T'_w)$$

$$+ g\beta' \sin \phi (C' - C'_w) + v \frac{\partial^2 u}{\partial y^2}$$

(6.7)

Introducing the following non-dimensional quantities

$$X = \frac{x u_0}{v} Gr^{5/2}, \quad Y = \frac{y u_0}{v} Gr^{5/4}, \quad U = \frac{u}{u_0},$$

$$V = \frac{V}{u_0} Gr^{-5/4}, \quad \tau = \frac{t' u_0^2}{v} Gr^{5/2},$$

$$T = \frac{T'_w - T'_m}{T'_w - T'_m}, \quad C = \frac{C' - C'_m}{C'_w - C'_m}$$

(6.8)

$$Gr = \frac{v g \beta (T'_w - T'_m)}{u_0^3}, \quad Gc = \frac{v g \beta' (C'_w - C'_m)}{u_0^3},$$

$$Pr = \frac{v}{\alpha}, \quad Sc = \frac{v}{D}, \quad N = \frac{Gc}{Gr}$$

Equations (6.1), (6.7), (6.4) and (6.5) are reduced to the following non-dimensional form.
\[ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \] \hspace{1cm} (6.9)

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = Gr^{-1/4} \cos \phi \frac{\partial}{\partial X} \int_0^\infty TdY \]

\[ + N Gr^{-1/4} \cos \phi \frac{\partial}{\partial X} \int Y \]

\[ + Gr^{-3/2} T \sin \phi + N Gr^{-3/2} C \sin \phi + \frac{\partial^2 U}{\partial Y^2} \] \hspace{1cm} (6.10)

\[ \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial Y^2} \] \hspace{1cm} (6.11)

\[ \frac{\partial C}{\partial t} + U \frac{\partial C}{\partial X} + V \frac{\partial C}{\partial Y} = \frac{1}{Sc} \frac{\partial^2 C}{\partial Y^2} \] \hspace{1cm} (6.12)

The corresponding initial and boundary conditions in non-dimensional quantities are given by

\[ t \leq 0: \quad U = 0, \quad V = 0, \quad T = 0, \quad C = 0 \]
\[ t > 0: \quad U = 1, \quad V = 0, \quad T = 1, \quad C = 1 \quad \text{at } Y = 0 \]
\[ U = 0, \quad T = 0, \quad C = 0 \quad \text{at } X = 0 \]
\[ U \to 0, \quad T \to 0, \quad C \to 0 \quad \text{as } Y \to \infty \] \hspace{1cm} (6.13)
Local and average skin friction in non-dimensional quantities are

\[ \tau_x = -Gr^{5/4} \left( \frac{\partial U}{\partial Y} \right)_{Y=0} \]  
(6.14)

\[ \bar{\tau} = -Gr^{-5/4} \int_0^1 \frac{\partial U}{\partial Y} \bigg|_{Y=0} dX \]  
(6.15)

Local and average Nusselt number in non-dimensional quantities are

\[ Nu_x = -X Gr^{-5/4} \left( \frac{\partial T}{\partial Y} \right)_{Y=0} \]  
(6.16)

\[ \bar{Nu} = -Gr^{-5/4} \int_0^1 \left[ \frac{\partial T}{\partial Y} \right]_{Y=0} dX \]  
(6.17)

Local and average Sherwood number in non-dimensional quantities are

\[ Sh_x = -X Gr^{-5/4} \left( \frac{\partial C}{\partial Y} \right)_{Y=0} \]  
(6.18)

\[ \bar{Sh} = -Gr^{-5/4} \int_0^1 \left[ \frac{\partial C}{\partial Y} \right]_{Y=0} dX \]  
(6.19)
6.3 NUMERICAL TECHNIQUE

A semi-implicit finite difference scheme of Crank-Nicolson type has been employed to solve these unsteady non-linear coupled Equations (6.9) to (6.12) under the conditions (6.13). The region of integration is considered as a rectangle with sides \( X_{\text{max}} = 1.0 \) and \( Y_{\text{max}} = 16.0 \) where \( Y_{\text{max}} \) corresponds to \( Y = \infty \) which lies very well outside the momentum, thermal and concentration boundary layers. The finite difference equations corresponding to Equations (6.9) to (6.12) are as follows

\[
\begin{align*}
\frac{U_{i,j}^{k+1} - U_{i,j}^{k}}{\Delta t} + U_{i,j}^{k} &= \frac{U_{i,j}^{k+1} - U_{i-1,j}^{k+1} + U_{i-1,j}^{k} - U_{i,j}^{k}}{2\Delta X} + \frac{V_{i,j}^{k+1} - V_{i,j-1}^{k+1} + V_{i,j}^{k} - V_{i,j-1}^{k}}{2\Delta Y} \quad (6.20)
\end{align*}
\]

\[
\begin{align*}
\frac{U_{i,j}^{k+1} - U_{i,j}^{k}}{\Delta t} + U_{i,j}^{k} &= \frac{U_{i,j}^{k+1} - U_{i-1,j}^{k+1} + U_{i-1,j}^{k} - U_{i,j}^{k}}{2\Delta X} + \frac{V_{i,j}^{k+1} - V_{i,j-1}^{k+1} + V_{i,j}^{k} - V_{i,j-1}^{k}}{2\Delta Y} \\
&= \text{Gr}^{-1/4} \cos \phi \frac{\partial}{\partial X} \int_{0}^{\infty} \left[ T_{i,j}^{k+1} + T_{i,j}^{k} \right] \, dY + \text{NGr}^{-1/4} \cos \phi \frac{\partial}{\partial X} \int_{0}^{\infty} \left[ C_{i,j}^{k+1} + C_{i,j}^{k} \right] \\
&+ \text{Gr}^{-3/2} \sin \phi \frac{T_{i,j}^{k+1} + T_{i,j}^{k}}{2} + \text{NGr}^{-3/2} \sin \phi \frac{C_{i,j}^{k+1} + C_{i,j}^{k}}{2} \\
&+ \frac{U_{i,j-1}^{k+1} + U_{i,j+1}^{k+1} + U_{i,j}^{k} - 2U_{i,j}^{k}}{2(\Delta Y)^2} \quad (6.21)
\end{align*}
\]
Here, the subscript $i$ designates the grid point along the $X$ direction, $j$ along the $Y$ direction and the superscript $k$ along the $t$-direction. Appropriate mesh sizes $\Delta X = 0.05$, $\Delta Y = 0.25$ and time step $\Delta t = 0.01$ are considered for calculations.

At a particular time level $k$, finite difference Equation (6.23) at every internal nodal point on a particular $i$-level constitutes a tri-diagonal system of equations. The system of equations is solved by Thomas algorithm as described in Carnahan et al (1969). Thus the values of $C$ are known at every nodal point on a particular $i$ at $(k+1)^{th}$ time level. Similarly the values of $T$ are calculated.
from Equation (6.22). Using the values of $C$ and $T$ at $(k+1)^{th}$ time level in the Equation (6.21), the values of $U$ at $(k+1)^{th}$ time level is found in a similar manner. Integrals in this equation are evaluated using Newton-Cotes closed integration formula. Then the values of $V$ are calculated explicitly by using Equation (6.20) at every nodal point on a particular $i$-level at $(k+1)^{th}$ time level. This process is repeated for various $i$-level. After computing values corresponding to each $i$ at a time level, the values at the next time level are determined in a similar fashion. Computations are repeated until steady state is reached. When the absolute differences between values of $U$ as well as temperature $T$ and concentration $C$ at two consecutive time steps are less than $10^{-5}$ at all grid points. Computations have been carried out for different values of parameters.

The derivatives involved in the Equations (6.14) to (6.19) are evaluated using five-point approximation formula and then the integrals are evaluated using Newton-Cotes closed integration formula.

The truncation error in the finite difference approximation is $O(\Delta t^2 + \Delta y^2 + \Delta x)$ and it tends to zero as $\Delta t$, $\Delta y$ and $\Delta x$ tend to zero. Hence the scheme is compatible. The finite difference scheme is unconditionally stable as explained in section 2.4. Therefore the scheme is convergent.
6.4 RESULTS AND DISCUSSION

In order to ascertain the accuracy of the numerical results, the present study is compared with available exact solution in the literature. The velocity and temperature profiles for \( Pr = 0.7, \ Sc = 0.6, \phi = 90^\circ, \ N = 1.0 \) and \( Gr = 1.0 \) (corresponding to \( \eta = y/2\sqrt{\nu} \)) are compared with the available exact solution of Soundalgekar (1979) at \( t = 0.2, 0.4 \) in Figure 6.1 and 6.2 respectively. It is observed that the present results are in good agreement with the exact solution.

The transient velocity profiles for different Grashof number and inclination angle \( \phi \) are shown in Figure 6.3. The velocity profiles presented are those at the leading edge of the plate i.e. at \( X = 1.0 \). It is observed that the velocity decreases with increasing the Grashof number. It is also observed that velocity decreases as the inclination angle \( \phi \) increases. More time is required to reach the steady state for higher value of \( Gr \).

Figure 6.4 shows that the effect of the buoyancy ratio parameter \( N \) on transient velocity. Velocity increases steadily as time advances and consequently it reaches steady state. When \( N \) increases the combined buoyancy ratio parameter increases. Therefore velocity increases with \( N \) near the plate. Steady state attained at an early stage for the higher value of \( N \).

In Figure 6.5 steady state velocity profiles are plotted for different values of Schmidt number and Prandtl number. It is observed that the velocity decreases with increasing Schmidt number or Prandtl number.

The profiles of steady state temperature for various values of \( Pr \) and \( Sc \) are shown in Figure 6.6. The temperature profile increases with the
Figure 6.1 Comparison of velocity profiles at X = 1.0

- Present Result
- Soundalgekar (1979)

\[ t = 0.2 \quad \text{Pr} = 0.7 \]
\[ t = 0.4 \quad \text{Sc} = 0.6 \]
\[ \phi = 90^0 \]
\[ N = 1.0 \]
\[ Gr = 1.0 \]
Figure 6.2 Comparison of temperature profiles at $X = 1.0$

- $\text{Present Result}$
- $\bullet$ $\text{Soundalgekar (1979)}$

- $\text{Pr} = 0.7$
- $\text{Sc} = 0.6$
- $\phi = 90^\circ$
- $\text{N} = 1.0$
- $\text{Gr} = 1.0$
Figure 6.3 Transient velocity profiles at $X = 1.0$
for different $Gr$ and $\phi$

(* - Steady state)

$Pr = 0.7$
$Sc = 0.5$
$N = 1.0$
Figure 6.4 Transient velocity profiles at $X = 1.0$ for different $N$

$Pr = 0.7$
$Sc = 0.6$
$Gr = 10^6$
$\phi = 45^\circ$

- $N = 2.0$, $t = 7.82^*$
- $N = 1.0$, $t = 8.21^*$
- $N = -0.5$, $t = 9.02^*$
- $N = 2.0$, $t = 0.20$
- $N = 1.0$, $t = 0.16$
- $N = -0.5$, $t = 0.12$

(* - Steady state)
Figure 6.5 Steady state velocity profiles at $X = 1.0$ for different $Pr$ and $Sc$
Figure 6.6 Steady state temperature profiles at $X = 1.0$ for different Pr and Sc

- $Gr = 10^6$
- $\phi = 30^\circ$
- $N = 2.0$

- $Pr = 0.7$  $Sc = 10.0$  $t = 7.79$
- $Pr = 0.7$  $Sc = 2.0$  $t = 7.59$
- $Pr = 0.7$  $Sc = 0.6$  $t = 7.39$
- $Pr = 7.0$  $Sc = 2.0$  $t = 5.65$
increasing Schmidt number and decreases with increasing Prandtl number of the fluid. The effect of the Prandtl number is very important in the temperature profiles. The thermal boundary layer thickness decreases with increasing Prandtl number.

The transient concentration profiles at $X=1.0$ for different values of $S_c$ and $\phi$ are shown graphically in Figure 6.7. It is observed that the concentration profile increases with decreasing the Schmidt number and decreases as $\phi$ decreases.

In Figure 6.8, transient concentration profiles at $X=1.0$ for different values of buoyancy ratio parameter $N$ are shown graphically. The buoyancy forces due to temperature and concentration are opposite in nature when $N$ is negative. The concentration decreases as $N$ increases.

The local skin friction values are plotted in Figure 6.9 as a function of the axial co-ordinate $X$. Local skin friction decreases as $X$ increases. The local wall shear stress increases with increasing values of Schmidt number. There is a fall in skin friction due to decrease in an inclination angle $\phi$ to the horizontal.

In Figure 6.10, local Nusselt number i.e., local heat transfer rate is plotted against the axial co-ordinate $X$ at the steady state level. Local Nusselt number increases with decreasing Schmidt number. Local Nusselt number decreases as $\phi$ increases.

Local Sherwood number is shown in Figure 6.11 for different parameters. Local Sherwood number increases as $S_c$ increases, because, the
Figure 6.7 Transient concentration profiles at $X = 1.0$
for different $Sc$ and $\phi$

(* - Steady state)

$Gr = 10^5$
$N = 1.0$
$Pr = 0.7$
Figure 6.8 Transient concentration profiles at $X = 1.0$ for different $N$

($* - $Steady state$)$

$Pr = 0.7$
$Sc = 0.6$
$Gr = 10^6$
$\phi = 45^0$
Figure 6.9  Local skin friction

\[ \frac{\tau_x}{Gr^{1/4}} \]

- Pr = 0.7
- Gr = 10^6
- N = 2.0

- Sc = 5.0  \( \phi = 60^\circ \)
- Sc = 2.0  \( \phi = 60^\circ \)
- Sc = 1.0  \( \phi = 60^\circ \)
- Sc = 2.0  \( \phi = 30^\circ \)
- Sc = 0.6  \( \phi = 60^\circ \)
- Sc = 1.0  \( \phi = 30^\circ \)
- Sc = 0.6  \( \phi = 30^\circ \)
Figure 6.10  Local Nusselt number

- \( \text{Sc} = 0.6 \quad \phi = 30^\circ \)
- \( \text{Sc} = 1.0 \quad \phi = 30^\circ \)
- \( \text{Sc} = 1.0 \quad \phi = 60^\circ \)
- \( \text{Sc} = 2.0 \quad \phi = 60^\circ \)
- \( \text{Sc} = 5.0 \quad \phi = 60^\circ \)

\( \text{Pr} = 0.7 \)
\( \text{Gr} = 10^5 \)
\( N = 2.0 \)
Figure 6.11  Local Sherwood number

$Pr = 0.7$
$Gr = 10^6$
$N = 2.0$

$Sc = 2.0$
$\phi = 30^0$
$\phi = 60^0$

$Sc = 1.0$
$\phi = 30^0$
$\phi = 60^0$

$Sc = 0.6$
$\phi = 30^0$
$\phi = 60^0$
concentration distribution is decreasing with increasing value of Sc. While it, decreases as $\phi$ increases.

Average skin friction, Nusselt number and Sherwood number are shown in Figure 6.12, 6.13 and 6.14 respectively for various values of Sc and $\phi$. Average skin friction increases with the increasing value of Schmidt number and $\phi$ throughout the transient period. Average Nusselt number increases with decreasing Schmidt number. Similarly, average Sherwood number increases with increasing Schmidt number. Average Nusselt number decreases as $\phi$ increases. The same trend is also noticed for average Sherwood number.
Figure 6.12 Average skin friction

\[ \frac{\tau}{Gr^{0.4}} \]

- \( Sc = 5.0 \ \phi = 60^\circ \)
- \( Sc = 2.0 \ \phi = 60^\circ \)
- \( Sc = 5.0 \ \phi = 30^\circ \)
- \( Sc = 1.0 \ \phi = 60^\circ \)
- \( Sc = 2.0 \ \phi = 30^\circ \)
- \( Sc = 1.0 \ \phi = 30^\circ \)
- \( Sc = 0.6 \ \phi = 30^\circ \)

\[ Pr = 0.7 \]
\[ N = 2.0 \]
\[ Gr = 10^6 \]
Figure 6.13  Average Nusselt number

$Pr = 0.7$
$Gr = 10^6$
$N = 2.0$
Figure 6.14  Average Sherwood number

$\text{Pr} = 0.7$
$\text{Gr} = 10^6$
$N = 2.0$

$\text{Sc} = 1.0$
$\phi = 30^\circ$
$\phi = 60^\circ$

$\text{Sc} = 0.6$
$\phi = 30^\circ$
$\phi = 60^\circ$