CHAPTER 7

SOME RESULTS ON A GENERALIZED $M/G/1$ FEEDBACK QUEUE WITH DISASTERS

7.1 INTRODUCTION

This chapter deals with a generalized $M/G/1$ feedback queueing system subject to disaster with two general service time distributions. The service time distribution of a customer who initiates a busy period is $G_e(x)$. All subsequent customers in the same busy period have service times drawn independent from the distribution $G_b(x)$ (Welch (1964)). There are many situations to which such models are appropriate. Consider, for instance, a service which is partially shut down during the idle periods resulting in an increased service time for the first customer of each busy period. Chang (1968) has applied two general service time distribution functions $G_b(x)$ and $G_e(x)$ to the priority processing problems associated with time-sharing computer systems. Further Lam (1977) has also discussed this problem for the delay performance of a Time Division Multiple Access channel for transmitting data messages.

In the area of optimal design and control of queues, the so called $N$-policy has received great attention, see Tijms (1986) and Takagi (1991) and references therein. According to the $N$-policy, the server idles until a fixed number $N$ of customers arrive to the queue at which moment, the server is “switched on” and serves exhaustively all the customers in the system. The server is then “switched off” and remains idle until $N$ customers accumulate again in the queue.

The $N$-policy is generalized to an $M/G/1$ queue with batch arrivals (Lee and Srinivasan (1989)). Kella (1989) has investigated an $M/G/1$ queue in which the server
takes vacations and is turned on depending on the number of customers present in the system at the end of the vacation periods. In this chapter, an another generalization of the $N$-policy for the $M/G/1$ queue is presented. At the end of the busy period, the server decides a random number $N$ of customers to accumulate in order to start the next busy period. This policy is called the random $N$-policy and its queueing analysis is the focus of this chapter.

Also, several researchers (Gelenbe (1991, 1993), Chao and Pinedo (1995), Henderson (1993) and Harrison and Pitel (1993)) have studied queueing systems subject to disasters. The disasters arrive as negative customers to the system and their characteristic is to remove some or all of the regular customers in the system. For example, in computer networks, if job infected with a virus arrives, it transmits virus to other processors inactivating them (Chao (1995)).

One more feature which has been widely studied in queueing systems is feedback. Feedback queues are useful for modelling many phenomena in computer communication systems. A classical example is found in computer time-sharing. Here jobs are served in a first-come-first-served fashion, and receive a fixed quantum of service. When that quantum expires and if the job needs more service, it then returns to the tail of the queue and repeats the cycle; In telecommunication, a telephone call may generate several tasks for processing, which may be considered as feedbacks. The study of queueing models with feedback goes back to a classical paper by Takacs (1963). In his model, customers who complete their services are feedback instantaneously to the tail of the queue with probability $1 - \theta$ ($0 < \theta \leq 1$) and leaves the system with probability $\theta$. This mechanism is called Bernoulli feedback. For an $M/G/1$ queue with Bernoulli feedback, Takacs has derived the joint transform of the distributions of queue length and remaining service time, from which the Laplace-Stieltjes transform of the total sojourn time distribution is obtained. Further studies on the queue length, the total sojourn time and waiting time are provided by Disney et al (1980, 1984) and Disney (1981). Fontana and Diaz Berzosa (1984, 1985) and Simon (1984) have extended some results
obtained for the $M/G/1$ model with Bernoulli feedback to a more general feedback model with priorities. Disney and Konig (1985) have given an overview of literature concerning Bernoulli feedback studies.

Although results have been reported separately on queueing systems with several types of service, vacation queueing models, queueing systems subject to disasters and feedback queueing models, no work has been found in literature which studies queueing systems taking together the above mentioned features. To fill this gap, in this chapter, $M/G/1$ queueing model subject to disasters with two types of service, and server vacation is generalized.

The organization of this chapter is as follows. In section 7.2, the model is described. In section 7.3, transient and limiting behaviours of the system length are obtained. Various interesting special cases and vacation models are discussed in section 7.4. Finally, in section 7.5, some performance measures are obtained. Also some numerical examples to illustrate the effect of the parameters on the system performance are provided.

7.2 MODEL DESCRIPTION

Consider a generalized $M/G/1$ feedback queueing system subject to disasters where the first customer in every busy cycle undergoes service different from that given to the rest of the customers. The customers arrive at the system in accordance with a Poisson process with intensity $\lambda$. The service times of customers are independent and non-negative random variables; if the $n^{th}$ customer arrives to find the queue empty, his service time $U_{e,n}$ has distribution function $G_e(x) = P\{U_{e,n} \leq x\}$ for $x \geq 0$ and $n = 1, 2, 3, \ldots$ with $G_e(0^+) = 0$; if, on the other hand, the $n^{th}$ customer arrives to find at least one customer in the system, his service time $U_{b,n}$ has distribution function $G_b(x) = P\{U_{b,n} \leq x\}$ for $x \geq 0$ and $n = 1, 2, 3, \ldots$ with $G_b(0^+) = 0$. In other words, it is assumed that the customer who initiates a busy period has service time distribution
function different from that of the customers who arrive during the busy period.

The server continues serving the customers until there is no customer waiting for service. The server then leaves the system to deal with some other job. When the first subsequent customer arrives, the server does not necessarily return to the counter, but in general he returns there to initiate a busy period upon the arrival of the $N$-th subsequent customer, where $N$ is a random variable assuming positive integer values with probability distribution $q_n = P\{N = n\}$, $n = 1, 2, 3, \ldots$. Further, it is assumed that disasters occur to the customer undergoing service according to a Poisson process with rate $\delta$. A customer survives a disaster independently of others with probability $p$ or it succumbs to disaster with probability $q (= 1 - p)$, $0 < p < 1$. When a customer completes his service, he departs from the system with probability $\theta_1$ or cycles back with probability $1 - \theta_1$ ($0 \leq \theta_1 < 1$). On the other hand, if a disaster occurs during the service time of a customer, it succumbs to the disaster with probability $q$ and such a customer either leaves the system with probability $\theta_2$ or is feedback to the end of the queue with probability $1 - \theta_2$ ($0 \leq \theta_2 < 1$). The inter-arrival times of customers and time between disasters are mutually independent random variables.

Under these assumptions, in the next section the generating function of the number of customers in the system either at a service completion epoch or at which the customer succumbs to disaster is obtained, when the queue works in statistical equilibrium. The queue discipline is first-come-first-served.

7.3 TRANSIENT AND LIMITING BEHAVIOUR OF THE QUEUE LENGTH

Let $\tau'_n = \text{either the epoch of service completion of the } n^{th} \text{ customer or that at which the } n^{th} \text{ customer succumbs to disaster} = 0$. 
Define $\xi(t) =$ number of customers in the system at time $t$ and $\xi_n = \xi(\tau_n' + 0)$.

It can be easily shown that the sequence $\{\xi_n; n = 0, 1, 2, \ldots\}$ constitutes a homogeneous Markov chain whose transition probabilities are given by

$$
P\{\xi_{n+1} = k | \xi_n = j\} = \begin{cases} P_{b,k+1-j} & \text{for } j > 0, k \geq j - 1 \\ 0 & \text{for } j > 0, k < j - 1 \end{cases}
$$

$$
P\{\xi_{n+1} = k | \xi_n = 0\} = \sum_{i=1}^{k+1} q_i P_{e,k+1-i} \text{ for } k \geq 0,
$$

where

$$
P_{b,k+1-j} = \theta_1 \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k+1-j}}{(k+1-j)!} \sum_{n=0}^{\infty} \frac{e^{-\delta x}(\delta x)^n}{n!} dG_b(x)
$$

$$
+ (1 - \theta_1) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k-j}}{(k-j)!} \sum_{n=0}^{\infty} \frac{e^{-\delta x}(\delta x)^n}{n!} dG_b(x)
$$

$$
+ \theta_2 \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k+1-j}}{(k+1-j)!} \sum_{n=0}^{\infty} \frac{\delta e^{-\delta x}(\delta x)^n}{n!} p^n dG_b(x)
$$

$$
+ (1 - \theta_2) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k-j}}{(k-j)!} \sum_{n=0}^{\infty} \frac{\delta e^{-\delta x}(\delta x)^n}{n!} p^n q [1 - G_b(x)] dx \text{, for } k \geq j - 1,
$$

$$
P_{e,k+1-j} = \theta_1 \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k+1-j}}{(k+1-j)!} \sum_{n=0}^{\infty} \frac{e^{-\delta x}(\delta x)^n}{n!} p^n dG_e(x)
$$

$$
+ (1 - \theta_1) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k-j}}{(k-j)!} \sum_{n=0}^{\infty} \frac{e^{-\delta x}(\delta x)^n}{n!} dG_e(x)
$$

$$
+ \theta_2 \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k+1-j}}{(k+1-j)!} \sum_{n=0}^{\infty} \frac{\delta e^{-\delta x}(\delta x)^n}{n!} p^n dG_e(x)
$$

$$
+ (1 - \theta_2) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{k-j}}{(k-j)!} \sum_{n=0}^{\infty} \frac{\delta e^{-\delta x}(\delta x)^n}{n!} p^n q [1 - G_e(x)] dx \text{, for } k \geq j - 1.
$$

The following recurrence relation holds between the probabilities of the system state at epochs $\tau_n + 0$ and the $\tau_{n+1} + 0$ :

$$
P(\xi_{n+1} = k) = P(\xi_n = 0) \sum_{i=1}^{k+1} q_i P_{e,k+1-i} + \sum_{i=1}^{k+1} P(\xi_n = i) P_{b,k+1-i}. \quad (7.1)
$$
Introducing the probability generating function of $P(\xi_n = k)$ as
\[ L_n(z) = \sum_{k=0}^{\infty} P(\xi_n = k)z^k \quad \text{for } |z| \leq 1, n = 0, 1, 2, ... \tag{7.2} \]
and the Laplace-Stieltjes transforms of the service time distributions $G_e(x)$ and $G_b(x)$ as
\[ \beta_e(s) = \int_0^\infty e^{-st} \, dG_e(t) \quad \text{for } \Re(s) > 0 \tag{7.3} \]
where $\ell$ stands either for $e$ or $b$ and expectations
\[ \beta_e = \int_0^\infty x \, dG_e(x) \quad \text{and} \quad \beta_b = \int_0^\infty x \, dG_b(x). \]
Define the probability generating function of $\{q_n\}$ as
\[ Q(z) = \sum_{n=1}^{\infty} q_n z^n, \quad \text{for } |z| \leq 1 \tag{7.4} \]
with $E(N)$ and $E(N^2)$ as the first and second moments of $N$.

The transient behaviour of this system size process is characterized by the following theorem.

**Theorem 7.3.1.** For the $M/G/1$ feedback queue with disasters defined as above the probability generating function of $L_n(z)$ is given by

\[
\sum_{n=0}^{\infty} L_n(z)\omega^n = \frac{zL_0(z)}{
\left\{ z - \omega \left[ \begin{array}{c}
\beta_b(\lambda(1 - z) + \delta q)(\theta_1 + (1 - \theta_1)z) \\
+ \delta q(\theta_2 + (1 - \theta_2)z)(1 - \beta_b(\lambda(1 - z) + \delta q)) \\
\end{array} \right]
\right\}
\]

\[ + \omega \left\{ (\theta_1 + (1 - \theta_1)z) [Q(z)\beta_e(\lambda(1 - z) + \delta q) - \beta_b(\lambda(1 - z) + \delta q)] \\
+ \delta q(\theta_2 + (1 - \theta_2)z) \left\{ Q(z) \left( \frac{1 - \beta_e(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} - \frac{1 - \beta_b(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} \right) \right\}
\right\}
\times \frac{L_0(g_b(\omega))}{\left\{ 1 - \frac{\omega Q(g_e(\omega))}{g_e(\omega)} [\beta_e(\lambda(1 - g_e(\omega)) + \delta q)(\theta_1 + (1 - \theta_1)g_e(\omega))] \\
+ \delta q(\theta_2 + (1 - \theta_2)g_e(\omega)) \left( \frac{1 - \beta_e(\lambda(1 - g_e(\omega)) + \delta q)}{\lambda(1 - g_e(\omega)) + \delta q} \right) \right\}} \tag{7.5} \]
for $|z| < 1$ and $|\omega| < 1$, where

$$g_b(\omega) = \sum_{n=1}^{\infty} \frac{\omega^n}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ [\beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right)]^n \right\}_{z=0}$$

is the unique root, with minimum absolute value, of the equation

$$z = \omega \left[ \beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right) \right].$$

**Proof.** From the recurrence relation (7.1), it can be obtained as

$$L_{n+1}(z) = \sum_{k=0}^{\infty} P(\xi_{n+1} = k) z^k$$

$$= P(\xi_n = 0) \sum_{i=k+1}^{\infty} z^k q_i P_{i+k-1} + \sum_{i=k+1}^{\infty} z^k P(\xi_n = i) P_{i+k-1}$$

$$= P(\xi_n = 0) \frac{Q(z)}{z} \left\{ \beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \frac{\delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right)}{z} \right\}$$

$$+ \frac{1}{z} \left[ L_n(z) - P(\xi_n = 0) \right] \left\{ \beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \frac{\delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right)}{z} \right\}$$

and hence

$$L_{n+1}(z) = \frac{L_n(0) Q(z)}{z} \left\{ \beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \frac{\delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right)}{z} \right\}$$

$$+ \left[ \frac{L_n(z) - L_n(0)}{z} \right] \left\{ \beta_b(\lambda(1-z) + \delta q)(\theta_1 + (1-\theta_1)z) + \frac{\delta q(\theta_2 + (1-\theta_2)z) \left( \frac{1 - \beta_b(\lambda(1-z) + \delta q)}{\lambda(1-z) + \delta q} \right)}{z} \right\}.$$
Now the generating function of $L_{n+1}(z)$, for $|\omega| < 1$ is given by

$$
\sum_{n=0}^{\infty} L_{n+1}(z)\omega^n = \sum_{n=0}^{\infty} L_n(0)\omega^n \frac{Q(z)}{z} \{\beta_0(\lambda(1-z) + \delta \eta)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\left(1 - \beta_0(\lambda(1-z) + \delta \eta)\right)\} \\
+ \delta q(\theta_2 + (1 - \theta_2)z)\left(1 - \beta_0(\lambda(1-z) + \delta \eta)\right)\} \\
+ \frac{1}{z} \left[ \sum_{n=0}^{\infty} L_n(z)\omega^n - \sum_{n=0}^{\infty} L_n(0)\omega^n \right] \left\{\beta_0(\lambda(1-z) + \delta \eta)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\left(1 - \beta_0(\lambda(1-z) + \delta \eta)\right)\right\}.
$$

Rewriting the left-hand side of the above equation in terms of $\sum_{n=0}^{\infty} L_n(z)\omega^n$ and solve for this quantity, the above reduces to

$$
\sum_{n=0}^{\infty} L_n(z)\omega^n = \{zL_n(z) + \omega \sum_{n=0}^{\infty} L_n(0)\omega^n \{Q(z)\beta_0(\lambda(1-z) + \delta \eta)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\left(1 - \beta_0(\lambda(1-z) + \delta \eta)\right)\} \}
$$

The left-hand side of (7.6) is an analytic function of $z$, for $|z| < 1$ and $|\omega| < 1$. By Rouche’s theorem it follows that for each $\omega$ with $|\omega| < 1$, the denominator of the right-hand side of (7.6) has a unique zero in the unit circle $|z| < 1$, call it $g_b(\omega)$. Then $g_b(\omega)$ must be a zero of the numerator. Setting the numerator equal to zero for $z = g_b(\omega)$, it is shown that

$$
\sum_{n=0}^{\infty} L_n(0)\omega^n = \frac{L_0(g_b(\omega))}{1 - \omega \delta \eta g_b(\omega)} \left\{1 - \frac{\beta_0(\lambda(1-g_b(\omega)) + \delta \eta)(\theta_1 + (1 - \theta_1)g_b(\omega)) + \delta q(\theta_2 + (1 - \theta_2)g_b(\omega))\left(1 - \beta_0(\lambda(1-g_b(\omega)) + \delta \eta)\right)\} \}
$$

Substituting (7.7) in (7.6), (7.5) is obtained and then apply Lagrange’s theorem to compute $g_b(\omega)$ explicitly. Hence the proof of the theorem is complete. □
Remark 7.3.2. If the service time distribution of all the customers is \( G(x) \), the busy period starts with a random number of customers in front of the server, then the generating function \( \sum_{n=0}^{\infty} L_n(z)\omega^n \) is

\[
\sum_{n=0}^{\infty} L_n(z)\omega^n = \frac{zL_0(z)}{z - \omega[\beta(\lambda(1-z) + \delta q)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q}]}
\]

\[
\omega(1 - Q(z))\beta(\lambda(1-z) + \delta q)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q]}
\]

\[
\left\{ \omega[\beta(\lambda(1-z) + \delta q)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q]} - z \right\}
\]

\( L_0(g(\omega)) \)

\[
\left\{ 1 - \frac{\omega g(g(\omega))}{g(\omega)} \beta(\lambda(1-g(\omega)) + \delta q)(\theta_1 + (1 - \theta_1)g(\omega)) + \delta q\frac{(1-\beta(\lambda(1-g(\omega)) + \delta q))}{\lambda(1-g(\omega)) + \delta q]}
\]

\[
\right\}.
\]

(7.8)

In particular, if the feedback parameter of all the customers is \( \theta \), i.e., \( \theta_1 = \theta_2 = \theta \), the generating function of \( L_n(z) \) is given by

\[
\sum_{n=0}^{\infty} L_n(z)\omega^n = \frac{zL_0(z)}{z - \omega(\theta + (1 - \theta)z)\beta(\lambda(1-z) + \delta q) + \delta q\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q]}}
\]

\[
+ \frac{\omega(1 - Q(z))(\theta + (1 - \theta)z)\beta(\lambda(1-z) + \delta q) + \delta q\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q]}}{\lambda(1-z) + \delta q]}
\]

\[
\left\{ \omega(\theta + (1 - \theta)z)\beta(\lambda(1-z) + \delta q) + \delta q\frac{(1-\beta(\lambda(1-z) + \delta q))}{\lambda(1-z) + \delta q]}
\]

\[
\right\}.
\]

(7.9)

Before discussing the limiting behaviour of system size, the necessary and sufficient condition for the system to be stable is carried out. It can be shown by using Foster's criteria that the homogeneous Markov chain \( \{\xi_n; n = 0, 1, 2, \ldots\} \) is ergodic, if

\[
\lim_{n \to \infty} E(\xi_{n+1} - \xi_n | \xi_n = i) = (1 - \theta_1)\beta_1(\delta q) + (1 - \theta_2)(1 - \beta_2(\delta q)) + \frac{\lambda(1 - \beta_2(\delta q))}{\delta q} - 1 < 0.
\]
This results in the stability condition
\[(1 - \theta_1)\beta_b(\delta q) + (1 - \theta_2)(1 - \beta_b(\delta q)) + \frac{\lambda(1 - \beta_b(\delta q))}{\delta q} < 1,\]
which is necessary as well.

The following theorem characterizes the limiting distribution of the system size.

**Theorem 7.3.3.** For the $M/G/l$ feedback queueing model with disasters defined as above, if \((1 - \theta_1)\beta_b(\delta q) + (1 - \theta_2)(1 - \beta_b(\delta q)) + \frac{\lambda(1 - \beta_b(\delta q))}{\delta q} < 1\), then independent of the initial distribution, \(\lim_{n \to \infty} P(\xi_n = k) = P_k\) where \(\{P_k\}\) is a probability distribution with probability generating function

\[L(z) = \sum_{k=0}^{\infty} P_k z^k\]  

(7.10)

with

\[L(0) = \left\{ \begin{array}{l}
1 - \frac{E(N)}{E(N) + \{(\theta_2 - \theta_1) - \frac{1}{\delta q}]} \end{array} \right\}.\]  

(7.11)

**Proof.** It is observed that the Markov chain \(\{\xi_n; n = 0, 1, 2, \ldots\}\) is irreducible and aperiodic and hence its limiting distribution exits. In this case \(L(z) = \lim_{n \to \infty} L_n(z)\) exists and so from (7.5), using Abel’s theorem on the continuity of a power series on the circle of convergence, then

\[L(z) = \lim_{n \to \infty} L_n(z)\]

\[= \lim_{n \to 1} (1 - \omega) \sum_{n=0}^{\infty} L_n(z)\omega^n\]

\[= \lim_{n \to 1} \frac{(1 - \omega)z L_0(z)}{z - \omega \beta_b(\lambda(1 - z) + \delta q)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z)\left(\frac{1 - \beta_b(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q}\right)},\]
\[
\left\{ \begin{align*}
    &\beta_b(\lambda(1 - z) + \delta q)(\theta_1 + (1 - \theta_1)z) + \delta q(\theta_2 + (1 - \theta_2)z) \\
    &\frac{(1 - \beta_b(\lambda(1 - z) + \delta q))}{\lambda(1 - z) + \delta q} - Q(z)|\beta_e(\lambda(1 - z) + \delta q)(\theta_1 + (1 - \theta_1)z) \\
    &+ \delta q(\theta_2 + (1 - \theta_2)z)\frac{(1 - \beta_e(\lambda(1 - z) + \delta q))}{\lambda(1 - z) + \delta q}
\end{align*} \right\}
\]

\[+
\lim_{\omega \to 1} \left\{ \omega|\beta_b(\lambda(1 - z) + \delta q)(\theta_1 + (1 - \theta_1)z) \\
+ \delta q(\theta_2 + (1 - \theta_2)z)\frac{(1 - \beta_b(\lambda(1 - z) + \delta q))}{\lambda(1 - z) + \delta q} - z \right\} \times \left( \frac{1 - \omega Q(g_b(\omega))}{g_b(\omega)} \right) \frac{(1 - \omega)\omega L_0(g_b(\omega))}{\lambda(1 - g_b(\omega) + \delta q)} . \quad (7.12)
\]

Now, if \((1 - \theta_1)\beta_b(\delta q) + (1 - \theta_2)\beta_b(\delta q) + \frac{\lambda(1 - \beta_b(\delta q)}{\delta q} < 1\), it can be concluded by the implicit function theorem that \(g_b(\omega)\) is analytic on \(|\omega| < 1\) and is continuous at \(\omega = 1\) and \(g_b(1) = 1\). Then

\[\lim_{\omega \to 1} g_b(\omega) = \frac{1}{1 - (1 - \theta_1)\beta_b(\delta q) - (1 - \theta_2)(1 - \beta_b(\delta q)) - \frac{\lambda(1 - \beta_b(\delta q))}{\delta q}} \]

and so

\[\lim_{\omega \to 1} \frac{1 - \omega}{1 - \omega Q(g_b(\omega))}\left\{ \beta_e(\lambda(1 - g_b(\omega)) + \delta q)(\theta_1 + (1 - \theta_1)g_b(\omega)) \\
+ \delta q(\theta_2 + (1 - \theta_2)g_b(\omega))\frac{(1 - \beta_e(\lambda(1 - g_b(\omega) + \delta q))}{\lambda(1 - g_b(\omega) + \delta q}
\right\}
\]

\[= \frac{1 - \{(1 - \theta_1)\beta_b(\delta q) + (1 - \theta_2)(1 - \beta_b(\delta q)) + \frac{\lambda(1 - \beta_b(\delta q))}{\delta q}\}}{E(N) + \{[(\theta_2 - \theta_1) - \frac{\lambda(1 - \beta_b(\delta q))}{\delta q}\}}
\]

where the mean value theorem of the differential calculus is used.

Substituting this in (7.12), Equation (7.10) is obtained. The proof of the theorem is complete.

\[\square\]
The moments of the limiting distribution can be obtained from the derivatives of the limiting generating function. For instance, let $E(\xi) = L'(1)$ be the mean of system size under steady-state condition, from Equation (7.10) the following expression for mean $E(\xi)$ is obtained:

$$E(\xi) = \left\{ [E(N^2) - E(N)] \left\{ 1 - \left[ \frac{(\lambda + \delta q(1 - \theta_2)) - \beta_0(\delta q)(\lambda + \delta q(\theta_1 - \theta_2))}{\delta q} \right] \right\} \right.$$  

$$+ \frac{2E(N)}{(\delta q)^2} \left\{ [\lambda + \delta q(\theta_1 - \theta_2)][(\lambda - \delta q \theta_2) \beta_\epsilon(\delta q) - (\lambda + \delta q(\theta_1 - \theta_2)) \beta_\epsilon(\delta q) \beta_\epsilon(\delta q) - \lambda \delta q q^2(\delta q)] + \delta q[\lambda + \delta q(1 - \theta_2)][\theta_2 + (\theta_1 - \theta_2) \beta_\epsilon(\delta q)] \right\}$$  

$$+ \frac{2\lambda}{(\delta q)^2} \left\{ [\beta_\epsilon(\delta q) - \beta_\epsilon(\delta q)][(\lambda + \delta q(1 - \theta_2)) \beta_\epsilon(\delta q)(\lambda + \delta q(\theta_1 - \theta_2)) - \lambda + \delta q \theta_2] \right\}$$  

$$\left\{ 2 \left[ 1 - \left( \frac{(\lambda + \delta q(1 - \theta_2)) - \beta_\epsilon(\delta q)(\lambda + \delta q(\theta_1 - \theta_2))}{\delta q} \right) \right] \right\}^{-1}$$

where $\beta_\epsilon(\delta q) = \int_0^\infty t e^{-s t} \beta_\epsilon(t) dt$ and $\ell$ stands either for $e$ or $b$. From Theorem 7.3.3, the following corollaries are deduced.

**Corollary 7.3.4.** Let $G(x)$ be the common service time distribution (i.e., $G_\epsilon = G_b = G$), $\beta(s)$ be its Laplace-Stieltjes transform with mean $\beta_1$ and the feedback parameters $\theta_1 = \theta_2 = \theta$. If $\lambda_\beta(\delta q) < 0$, then independent of the initial distribution, the limiting distribution $\{P_k\}$ of the number of customers in the system either at the service completion epoch of the customer or at which the customer succumbs to disaster, has probability generating function $L(z)$ given by

$$L(z) = \sum_{k=0}^{\infty} P_k z^k = \frac{[1 - Q(z)]}{E(z)(1 - z)} \left\{ \frac{(\theta - \frac{\lambda(1 - \beta(\delta q))}{\delta q}) (1 - z) [\beta(\lambda(1 - z) + \delta q) + \frac{\delta q (1 - \beta(\lambda(1 - z) + \delta q))}{\lambda(1 - z) + \delta q}]}{(\theta + (1 - \theta) z) [\beta(\lambda(1 - z) + \delta q) + \frac{\delta q (1 - \beta(\lambda(1 - z) + \delta q))}{\lambda(1 - z) + \delta q} - z] \right\}$$

(7.14)
and the mean $E(\xi)$ as

$$E(\xi) = \frac{1}{2} \left[ \frac{E(N^2)}{E(N)} - 1 \right] + \frac{\lambda \left( \lambda - \delta_q \theta \right) \beta (\delta_q - \omega^2 \delta_q) - \lambda \delta_q \beta (\delta_q) + \delta_q (\lambda + \delta_q (1 - \theta)) \right]}{\delta_q \left( \delta_q - \left( \lambda + \delta_q (1 - \theta) \right) - \lambda \beta (\delta_q) \right)}.$$

Further, if $q \to 0$ and $\lambda \beta_1 < 0$, then independently of the initial distribution the limiting distribution $\{P_k\}$ of the number of customers in the system just after a service completion epoch of a customer has probability generating function $L(z)$ given by

$$L(z) = \frac{(1 - Q(z))}{E(N)(1 - z)} \left\{ \frac{(\theta + (1 - \theta)z)(\theta - \lambda \beta_1)(1 - z)\beta(\lambda(1 - z))}{(\theta + (1 - \theta)z)\beta(\lambda(1 - z)) - z} \right\}.$$  

(7.15)

In particular, if the busy period starts with one customer i.e., $Q(z) = z$, then (15) becomes

$$L(z) = \frac{(\theta + (1 - \theta)z)(\theta - \lambda \beta_1)(1 - z)\beta(\lambda(1 - z))}{(\theta + (1 - \theta)z)\beta(\lambda(1 - z)) - z}.$$  

(7.16)

which is in accordance with Equation (3.9) of Disney et al (1984).

Equation (7.16) can be written as

$$L(z) = \theta H(z) + (1 - \theta)z H(z)$$

where

$$H(z) = \frac{(\theta - \lambda \beta_1)(1 - z)\beta(\lambda(1 - z))}{(\theta + (1 - \theta)z)\beta(\lambda(1 - z)) - z}$$

is the probability generating function of the number of customers in the $M/G/1$ Bernoulli feedback queueing model when a customer leaves the system just after his service completion (Takacs (1963)).

**Corollary 7.3.5.** In Corollary 7.3.4 if the service time distribution $G(x) = 1 - e^{-\mu x}, x \geq 0$, and the busy period starts with one customer i.e. $Q(z) = z$, then the limiting distribution $\{P_k\}$ is given by

$$P_0 = \theta(1 - \rho)$$

and

$$P_n = (1 - \rho)\rho^{n-1}[(\theta + (1 - \theta)]}, \quad n = 1, 2, 3, \ldots$$  

(7.17)

where $\rho = \frac{\lambda}{\theta(\mu + 4q)}$. 

Proof. From (7.14),
\[
L(z) = \frac{[\theta + (1 - \theta)z][\theta(\mu + \delta q) - \lambda]}{[\theta(\mu + \delta q) - \lambda z]}
\]
and hence
\[
L(z) = \left(1 - \frac{\lambda}{\theta(\mu + \delta q)}\right)\left(\theta + (1 - \theta)z\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{\theta(\mu + \delta q)}\right)^n z^n.
\]
Equating the coefficients of \(z^n\) for \(n = 0, 1, 2, \ldots\) on both sides of the above equation, the expression for \(P_n\) are obtained. □

Corollary 7.3.6. Let the common service time distribution \(G(x)\) be exponential with parameter \(\mu\) i.e. \(G(x) = 1 - e^{-\mu x}, x \geq 0\) and the busy period starts with one customer i.e. \(Q(z) = z\). If \(\theta_1 \to 1\), then the limiting distribution \(\{P_k\}\) is given by
\[
P_0 = \left(1 - \frac{(\lambda + (1 - \theta_2)\delta q)}{\mu + \delta q}\right)
\]
\[
P_n = \left(1 - \frac{(\lambda + (1 - \theta_2)\delta q)}{\mu + \delta q}\right) \left(\frac{\lambda}{\mu + \delta q\theta_2}\right)^{n-1} \left[\frac{\lambda + \delta q(1 - \theta_2)}{\mu + \delta q\theta_2}\right], \quad n = 1, 2, 3, (7.18)
\]
On the other hand, if \(\theta_2 \to 1\) then the limiting distribution \(\{P_k\}\) is given by
\[
P_0 = \left(1 - \frac{(\lambda + \mu(1 - \theta_1))}{\mu + \delta q}\right)
\]
\[
P_n = \left(1 - \frac{(\lambda + \mu(1 - \theta_1))}{\mu + \delta q}\right) \left(\frac{\lambda}{\mu\theta_1 + \delta q}\right)^{n-1} \left[\frac{\lambda + \mu(1 - \theta_1)}{\mu\theta_1 + \delta q}\right], \quad n = 1, 2, 3, (7.19)
\]
Proof. Substituting \(\beta_k(s) = \beta_k(s) = \frac{\mu}{\mu + s}\), \(Q(z) = z\) and \(\theta_1 = 1\) in Equation (7.10), the probability generating function \(L(z)\) is given as
\[
L(z) = \left(1 - \frac{(1 - \theta_2)\delta q + \lambda}{\mu + \delta q}\right) \left[\frac{\mu + \delta q\theta_2 + \delta q(1 - \theta_2)z}{(\mu + \theta_2\delta q) - \lambda z}\right]
\]
\[
= \left(1 - \frac{(1 - \theta_2)\delta q + \lambda}{\mu + \delta q}\right) \left[\frac{1 + \frac{\delta q(1 - \theta_2)z}{1 - \frac{1}{\mu + \delta q\theta_2}}}{\frac{1}{\mu + \theta_2\delta q}}\right].
\]
The above can be written in the series form as
\[
L(z) = \left[1 - \frac{(1 - \theta_2)\delta q + \lambda}{\mu + \delta q}\right] \left[1 + \frac{\delta q(1 - \theta_2)z}{\mu + \delta q\theta_2}\right] \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu + \delta q\theta_2}\right)^n z^n.
\]
Equating the coefficient of $z^n, n = 0, 1, 2, \ldots$, on both sides of the above equation, explicit expressions for the probabilities $P_n$ are obtained.

In a similar manner, substituting \( \beta_h(s) = \beta_u(s) = \frac{\mu}{\mu + \theta} \), \( Q(z) = z \) and \( \theta_2 = 1 \) in Equation (7.10), it is shown that

\[
L(z) = \left( \frac{\delta q + \theta \mu - \lambda}{\mu + \delta q} \right) \left( \frac{\mu \theta + \delta q + \mu(1 - \theta) z}{(\delta q + \theta \mu)(1 - \frac{\lambda}{\delta q + \mu \theta}) z} \right)
\]

so that

\[
L(z) = \left[ 1 - \frac{(\lambda + \mu(1 - \theta))}{\mu + \delta q} \right] \left[ 1 + \frac{\mu(1 - \theta)}{\delta q + \mu \theta} z \right] \sum_{n=0}^{\infty} \left( \frac{\lambda}{\delta q + \mu \theta} \right)^n z^n
\]

and equating the coefficient of $z^n, n = 0, 1, 2, \ldots$, on both sides of the above equation, Equation (7.19) is obtained. \( \square \)

**Corollary 7.3.7.** Let the common service time distribution be \( G(x) = 1 - e^{-\mu x}, x \geq 0 \).

Further, it is assumed that each time when the system becomes empty, the server waits until exactly \( N \) customers are accumulated. That is, the busy period starts only when a system of size \( N \) has been built up. For \( Q(z) = z^N \) and \( \theta = \theta_1 = \theta_2 \), the limiting distribution \( \{P_n\} \) of the number of customers in the system either at service completion epoch of a customer or at which the customer succumbs to disaster is given as

\[
P_n = \begin{cases} \frac{1}{N} \theta(1 - \rho^{n+1}) + \frac{(1-\theta)}{N} (1 - \rho^{n+2}), & \text{for } 0 \leq n \leq N - 1 \\ \frac{1}{N} \theta(\rho^{n-N+1} - \rho^{n+1}) + \frac{(1-\theta)}{N} (\rho^{n-N+2} - \rho^{n+2}), & \text{for } n \geq N \end{cases}
\]

where \( \rho = \frac{\lambda}{\theta(\mu + \delta q)} \).

**Proof.** From Equation (7.10), after some algebraic manipulation, \( L(z) \) ism given as

\[
L(z) = \left[ \theta + (1 - \theta)z \right] \left( \frac{1 - z^N}{N(1 - z)} \right) \left[ \frac{1 - \rho}{1 - \rho z} \right]
\]
and hence

\[ L(z) = \left( \frac{1 - \rho}{N} \right) \sum_{j=0}^{N-1} z^j (\theta + (1 - \theta)z) \sum_{n=0}^{\infty} \rho^n z^n \]

\[ = \left( \frac{1 - \rho}{N} \right) \left\{ \theta \sum_{j=0}^{N-1} z^j \sum_{n=0}^{\infty} \rho^n z^n + (1 - \theta) \sum_{j=0}^{N-1} \sum_{n=0}^{\infty} \rho^n z^n \right\} \]

\[ = \left( \frac{1 - \rho}{N} \right) \left\{ \theta \sum_{n=0}^{\infty} \sum_{j=0}^{N-1} \rho^n z^{n+j} + (1 - \theta) \sum_{n=0}^{\infty} \sum_{j=0}^{N-1} \rho^n z^{n+j+1} \right\} \]

\[ = \left( \frac{1 - \rho}{N} \right) \left\{ \theta \sum_{n=0}^{\infty} \sum_{i=0}^{i+N+1} \rho^i z^n + (1 - \theta) \sum_{n=0}^{\infty} \sum_{i=0}^{i+N} \rho^i z^n \right\} \]

\[ = \frac{1}{N} \left\{ \theta \sum_{n=0}^{N-1} (1 - \rho^{n+1}) z^n + \theta \sum_{n=N}^{\infty} (\rho^{n-N+1} - \rho^{n+1}) z^n 
+ (1 - \theta) \sum_{n=0}^{N} (1 - \rho^{n+2}) z^n + (1 - \theta) \sum_{n=N+1}^{\infty} (\rho^{n-N+2} - \rho^{n+2}) z^n \right\} \quad (7.21) \]

and equating the coefficient of \( z^n \) of first and last terms of (7.21), Equation (7.20) is obtained.

In the next section, various vacation models are examined as special cases for this model.

### 7.4 VACATION QUEUEING SYSTEMS

The significant developments in queueing theory with server vacation have lead to the new branch of queueing systems, namely, "Vacation Queueing Systems". In vacation queueing systems, the vacation duration corresponds to the length of times the server or system spends on secondary jobs. The utilization of server idle time in a vacation queueing system corresponds to performing other tasks like server taking a rest in a queueing system, attending preventive maintenance jobs in a production system, testing and maintenance jobs in computer and communication systems and serving
secondary jobs in some other queueing systems (Levy and Yechiali (1975)). An excellent survey of queueing system with server vacations could be found in Doshi (1986, 1990) and Takagi (1991). Queueing systems with threshold policy and multiple vacations including some applications, have been studied by Lee and Srinivasan (1989) and Kella (1989). They have dealt with the batch arrival $M^X/G/1$ and single unit arrival $M/G/1$ queueing systems respectively, and examined the system performances and obtained the optimal threshold under a stationary cost function. Lee et al (1994a, b) have analyzed in detail Lee and Srinivasan system with a single vacation and multiple vacations, respectively. They also have confirmed the stochastic decomposition property given by Fuhrmann and Cooper (1985).

The server setup corresponds to the preparatory work of the server before starting the service. In some actual situations, the server often requires a startup time before commencing each service period. Baker (1973) has proposed the single threshold policy $M/M/1$ queueing system with exponential startup time. Borthakar et al (1987) have extended Baker’s results to the general startup time. The single threshold policy $M/G/1$ queueing system with startup time has been studied by Minh (1988), Medhi and Templeton (1992), Takagi (1993) and Lee and Park (1997). Recently, Hur and Paik (1999) have examined the operation characteristics of the threshold policy $M/G/1$ queueing system with server startup and obtained the system’s optimal policy and cost behaviour for various arrival rates. In this section, the various vacation models, threshold policy and setup time models for the generalized feedback queueing system with disasters has been analyzed as discussed in section 7.3.

7.4.1 MULTIPLE VACATION QUEUEING SYSTEM

Assume that the server begins a vacation each time the queue becomes empty. If the server returns from a vacation to find the queue not empty, he starts serving immediately those customers accumulated there during his vacation and continues
until the queue becomes empty again (exhaustive service) and then begins the next vacation. If the server returns from a vacation and finds no customer waiting, he begins another vacation immediately and continues in this fashion until he finds at least one customer waiting upon returning from a vacation (multiple vacations). The length of each vacation is assumed to be independent of the arrival process (Cooper (1970), Levy and Yechiali (1975) and Scholl (1976)).

Further assume that the length $V$ of each vacation is independent and identically distributed with its Laplace-Stieltjes transform of the distribution function given by $V^*(s)$. Then, because of the independence of $V$ and the arrival process,

$$F(z) = V^*(\lambda(1 - z)) \quad \text{and} \quad E(X) = \lambda E(V)$$

where $X$ is the number of customers that arrive during each vacation and $F(z)$ is the generating function for $X$. Thus $F(0) = V^*(\lambda)$ is the probability that no customers arrive during a vacation. As the vacation period terminates if and only if at least one customer arrives during a vacation, then for this model,

$$Q(z) = \frac{V^*(\lambda(1 - z)) - V^*(\lambda)}{1 - V^*(\lambda)} \quad \text{and} \quad E(N) = \frac{\lambda E(V)}{1 - V^*(\lambda)}$$

so that

$$\frac{1 - Q(z)}{(1 - z)E(N)} = \frac{1 - V^*(\lambda(1 - z))}{\lambda E(V)(1 - z)},$$

which is the probability generating function of the number of customers who arrive before an arbitrary time point during the vacation period.

From (7.14), the probability generating function $L(z)$ of the number of customers in the system either after the service completion epoch of a customer or at which the customer succumbs to disaster is

$$L(z) = \frac{1 - V^*(\lambda(1 - z))}{\lambda E(V)(1 - z)} [\theta + (1 - \theta)z]$$

$$\left\{ \frac{(\theta - \frac{\lambda(1 - \theta)\delta q}{\beta q})(1 - z)[\beta(\lambda(1 - z) + \delta q) + \delta q (1 - \beta(\lambda(1 - z) + \delta q))]}{\lambda(1 - z) + \delta q} \right\} \left\{ \frac{(\theta + (1 - \theta)z)[\beta(\lambda(1 - z) + \delta q) + \delta q (1 - \beta(\lambda(1 - z) + \delta q)]}{\lambda(1 - z) + \delta q} \right\}.$$
7.4.2 SINGLE VACATION QUEUEING MODEL

This subsection is devoted to the single vacation process (see Levy and Yechiali (1975) and Doshi (1986)) where the server takes only a single vacation at the end of a busy period. If, upon return from a vacation, there are customers present, their service starts (one at a time), as in the previous model, with no delay and the server serves exhaustively during the busy period until the system becomes empty when he goes on vacation. However, if upon return from a vacation the server finds an empty system he waits idly until for the first customer arrives, upon which the next busy period starts. Note that the vacation here can be thought of as a post-processing time needed after clearing the jobs.

For this model, the number of customers in the system when the service actually begins is 1 if no customers arrive during the vacation (with probability \(V^*(X)\)) or ‘\(n\)’ customers with probability \(p_m(m \geq 1)\). Thus,

\[Q(z) = V^*(\lambda)z + V^*(\lambda(1 - z)) - V^*(\lambda) \quad \text{and} \quad E(N) = \lambda E(V) + V^*(\lambda)\]

and hence

\[
1 - Q(z) = \frac{1 - V^*(\lambda(1 - z)) + (1 - z)V^*(\lambda)}{(1 - z)[\lambda E(V) + V^*(\lambda)]}.
\]

Using these results in (7.14), the probability generating function of the number of customers in the system either after the service completion epoch of a customer or at which the customer succumbs to disaster is obtained as

\[
L(z) = \frac{[1 - V^*(\lambda(1 - z)) + (1 - z)V^*(\lambda)]}{(1 - z)[\lambda E(V) + V^*(\lambda)]} \left[ \theta + (1 - \theta)z \right]
\]

\[
\left\{ \frac{(\theta - \frac{\lambda(1-\lambda(1-z))}{\delta q})(1-z)[\beta(1-z)+\delta q]}{(\theta + (1-\theta)z)[\beta(1-z)+\delta q] + \delta q \frac{1-\beta(1-z)+\delta q}{\lambda(1-z)+\delta q} - z} \right\}.
\]
7.4.3 QUEUES WITH N-POLICY

Now consider an $M/G/1$ queue with $N$-policy, which has been studied by Yadin and Naor (1963). The behaviour of the system is controlled by the $N$-policy. In this policy, the server is turned off when the system becomes empty and turned on when the number of customers reaches the size exactly $N$. Thus, it is clear that the particular case $N = 1$ reduces to the standard $M/G/1$ queue. In this case,

$$Q(z) = z^N \quad \text{and} \quad E(N) = N$$

so that

$$\frac{1 - Q(z)}{(1 - z)E(N)} = \frac{1 - z^N}{N(1 - z)}.$$

Using the above results in (7.14) we obtain the probability generating function of the number of customers in the system is obtained as

$$L(z) = \frac{1 - z^N}{N(1 - z)} \left[ \theta + (1 - \theta)z \right] \left\{ \frac{\theta - \lambda(1 - \beta(\delta q))}{\beta(1 - z) + \delta q} (1 - z) \beta(1 - z) + \delta q \frac{1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} \right\} + \left( \theta + (1 - \theta)z \right) \beta(1 - z) + \delta q \frac{1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} - z.$$

7.4.4 M/G/1 QUEUEING SYSTEM WITH SETUP TIMES

Consider another variant of an $M/G/1$ queue where the first customer in each busy period needs a random setup time $S$, before its service starts, whose Laplace-Stieltjes transform of the distribution function is given by $S^*(s)$. This model has been studied by Doshi (1985, 1986), Scholl and Kleinrock (1983), Levy and Kleinrock (1986) and Scholl (1976).

For this model,

$$Q(z) = zS^*(\lambda(1 - z)) \quad \text{and} \quad E(N) = 1 + \lambda E(S)$$
and then it follows that
\[
\frac{1 - Q(z)}{(1 - z)E(N)} = \frac{1 - zS^*(\lambda(1 - z))}{(1 + \lambda E(S))(1 - z)}
\]
for the number of customers found in the system at the end of the setup time. The probability generating function of the number of customers in the system for this queueing system is given as

\[
L(z) = \frac{1 - zS^*(\lambda(1 - z))}{(1 + \lambda E(S))(1 - z)} [\theta + (1 - \theta)z] \\
\left\{ \frac{\left(\theta - \frac{\lambda(1 - \theta)\delta}{\delta q}\right)(1 - z)\beta(\lambda(1 - z) + \delta q) + \delta q \frac{1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q}}{\theta + (1 - \theta)z}[\beta(\lambda(1 - z) + \delta q) + \delta q \frac{1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q}] - z \right\}
\]

**7.4.5 QUEUEING SYSTEM UNDER \textit{N}-POLICY WITH SETUP AND CLOSE-DOWN TIMES**

As an extension of the \textit{N}-policy, assume that a setup time, whose Laplace-Stieltjes transform of the distribution function is given by \(S^*(s)\), is needed before starting each busy period. It is further assumed that, after each busy period, the server needs a close-down time, before going on vacation, whose distribution function has Laplace-Stieltjes transform \(D^*(s)\).

If a customer arrives during the close-down time, the service is immediately started without waiting for the accumulation of \(N\) customers and without a setup time. For such a model, \(Q(z)\) is expressed as

\[
Q(z) = D^*(\lambda)z^NS^*(\lambda(1 - z)) + [1 - D^*(\lambda)]z
\]

\[
E(N) = [N + \lambda E(S)]D^*(\lambda) + [1 - D^*(\lambda)]
\]

and hence

\[
\frac{1 - Q(z)}{(1 - z)E(N)} = \frac{1 - [D^*(\lambda)z^NS^*(\lambda(1 - z)) + (1 - D^*(\lambda))]z}{(1 - z)[N + \lambda E(S)]D^*(\lambda) + [1 - D^*(\lambda)]}
\]
This leads to the probability generating function as

\[
L(z) = \frac{1 - [D*(\lambda)zN S*(\lambda(1 - z)) + [1 - D*(\lambda)]z]}{(1 - z)[(N + \lambda E(S)*D*(\lambda) + [1 - D*(\lambda)]]} \left[ \theta + (1 - \theta)z \right]
\]

\[
\left\{ \frac{(\theta - \lambda(1 - \beta(\delta q)))(1 - z)\beta(\lambda(1 - z) + \delta q) + \delta q(1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} \right\}.
\]

\[
L(z) = 1 - \left[ D*(\lambda)zN S*(\lambda(1 - z)) + [1 - D*(\lambda)]z \right]
\]

\[
(1 - z)[(N + \lambda E(S)*D*(\lambda) + [1 - D*(\lambda)]] \theta + (1 - \theta)z \right]\left\{ \frac{(\theta - \lambda(1 - \beta(\delta q)))(1 - z)\beta(\lambda(1 - z) + \delta q) + \delta q(1 - \beta(\lambda(1 - z) + \delta q)}{\lambda(1 - z) + \delta q} \right\}.
\]

7.5 NUMERICAL RESULTS

In this section, numerical behaviour of the quantities \(E(\xi)\) and \(L(0)\), are examined. The main interest is to illustrate the effect of the parameters on the expected value \(E(\xi)\) of the number of customers in the system and \(L(0)\) the probability that the system is empty.

This numerical analysis deals with the case of exponential distribution for service time distributions denoted by \(G_\mu(x) = 1 - e^{-\mu x}\) and \(G_\delta(x) = 1 - e^{-\delta x}\) for the feedback \(N\)-policy problem subject to disasters. Under steady-state condition, from Equation (7.13), the effect of varying feedback parameters \(\theta_1\) and \(\theta_2\) on the mean of the system size \(E(\xi)\) is shown as the surface in Figure 7.1 for \((\lambda, q, \delta, \mu_1, \mu_2, N) = (0.5,0.7,8,20,15,7)\). The surface displays, as expected, a downward trend for \(E(\xi)\) against increasing \(\theta_1\) and \(\theta_2\) values. In Figure 7.2, the variation of \(E(\xi)\) with respect to \(\delta\) and \(N\) is plotted for \((\lambda, q, \delta, \mu_1, \mu_2, \theta_1, \theta_2) = (0.5,0.5,20,15,0.5,0.5)\). Here the surface displays a sharp increase in \(E(\xi)\) with increasing values of \(\delta\) and \(N\).

The Figures 7.3 and 7.4 display the surface for variation of the probability of an empty system \(L(0)\) as in Equation (7.11) under steady-state condition, for chosen parametric values.

Figure 7.3 : \((\lambda, q, \delta, \mu_1, \mu_2, N) = (0.5,0.7,8,20,15,7)\),

Figure 7.4 :\((\lambda, q, \delta, \mu_1, \mu_2, \theta_1, \theta_2) = (0.5,0.5,20,15,0.5,0.5)\).
Figure 7.1 $E(\xi)$ versus $(\theta_1, \theta_2)$ for a queue with Exponential distribution $(N, \delta, \lambda, \mu_1, \mu_2, q) = (7, 8, 0.5, 20, 15, 0.7)$.

Figure 7.2 $E(\xi)$ versus $(N, \delta)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \theta_1, \theta_2, q) = (0.5, 20, 15, 0.5, 0.5, 0.5)$. 
Figure 7.3 $L(0)$ versus $(\theta_1, \theta_2)$ for a queue with Exponential distribution $(N, \delta, \lambda, \mu_1, \mu_2, q) = (7, 8, 0.5, 20, 15, 0.7)$.

Figure 7.4 $L(0)$ versus $(N, \delta)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \theta_1, \theta_2, q) = (0.5, 20, 15, 0.5, 0.5, 0.5)$. 
Figure 7.5 $E(\xi)$ versus $N$ with $(\lambda, \delta, \mu_1, \mu_2, \theta_1, q) = (2, 9, 20, 15, 0.5, 0.4)$.

Figure 7.6 $E(\xi)$ versus $N$ with $(\lambda, \delta, \mu_1, \mu_2, \theta_1, q) = (2, 9, 20, 15, 0.5, 0.4)$. 
Figure 7.7 $L(0)$ versus $N$ with $(\lambda, \delta, \mu_1, \mu_2, \theta_2, q) = (2, 9, 20, 15, 0.5, 0.5)$.

Figure 7.8 $L(0)$ versus $N$ with $(\lambda, \delta, \mu_1, \mu_2, \theta_2, q) = (2, 9, 20, 15, 0.5, 0.5)$. 
Figure 7.9 The steady-state system size probabilities for different values of $N$ for the $N$-policy feedback queueing system subject to the disasters with $\lambda = 0.5, \delta = 9, \mu = 25, \theta = 0.5, q = 0.5$. 

Figure 7.9 The steady-state system size probabilities for different values of $N$ for the $N$-policy feedback queueing system subject to the disasters with $\lambda = 0.5, \delta = 9, \mu = 25, \theta = 0.5, q = 0.5$. 
Observe that probability of the system size being zero increases gradually as in the Figure 7.3 with increasing parametric values of $\theta_1$ and $\theta_2$, while, the Figure 7.4 shows decreasing trend of the probability of the system size being zero, against increasing values of $\delta$ and $N$.

The graphs illustrated in Figures 7.4 and 7.5 compare the behaviour of $E(\xi)$ against the parameter $N$ for varying values of $\theta_1$ and $\theta_2$ respectively for increasing values of $N$. Similarly the graphs 7.6 and 7.7 compare the trend of $L(0)$ for the chosen parametric values and distributions.

Expressions for the steady-state system size probabilities are given analytically in Equation (7.20). However, it is important to visualize the solutions in practical situations. For this reason numerical illustrations of the steady-state probabilities of the system are given. In Figure 5, the steady-state probabilities $P_1, P_2, P_3, P_4$ and $P_5$ for the system with the values of the parameters $\lambda = 0.5, \mu = 25, \theta = 0.5, q = 0.5$ and $\delta = 9$ are plotted. It is observed that the steady-state probability curves increase initially and then decrease gradually for increasing values of $N$.

7.6 CONCLUSION

In this chapter, a generalized $M/G/1$ feedback queue in which a customer in service is subject to disasters is analyzed. It is assumed that the service time distributions of a customer which initiates a busy period is $G_e(x)$ and all subsequent customers in the same busy period have service time drawn independently from the distribution $G_b(x)$. The server is idle until a random number $N$ of customers accumulated in the queue.