CHAPTER 5

NATURAL CONVECTION EFFECTS ON IMPULSIVELY STARTED ISOTHERMAL INCLINED PLATE

5.1 INTRODUCTION

Free convection is a very common phenomenon in nature. Free convection along an inclined plate has received less attention than the cases of vertical and horizontal plates. However, free convection heat transfer from an inclined surface is very frequently encountered in engineering devices and natural environment. The heat treatment of materials, travelling between a feed roll and wind-up roll or conveyor belts, the hot extrusion of steel, the lamination and melt-spinning process in the extrusion of polymers, all possess the characteristics of moving continuous surfaces.

Flow of viscous incompressible fluid past an impulsively started infinite horizontal plate was first studied by Stokes (1851). Stewartson (1951) presented analytic solution to the viscous flow past an impulsively started semi-infinite horizontal plate. Hall (1969) solved the problem of Stewartson (1951) by finite difference method. Soundalgekar (1977) first presented an exact solution to the flow of a viscous incompressible fluid past an impulsively started infinite vertical plate by the Laplace transform technique. Moutsoglou and Chen (1980) considered the effects of buoyancy in the case of inclined, continuous moving sheets. However, the problem of natural convection flow over an impulsively started semi-infinite inclined plate has not received the
attention of any researcher. But the problem of heat transfer in boundary layer on an inclined moving surface has many practical applications in manufacturing process in industry. Hence, in this chapter, an attempt is made to study the problem of natural convection flow over an impulsively started isothermal semi-infinite inclined plate. Section 5.2 describes the Mathematical Formulation of the problem. Numerical Technique is discussed in section 5.3 and Section 5.4 deals with Results and Discussion.

5.2 MATHEMATICAL FORMULATION

A problem of laminar, two-dimensional, unsteady flow of a viscous incompressible fluid past an impulsively started inclined plate is analysed mathematically in this section. The plate makes an inclination angle $\phi$ with the horizontal. The $x$ - axis is measured along the plate and $y$ - axis is measured along upward normal to the plate at the leading edge. Initially, the temperature of the plate and the fluid are assumed to be the same. At time $t' > 0$, the plate temperature is changed to $T'_w$ causing convection current to flow near the plate and the plate starts moving in an inclined direction due to impulsive motion gaining a velocity $u_0$. The viscous dissipation effects are also assumed to be negligible in the energy equation. Then under the usual Boussinesq's approximation, the flow can be governed by the following equations.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.1}
\]
\[
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + g \beta \sin \phi (T' - T_{\infty}') + v \frac{\partial^2 u}{\partial y^2} \quad (5.2)
\]

\[
0 = - \frac{1}{\rho} \frac{\partial P}{\partial y} + g \beta \cos \phi (T' - T_{\infty}') \quad (5.3)
\]

\[
\frac{\partial T'}{\partial t'} + u \frac{\partial T'}{\partial x} + v \frac{\partial T'}{\partial y} = \alpha \frac{\partial^2 T'}{\partial y^2} \quad (5.4)
\]

The initial and boundary conditions are

\[
t' \leq 0 : u = 0, \quad v = 0, \quad T' = T_{\infty}',
\]
\[
t' > 0 : u = u_0, \quad v = 0, \quad T' = T_{\infty}', \quad \text{at} \quad y = 0
\]
\[
u = 0, \quad T' = T_{\infty}', \quad \text{at} \quad x = 0 \quad (5.5)
\]
\[
u \to 0, \quad T' \to T_{\infty}', \quad \text{as} \quad y \to \infty
\]

Here, \(u\) and \(v\) are the velocity components in \(x\) and \(y\) directions respectively, \(u_0\) - the velocity of the plate, \(\rho\) - the density of the fluid, \(t'\) - the time, \(v\) - the kinematic viscosity, \(g\) - the acceleration due to gravity, \(\beta\) - the volumetric coefficient of thermal expansion, \(T'\) - the temperature of the fluid in the boundary layer, \(T_{\infty}'\) - the temperature of the fluid far away from the plate and \(\alpha\) - the thermal diffusivity.

After eliminating \(P\) from Equation (5.2) using Equation (5.3), one can get

\[
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \beta \cos \phi \frac{\partial}{\partial x} \int_y (T' - T_{\infty}') \, dy
\]

\[
+ g \beta \sin \phi (T' - T_{\infty}') + v \frac{\partial^2 u}{\partial y^2} \quad (5.6)
\]
Local and average skin friction are given by

\[ \tau_x = - \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} \]  (5.7)

\[ \tau_L = - \frac{1}{L} \int_0^L \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} \, dx \]  (5.8)

Local and average Nusselt number are given by

\[ \text{Nu}_x = - \frac{x \left( \frac{\partial T'}{\partial y} \right)_{y=0}}{(T'_w - T'_\infty)} \]  (5.9)

\[ \overline{\text{Nu}}_L = - \int_0^L \frac{\left( \frac{\partial T'}{\partial y} \right)_{y=0}}{(T'_w - T'_\infty)} \, dx \]  (5.10)

Introducing the following non-dimensional quantities

\[ X = \frac{x u_0}{v} \text{Gr}^{5/4}, \quad Y = \frac{y u_0}{v} \text{Gr}^{5/4}, \quad U = \frac{u}{u_0}, \]

\[ V = \frac{v}{u_0} \text{Gr}^{-5/4}, \quad t = \frac{t' u_0^2}{v} \text{Gr}^{3/2}, \]  (5.11)

\[ T = \frac{T' - T'_\infty}{T'_w - T'_\infty}, \quad \text{Gr} = \frac{v \beta (T'_w - T'_\infty)}{u_0^3}, \quad \text{Pr} = \frac{v}{\alpha} \]
Equations (5.1), (5.6) and (5.4) are reduced to the following non-dimensional form.

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{5.12}
\]

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \text{Gr}^{-1/4} \cos \phi \frac{\partial}{\partial x} \int_T dY + \text{Gr}^{-3/2} T \sin \phi + \frac{\partial^2 U}{\partial Y^2} \tag{5.13}
\]

\[
\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \frac{1}{\text{Pr}} \frac{\partial^2 T}{\partial Y^2} \tag{5.14}
\]

The corresponding initial and boundary conditions in non-dimensional quantities are given by

\[
t \leq 0 : \quad U = 0, \quad V = 0, \quad T = 0, \quad U \rightarrow 0, \quad T \rightarrow 0, \quad \text{at} \ Y \rightarrow \infty
\]

\[
t > 0 : \quad U = 1, \quad V = 0, \quad T = 1, \quad \text{at} \ Y = 0
\]

\[
\quad U = 0, \quad T = 0, \quad \text{at} \ X = 0
\]

\[
\quad U \rightarrow 0, \quad T \rightarrow 0, \quad \text{as} \ Y \rightarrow \infty
\]
Local and average skin friction in non-dimensional quantities are

$$\tau_x = - \text{Gr}^{5/4} \left( \frac{\partial U}{\partial Y} \right)_{Y=0}$$  (5.16)

$$\bar{\tau} = - \text{Gr}^{-5/4} \int_0^1 \frac{\partial U}{\partial Y} \left. \right|_{Y=0} \ dX$$  (5.17)

Local and average Nusselt number in non-dimensional quantities are

$$N_{u,x} = -X \text{Gr}^{-5/4} \left( \frac{\partial T}{\partial Y} \right)_{Y=0}$$  (5.18)

$$\overline{N_u} = - \text{Gr}^{-5/4} \int_0^1 \left( \frac{\partial T}{\partial Y} \right)_{Y=0} \ dX$$  (5.19)

5.3 NUMERICAL TECHNIQUE

The unsteady, coupled and non-linear Equations (5.12) to (5.14) with the conditions (5.15), are solved by employing a semi-implicit finite difference scheme of Crank-Nicolson type. The finite difference equations corresponding to Equations (5.12) to (5.14) are given by

$$\frac{U_{i,j}^{k+1} - U_{i-1,j-1}^{k+1} + U_{i,j}^{k+1} - U_{i-1,j}^{k+1}}{4\Delta X} = \frac{U_{i-1,j-1}^{k} - U_{i-1,j}^{k} + U_{i,j}^{k} - U_{i-1,j}^{k}}{4\Delta X}$$
\[ U_{i,j}^{k+1} - U_{i,j}^k = \frac{U_{i,j}^{k+1} - U_{i-1,j}^{k+1} + U_{i,j}^k - U_{i-1,j}^k}{2\Delta X} + \frac{(U_{i,j}^{k+1} - U_{i,j-1}^{k+1} + U_{i,j}^k - U_{i,j-1}^k)}{4\Delta Y} \]

\[ + \frac{(U_{i,j-1}^{k+1} - 2U_{i,j}^{k+1} + U_{i,j-1}^k + U_{i,j+1}^k - 2U_{i,j}^k + U_{i,j+1}^k)}{2(\Delta Y)^2} \]

\[ T_{i,j}^{k+1} - T_{i,j}^k = \frac{U_{i,j}^k}{2\Delta X} \left( \frac{T_{i,j}^{k+1} - T_{i-1,j}^{k+1} + T_{i,j}^k - T_{i-1,j}^k}{2} \right) \]

\[ + \frac{(T_{i,j-1}^{k+1} - T_{i,j-1}^{k+1} + T_{i,j}^k - T_{i,j-1}^k)}{4\Delta Y} \]

\[ = \frac{(T_{i,j-1}^{k+1} - 2T_{i,j}^{k+1} + T_{i,j-1}^k + T_{i,j+1}^k - 2T_{i,j}^k + T_{i,j+1}^k)}{2Pr(\Delta Y)^2} \]

The region of integration is considered as a rectangle with sides \( X_{\text{max}} (= 1.0) \) and \( Y_{\text{max}} (= 14.0) \) where \( Y_{\text{max}} \) corresponds to \( Y = \infty \) which lies very well outside the momentum and thermal boundary layers. The maximum value of \( Y \) is chosen as 14 after some preliminary investigations so that the last two of the boundary conditions (5.15) are satisfied. Here, the subscript
i-designates the grid point along the X-direction, j - along the Y-direction and the superscript k along the t-direction. Appropriate mesh sizes $\Delta X = 0.05$, $\Delta Y = 0.25$ and time step $\Delta t = 0.01$ are considered for calculations.

Equation (5.22) at every internal nodal point on a particular i-level constitutes a tri-diagonal system of equations, which is solved by Thomas algorithm, described by Carnahan et al (1969). Thus the values of $T$ are known at every nodal point on a particular i-level at $t = \Delta t$. Similarly the values of $U$ are calculated from Equation (5.21). Integral involved in this equation is evaluated using Newton-Cotes closed integration formula. Finally values of $V$ are calculated directly from Equation (5.20) at every nodal point on a particular i-level. By moving along i-direction in a similar manner all the values of $T$, $U$ and $V$ at all nodal points are calculated at $t = \Delta t$. This process is repeated until the steady state is reached. Steady state solutions are assumed to have been obtained when the absolute differences between values of $U$ as well as temperature $T$ at two consecutive time steps are less than $10^{-5}$ at all grid points. Computations have been carried out for different values of parameters.

The derivatives involved in the Equations (5.16) to (5.19) are evaluated using five-point approximation formula and then the integrals are evaluated using Newton-Cotes closed integration formula.

The truncation error in the finite difference approximation is $O(\Delta t^2 + \Delta Y^2 + \Delta X)$ and it tends to zero as $\Delta t$, $\Delta Y$ and $\Delta X$ tend to zero. Hence the scheme is compatible. The finite difference scheme is unconditionally stable as explained in section 2.4. Therefore the scheme is convergent.
5.4 RESULTS AND DISCUSSION

In order to assess the accuracy of the numerical results, the present study is compared with previous study. The velocity and temperature profiles are found to be in good agreement. The velocity and temperature profiles for $Pr = 0.7$, $Gr = 1.0$, $\phi = 90^\circ$ and $t = 0.2, 0.4$ (corresponding to $\eta = y/2\sqrt{t}$) are compared with the available exact solution of Soundalgekar (1977) in Figure 5.1 and 5.2 respectively. It is observed that the present results are in good agreement with exact solution.

The transient velocity profiles for different values of Grashof number are shown in Figure 5.3. Velocity increases steadily as time advances and consequently reaches the steady state. It is observed that velocity decreases as Gr increases. It is observed that the system reaches the steady state early when the Grashof number is large.

The transient velocity profiles for different inclination angle $\phi$ are shown in Figure 5.4. Due to rise in the inclination of the plate with the horizontal, the steady state velocity is found to decrease. Time required to reach steady state value is less with an increase in the inclination angle $\phi$ to the horizontal.

The transient temperature profiles for different values of Prandtl number and inclination angle $\phi$ are shown in Figure 5.5. It is clear that temperature profiles increases with decreasing Prandtl number of the fluid. We observe from this figure that there is an increase in temperature of the air due to rise in the inclination of the plate.
Figure 5.1 Comparison of velocity profiles at $X = 1.0$

- Soundalgekar (1977)
- Present Result

$t = 0.2$
$t = 0.4$

$Pr = 0.7$
$Gr = 1.0$
$\phi = 90^0$

Figure 5.1 Comparison of velocity profiles at $X = 1.0$
Figure 5.2 Comparison of temperature profiles at X = 1.0

- Soundalgekar (1977)
- Present Result

Pr = 0.7
Gr = 1.0
\( \phi = 90^\circ \)

\( t = 0.2 \)
\( t = 0.4 \)
Figure 5.3 Transient velocity profiles at $X = 1.0$ for different Grashof number
(* - Steady state)

$Pr = 0.7$
$\phi = 30^0$

$\text{Gr} = 10^{6}$ $t = 7.77^*$
$\text{Gr} = 10^{8}$ $t = 7.50^*$
$\text{Gr} = 10^{6}$ $t = 0.38$
$\text{Gr} = 10^{8}$ $t = 0.33$
\[ \text{Pr} = 0.7 \]
\[ \text{Gr} = 10^6 \]

Figure 5.4 Transient velocity profiles at \( X = 1.0 \) for different \( \phi \)

(* - Steady state)
Figure 5.5 Transient temperature profiles at X = 1.0 for different Pr and $\phi$ ($\ast$ - Steady state)

$Gr = 10^6$

- $Pr = 0.7$
- $Pr = 7.0$

$\phi = 60^0$ $t = 7.29^*$
$\phi = 30^0$ $t = 7.77^*$
$\phi = 60^0$ $t = 5.72^*$
$\phi = 30^0$ $t = 0.16$
$\phi = 60^0$ $t = 0.24$
Local skin friction is shown in Figure 5.6. It is interesting to note here that due to the inclination of the plate there is a fall in local skin friction, because of the fact that the velocity gradient decreases as $\phi$ increases.

Local Nusselt number for different values of $\phi$ are shown in Figure 5.7. It increases as $X$ increases. It is observed that the local Nusselt number decreases by the increasing value of inclination angle $\phi$.

Average values of skin friction and Nusselt number are shown in Figures 5.8 and 5.9 respectively. In Figure 5.8, it is observed that Average skin friction decreases with time and become steady. Average skin friction gets reduced with decreasing value of $\phi$. Also, we observed from this figure that average skin friction decreases as Pr decreases. Average Nusselt number decreases as $\phi$ increases.
Pr = 0.7
Gr = 10^6

Figure 5.6  Local skin friction
Figure 5.7  Local Nusselt number

\[ Pr = 0.7 \]
\[ Gr = 10^6 \]

\( \phi = 30^\circ \)
\( \phi = 45^\circ \)
\( \phi = 60^\circ \)
\( \phi = 90^\circ \)
Figure 5.8 Average skin friction
Figure 5.9 Average Nusselt number

\[ Pr = 0.7 \]
\[ Gr = 10^6 \]