5.1 INTRODUCTION

Retrial queueing systems with general service times and non-exponential retrial time distributions have been received little attention. The first work on the $M/G/1$ retrial queue with general retrial times is due to Kapyrin (1997) who assumed that each customer in orbit generates a sequence of repeated attempts that are independent of the customers in the orbit and the server state. However, this methodology was found to be incorrect by Falin (1990). Subsequently, Yang et al (1994) have developed an approximation method to obtain the steady-state performance measures for the model of Kapyrin.

Fayolle (1986) has investigated an $M/M/1$ retrial queue where the customers in the retrial group form a queue and only the customer at the head of the queue can request for service from the server after exponentially distributed retrial time with constant rate $\alpha$. Farahmand (1990) calls this discipline as a retrial queue with FCFS orbit. This kind of retrial control policy is well known for the stability of the ALOHA protocol in communication systems (see, Bertsekas and Gallager (1987)). A retrial queueing system with FCFS discipline and general retrial times has been extensively discussed by Gomez-Corral (1999).

In some of the service stations, an arriving job of higher priority may push-out the job of lower priority whose service is ongoing, to the orbit to commence its own service.
A push-out scheme with preemptive resume for a shared buffer (see, Kroener et al (1991) and Sumita (1989)) is one of the control methods adopted for problems involving the sharing of common resources in the network. The model under consideration in this chapter in a way relates to the delay control in the communication networks with a priority component.

One of the important characteristics of a queueing system is the service process. The service of jobs may be provided by the server in two phases. In the first phase (preliminary processing) the server may process the incoming jobs and if they qualify, they are routed to the second phase (primary processing) for actual service. Such problems arising in distributed system control requiring two-phase execution at a central server have been discussed by Doshi (1991). The $M/G/1$ queueing system where the customer after first (regular) service may opt for the second optional service or may leave the system has been studied by Madan (2000).

Further, in computer and communication networks, messages of variable length arrive at a service station which processes them in two stages by a single server. At the preprocessing stage (preliminary service) the server may preempt the current message from the ongoing process to accommodate the new priority message upon arrival. After the completion of preprocessing, if the message qualifies, the primary processing will commence and otherwise, it has to leave the system. The server gets locked during the primary processing so that an arriving message is routed to the orbit for retrial.

To model such a system, in this chapter, consider an $M/G/1$ retrial queueing system with additional phase of service with a push-out scheme under possible Preemptive Resume (PR) service discipline. It is assumed that the retrial time is governed by an arbitrary distribution and that the customer at the head of the orbit queue is allowed for access to the server. The organization of the chapter is as follows: The model under consideration is described in section 5.2 along with the necessary and sufficient conditions for the system to be stable. The steady-state distribution of the server state
and the orbit length are discussed in section 5.3. In section 5.4, some performance measures are obtained along with some numerical examples to illustrate the effect of the parameters on the system performance. Finally, in section 5.5, a general stochastic decomposition law for the present model is established.

5.2 MODEL DESCRIPTION AND STABILITY CONDITION

Consider an \(M/G/1\) retrial queue with second phase of service provided after the first phase of (preliminary) service. There is a single server who provides this preliminary service to all arriving customers. Let \(B(x)\) and \(b(x)\) be the cumulative distribution function and the probability density function of the preliminary service time respectively with Laplace-Stieltjes transform \(\beta^*(\theta)\) and \(\beta_1, \beta_2\) as the first two moments. As soon as the preliminary service of a customer is completed, then with probability \(p\), the customer may be provided primary service in second phase or else with probability \(q(= 1 - p)\) he has to leave the system. The primary service times of customers are independent random variables with common distribution function \(H(x)\), probability density function \(h(x)\), Laplace-Stieltjes transform \(\nu^*(\theta)\) and first two moments \(\nu_1\) and \(\nu_2\).

New customers arrive from outside the system according to a Poisson stream with rate \(\lambda\). It is assumed that the preliminary service commences for an arriving customer, if the server is free. While at the preliminary service, the server may push-out (with probability \(\alpha\)) the customer undergoing such service to the orbit, to commence preliminary service of an arriving customer or continue the ongoing service (with probability \((1 - \alpha)\)) so that the arriving customer leaves the service area to join the orbit in accordance with FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to the server. We further assume that such push-out facility is not available while the server is providing primary service. Successive inter-retrial times of any customer are governed by an arbitrary probability distribution function \(A(x)\)
with corresponding density function \( a(x) \) and Laplace-Stieltjes transform \( \phi^*(\theta) \). It is observed that for \( \alpha = 0 \), the system becomes \( M/G/1 \) retrial queue (with no push-out) having FCFS discipline, while for \( \alpha = 1 \), the model leads to LCFS discipline with preemptive resume service.

Further assume that the input flow of new arrivals (i.e., arrival epochs), intervals between repeated trials, preliminary and primary service times are mutually independent. Note that the server becomes free when the customer quits after preliminary service or after completion of his primary service. In such a case both the new arrival and the one (if any) at the head of the orbit queue compete for preliminary service.

The stochastic behaviour of this retrial queueing system can be described by the Markov process \( \{N(t); t \geq 0\} = \{(C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t)); t \geq 0\} \) where \( C(t) \) denotes the server state (0, 1, or 2, according as the server being free, providing the preliminary service or providing the primary service respectively) and \( X(t) \) corresponds to the number of customers in the orbit at any time \( t \). If \( C(t) = 0 \) and \( X(t) > 0 \), then \( \xi_0(t) \) represents the elapsed retrial time; if \( C(t) = 1 \), and \( X(t) \geq 0 \), then \( \xi_1(t) \) corresponds to the elapsed time of the customer being provided preliminary service; if \( C(t) = 2 \) and \( X(t) \geq 0 \), then \( \xi_2(t) \) represents the elapsed time of the customer being provided primary service at time \( t \). In what follows, we neglect \( \xi_i(t), \ i = 0, 1, 2 \) and consider only the pair \( (C(t), X(t)) \) whose state space is \( S = \{0, 1, 2\} \times \{0, 1, 2, 3, \ldots\} \). The functions \( \eta(x), \mu_1(x) \) and \( \mu_2(x) \) are the conditional completion rates (at time \( x \)) for repeated attempts, for preliminary service and for primary service respectively, i.e., \( \eta(x) = \frac{a(x)}{1 - A(x)}, \mu_1(x) = \frac{b(x)}{1 - B(x)} \) and \( \mu_2(x) = \frac{c(x)}{1 - C(x)} \).

The following theorem investigates the ergodicity of the embedded Markov chain at the customer departure epochs. Let \( \{t_n; n \in N\} \) be the sequence of epochs of the end of the service completion times at which the server is idle. The sequence \( \{X_n = X(t_n +)\} \) forms a Markov chain which is embedded in our retrial queueing system on the state space \( N \).
Theorem 5.2.1. Let $X_n$ be the orbit length at the time of the $n^{th}$ customer's departure, $n \geq 1$. Then $\{X_n; n \geq 1\}$ is ergodic if and only if $\frac{1-\beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} + p \lambda \nu_1 < \phi^*(\lambda)$.

The theorem can be proved along similar lines as in Gomez-Corral (1999). As the arrival stream is a Poisson process, Burke's Theorem [see Cooper (1981)] establishes the existence of the steady-state probabilities of $\{(C(t), X(t)); t \geq 0\}$ and they are positive if and only if $\frac{1-\beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} + p \lambda \nu_1 < \phi^*(\lambda)$.

From the mean drift $\chi_j = \frac{1-\beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} + p \lambda \nu_1 - \phi^*(\lambda)$ for all $j \geq 1$, where $j$ denotes the number of customers in the orbit. It can be reasonably concluded that term $\frac{1-\beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)}$ represents the contribution for the mean number of customers leaving for orbit due to the decision of the server to push-out or continue the ongoing service. The second term $p \lambda \nu_1$ corresponds to the mean number of customers leaving for orbit due to the server being busy with primary service. Further $\phi^*(\lambda)$ provides the expected number of orbiting customers who enter service successfully, given that the previous service time leaves $j$ customers in the orbit. For stability, it is required that the new customers arrive during a service time more slowly than customers from the orbit who enter service successfully. That is

$$\frac{1-\beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} + p \lambda \nu_1 < \phi^*(\lambda)$$

implying $\chi_j < 0$ for $j \geq 1$.

5.3 STEADY-STATE DISTRIBUTION

In this section, the steady-state distribution for the system under consideration is analyzed. For the process $\{N(t); t \geq 0\}$, define the probability $R_0(t) = P\{C(t) = 0, X(t) = 0\}$.
and the probability densities

\[ R_n(x,t)dx = P\{C(t) = 0, X(t) = n, x \leq \xi_0(t) < x + dx\}, \]
for \( t \geq 0, x \geq 0 \) and \( n \geq 1 \),

\[ P_n(x,t)dx = P\{C(t) = 1, X(t) = n, x \leq \xi_1(t) < x + dx\}, \]
for \( t \geq 0, x \geq 0 \) and \( n \geq 0 \)

and

\[ S_n(x,t)dx = P\{C(t) = 2, X(t) = n, x \leq \xi_2(t) < x + dx\}, \]
for \( t \geq 0, x \geq 0 \) and \( n \geq 0 \).

By supplementary variable technique, the following system of equations are obtained:

\[
\frac{dR_0(t)}{dt} = -\lambda R_0(t) + q \int_0^\infty P_0(x,t)\mu_1(x)dx \\
+ \int_0^\infty S_0(x,t)\mu_2(x)dx 
\]
(5.1)

\[
\frac{\partial R_n(x,t)}{\partial t} + \frac{\partial R_n(x,t)}{\partial x} = -(\lambda + \eta(x)) \ R_n(x,t), n = 1, 2, 3, ... 
\]
(5.2)

\[
\frac{\partial P_n(x,t)}{\partial t} + \frac{\partial P_n(x,t)}{\partial x} = -(\lambda + \mu_1(x)) \ P_n(x,t) \\
+ (1 - \delta_{n0})(1 - \alpha)\lambda P_{n-1}(x,t), n = 0, 1, 2, ... 
\]
(5.3)

\[
\frac{\partial S_n(x,t)}{\partial t} + \frac{\partial S_n(x,t)}{\partial x} = -(\lambda + \mu_2(x)) \ S_n(x,t) \\
+ (1 - \delta_{n0})\lambda S_{n-1}(x,t), n = 0, 1, 2, ... 
\]
(5.4)

where \( \delta_{nm} \) is the Kronecker delta. The boundary conditions are

\[
R_n(0,t) = q \int_0^\infty P_n(x,t)\mu_1(x)dx \\
+ \int_0^\infty S_n(x,t)\mu_2(x)dx, n = 1, 2, 3, ... 
\]
(5.5)
\[ P_0(0,t) = \int_0^\infty R_1(x,t)\eta(x)dx + \lambda R_0(t) \]  
(5.6)

\[ P_n(0,t) = \alpha \lambda \int_0^\infty P_{n-1}(x,t)dx + \int_0^\infty R_{n+1}(x,t)\eta(x)dx + \lambda \int_0^\infty R_n(x,t)dx, \ n = 1, 2, 3, ... \]  
(5.7)

\[ S_n(0,t) = p \int_0^\infty P_n(x,t)\mu_1(x)dx, \ n = 0, 1, 2, ... \]  
(5.8)

Assuming that the condition \( 1 - \phi'(\lambda) + p \lambda \nu_1 < \phi'(\lambda) \) is fulfilled, it is seen that the limiting probability \( R_0 = \lim_{t \to \infty} R_0(t) \) and limiting densities \( R_n(x) = \lim_{t \to \infty} R_n(x,t) \) for \( x \geq 0 \) and \( n \geq 1 \), \( P_n(x) = \lim_{t \to \infty} P_n(x,t) \) for \( x \geq 0 \) and \( n \geq 0 \) and \( S_n(x) = \lim_{t \to \infty} S_n(x,t) \) for \( x \geq 0 \) and \( n \geq 0 \) exist. Letting \( t \to \infty \) in Equations (5.1)-(5.8),

\[ \lambda R_0 = q \int_0^\infty P_0(x)\mu_1(x)dx + \int_0^\infty S_0(x)\mu_2(x)dx \]  
(5.9)

\[ \frac{dR_n(x)}{dx} = -(\lambda + \eta(x))R_n(x), \ n = 1, 2, 3, ... \]  
(5.10)

\[ \frac{dP_n(x)}{dx} = -(\lambda + \mu_1(x))P_n(x) \]  
(5.11)

\[ \frac{dS_n(x)}{dx} = -(\lambda + \mu_2(x))S_n(x) \]  
(5.12)

The steady-state boundary conditions are

\[ R_n(0) = q \int_0^\infty P_n(x)\mu_1(x)dx + \int_0^\infty S_n(x)\mu_2(x)dx, \ n = 1, 2, 3, ... \]  
(5.13)

\[ P_0(0) = \int_0^\infty R_1(x)\eta(x)dx + \lambda R_0 \]  
(5.14)

\[ P_n(0) = \alpha \lambda \int_0^\infty P_{n-1}(x)dx + \int_0^\infty R_{n+1}(x)\eta(x)dx + \lambda \int_0^\infty R_n(x)dx, \ n = 1, 2, 3, ... \]  
(5.15)

\[ S_n(0) = p \int_0^\infty P_n(x)\mu_1(x)dx, \ n = 1, 2, 3, ... \]  
(5.16)

and the normalizing condition is

\[ R_0 + \sum_{n=0}^{\infty} \int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty} \int_0^\infty S_n(x)dx + \sum_{n=1}^{\infty} \int_0^\infty R_n(x)dx = 1. \]  
(5.17)
In order to solve the Equations (5.9)-(5.16), define the generating functions as:

\[ P(x, z) = \sum_{n=0}^{\infty} P_n(x)z^n, \quad S(x, z) = \sum_{n=0}^{\infty} S_n(x)z^n \quad \text{and} \quad R(x, z) = \sum_{n=1}^{\infty} R_n(x)z^n. \]

The following theorem discusses the steady-state distribution of the system.

**Theorem 5.3.1.** If \( \frac{1 - p(x)}{\partial p(x)/\partial x} + p\lambda \nu_1 \neq \phi^*(\lambda) \), then the joint steady-state distributions of \( \{N(t); t \geq 0\} \) under different server states are obtained as

\[ P(x, z) = \lambda R_0 \left\{ \frac{(1 - z)(1 - z + \alpha z)\phi^*(\lambda)}{\beta^*(\lambda(1 - z) + \alpha \lambda z)} \right\} \]
\[ S(x, z) = \lambda R_0 \left\{ \frac{p(1 - z)(1 - z + \alpha z)\beta^*(\lambda(1 - z) + \alpha \lambda z)}{\phi^*(\lambda) e^{-\beta(1-z)+\alpha \lambda z}[1 - H(x)]} \right\} \]
\[ R(x, z) = \lambda R_0 \left\{ \frac{z\phi^*(\lambda(1 - z) + \alpha \lambda z)[(1 - z + \alpha z)q + p\nu^*(\lambda(1 - z))]}{\beta^*(\lambda(1 + \alpha z)) + (1 - z)e^{-\beta(1-z)+\alpha \lambda z}[1 - A(x)]} \right\} \]

where

\[ D(z) = \beta^*(\lambda(1 - z) + \alpha \lambda z)\{(1 - z + \alpha z) \]
\[ [(1 - z)\phi^*(\lambda) + z][q + p\nu^*(\lambda(1 - z))] - \alpha z^2 \} - z(1 - z) \]

and the probability \( R_0 \) is to be determined from the normalization condition.

**Proof.** Multiplying Equations (5.9)-(5.16) by \( z^n \) and summing over all possible values of \( n \),

\[ \frac{\partial R(x, z)}{\partial x} = -[\lambda + \eta(x)]R(x, z) \]  
\[ \frac{\partial P(x, z)}{\partial x} = -[(\lambda(1 - z) + \lambda \alpha z) + \mu_1(x)]P(x, z) \]
\[ \frac{\partial S(x, z)}{\partial x} = -[(\lambda(1 - z) + \mu_2(x))S(x, z) \]
The solutions of the partial differential Equations (5.21)-(5.23) are given by

\[ R(0, z) = q \int_0^\infty P(x, z) \mu_1(x) \, dx + \int_0^\infty S(x, z) \mu_2(x) \, dx - \lambda R_0 \]  
(5.24)

\[ P(0, z) = \lambda \alpha z \int_0^\infty P(x, z) \, dx + \frac{1}{z} \int_0^\infty R(x, z) \eta(x) \, dx + \lambda \int_0^\infty R(x, z) \, dx + \lambda R_0 \]  
(5.25)

\[ S(0, z) = p \int_0^\infty P(x, z) \mu_1(x) \, dx. \]  
(5.26)

The solutions of the partial differential Equations (5.21)-(5.23) are given by

\[ R(x, z) = R(0, z) e^{-\lambda x}[1 - \beta(x)] \]  
(5.27)

\[ P(x, z) = P(0, z) e^{-\lambda(1-z)\alpha z}[1 - \beta(1 - z)] \]  
(5.28)

\[ S(x, z) = S(0, z) e^{-\lambda(1-z)\beta(x)}[1 - H(x)]. \]  
(5.29)

Using (5.28) in (5.26),

\[ S(0, z) = p \beta'(\lambda(1 - z) + \alpha \lambda z) P(0, z). \]  
(5.30)

Combining (5.24)-(5.30) and on simplification, it can be obtained as

\[ R(0, z) = P(0, z)[q + \phi'(\lambda(1 - z))] - \beta'(\lambda(1 - z) + \alpha \lambda z) - \lambda R_0. \]  
(5.31)

Again, substituting (5.27)-(5.31) in (5.25) and solving for \( P(0, z) \), after some algebraic manipulation, it can be shown that

\[ P(0, z) = \lambda R_0 \frac{(1 - z)(1 - z + \alpha z)\phi'(\lambda)}{D(z)}. \]  
(5.32)

Finally, combining (5.30)-(5.32), the required results (5.18)-(5.20) are derived. \( \square \)

Define the partial generating function \( \psi(z) = \int_0^\infty \psi(x, z) \, dx \) for any generating function \( \psi(x, z) \). Then, from (5.18)-(5.20), the following partial generating functions under steady-state conditions are given as

\[ R(z) = R_0 \frac{z(1 - \phi'(\lambda))\{(1 - z) - \beta'(\lambda(1 - z) + \alpha \lambda z)\}}{|(1 - z + \alpha z)q + \phi'(\lambda(1 - z)) - \alpha z|} \]  
(5.33)

\[ P(z) = R_0 \frac{\phi'(\lambda)(1 - z)[1 - \beta'(\lambda(1 - z) + \alpha \lambda z)]}{D(z)} \]  
(5.34)
and

\[ S(z) = R_0 \frac{p \phi^*(\lambda)(1 - z + \alpha z) \beta^*(\lambda(1 - z) + \alpha \lambda z)}{[1 - \nu^*(\lambda(1 - z))] D(z)} \]  

(5.35)

where

\[ R_0 = 1 - \left( \frac{p \lambda \nu}{\phi^*(\lambda)} + \frac{1 - \beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda) \phi^*(\lambda)} \right) \]  

(5.36)

which is determined using the normalizing condition \( R_0 + R(1) + P(1) + S(1) = 1 \) and \( K(z) = R_0 + R(z) + zP(z) + zS(z) \), the probability generating function for the number of customers in the system is given by

\[ K(z) = R_0 \frac{\phi^*(\lambda)(1 - z)(1 - z + \alpha z) \beta^*(\lambda(1 - z) + \alpha \lambda z)}{[q + p \nu^*(\lambda(1 - z))] D(z)} \]  

(5.37)

Note that \( R(z) \) is the probability generating function of orbit size when the server is idle, \( P(z) \) is the probability generating function of the orbit size when the server is busy with preliminary service, \( S(z) \) is the probability generating function of the orbit size when the server is busy with primary service and \( R_0 \) is the probability that the server is idle in the system, i.e. no customer in the system. These expressions are used in the next section for obtaining performance measures.

**Remark 5.3.2.** The case of \( p = 1 \) may be viewed as a system with the preliminary service being equivalent to general setup time (Takagi (1991)) with possible preemptive resume during setup time to accommodate a new arrival while the primary service remains undisturbed. Such setup time models are common in communication networks involving delay control and have not been explored in the retrial context.

**Remark 5.3.3.** For the case \( p = 0 \) and \( \alpha = 0 \), the results agree with Atencia (2001).

### 5.4 Performance Measures

In this section, some performance measures for the system under steady-state are derived. Let \( U_1 \) be the steady-state probability that the server is busy for providing
the preliminary service, \( U_2 \), the steady-state probability that the server is busy for providing the primary service and \( I \), the steady-state probability that the server is idle during retrial time. From the results of the earlier section, it is seen that

\[
U_1 = P(1) = \frac{1 - \beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} \\
U_2 = S(1) = p \lambda \nu_1
\]

and

\[
I = R(1) = \frac{[1 - \phi^*(\lambda)][1 - \beta^*(\alpha \lambda) + p \alpha \lambda \nu_1 \beta^*(\alpha \lambda)]}{\alpha \beta^*(\alpha \lambda) \phi^*(\lambda)}.
\]

The mean number of customers in the system \( L_s \) under steady-state condition is obtained by differentiating (5.37) with respect to \( z \) and evaluating at \( z = 1 \).

\[
L_s = K'(1)
\]

\[
= \frac{2\{(\beta^*(\alpha \lambda))^2[p\alpha \lambda \nu_1(1 + \alpha - p \alpha \lambda \nu_1) - 1] + \beta^*(\alpha \lambda)[1 - p \alpha \lambda \nu_1]\}}{2\alpha \beta^*(\alpha \lambda)[\beta^*(\alpha \lambda)[1 + \alpha \phi^*(\lambda) - p \alpha \lambda \nu_1] - 1}.
\]

Define \( H(z) = R_0 + R(z) + P(z) + S(z) \). Then \( H(z) \) represents the probability generating function for the number of customers in the orbit. Using (5.33)-(5.36) and simplifying, it is obtained as

\[
H(z) = \left\{ \begin{align*}
\beta^*(\alpha \lambda)[1 + \alpha \phi^*(\lambda) - p \alpha \lambda \nu_1] - 1 \\
(1 - z)[(1 - z) + \alpha z \phi^*(\lambda(1 - z) + \alpha z)]
\end{align*} \right\} \]

\[
D(z)
\]

Hence, the mean number of customers in the orbit is given by

\[
L_q = H'(1)
\]

\[
= \left\{ \begin{align*}
2\{(\beta^*(\alpha \lambda))^2[\alpha \phi^*(\lambda) - p \alpha \lambda \nu_1 + \alpha \lambda^2 \nu_1(1 - \phi^*(\lambda))] - \beta^*(\alpha \lambda)[1 \\
+ \alpha \phi^*(\lambda) - p \alpha \lambda \nu_1 + 1 - \alpha \lambda(1 - \alpha)\beta^*(\alpha \lambda)] + (\beta^*(\alpha \lambda))^2 p \alpha \lambda^2 \nu_2 \}
\end{align*} \right\} \]

\[
2\alpha \beta^*(\alpha \lambda)[\beta^*(\alpha \lambda)[1 + \alpha \phi^*(\lambda) - p \alpha \lambda \nu_1] - 1}
\]

Further, let \( M_0^q \) and \( M_1^q \) denote the moments defined by

\[
M_0^q = \sum_{n=0}^{\infty} n \int_0^\infty R_n(x)dx
\]

and

\[
M_1^q = \sum_{n=0}^{\infty} n \int_0^\infty (P_n(x) + S_n(x))dx.
\]

If \( \frac{1 - \beta^*(\alpha \lambda)}{\alpha \beta^*(\alpha \lambda)} + p \lambda \nu_1 < \phi^*(\lambda) \), then by routine
differentiation of the partial probability generating functions $R(z), P(z)$ and $S(z)$ yield

$$M_1^0 = \frac{(1 - \phi^*(\lambda)) \left\{ 2 \left\{ (\beta^*(\alpha\lambda))^2 [\rho \alpha \nu_1 (1 + \alpha - \rho \alpha \nu_1)] - \alpha \lambda (1 - \alpha) \beta^*(\alpha \lambda) \right\} + \beta^*(\alpha \lambda) [1 - \rho \alpha \nu_1] + (\beta^*(\alpha \lambda))^2 \rho \alpha^2 \nu_2 \right\}}{2 \alpha \beta^*(\alpha \lambda) [\beta^*(\alpha \lambda) [1 + \alpha \phi^*(\lambda) - \rho \alpha \nu_1] - 1]}$$

and

$$M_1^1 = \frac{(\beta^*(\alpha \lambda))^2 \left( 1 + \rho \alpha \nu_1 [\rho \alpha \nu_1 (1 - \phi^*(\lambda)) - 2 + \phi^*(\lambda)] \right) + (\alpha - 1) \phi^*(\lambda) + 1 - \alpha \lambda (1 - \alpha) \beta^*(\alpha \lambda) \phi^*(\lambda)}{-\beta^*(\alpha \lambda) \left( 1 - \phi^*(\lambda) \right) \left( 1 - \alpha - \rho \alpha \nu_1 \right) - 4 \rho \alpha \nu_1} \right\} \frac{\alpha \beta^*(\alpha \lambda) [\beta^*(\alpha \lambda) [1 + \alpha \phi^*(\lambda) - \rho \alpha \nu_1] - 1]}{1 - \alpha \lambda (1 - \alpha) \beta^*(\alpha \lambda) [\beta^*(\alpha \lambda) [1 + \alpha \phi^*(\lambda) - \rho \alpha \nu_1] - 1]}.$$  

It is easy to verify that the mean number of customers in the system in steady-state is

$$L_s = M_1^0 + M_1^1 + \lim_{t \to \infty} P(C(t) = 1) + \lim_{t \to \infty} P(C(t) = 2).$$

### 5.5 NUMERICAL RESULTS

First set of numerical examples deal with the case of exponential distribution for preliminary, primary and retrial time distributions denoted by $b(x) = \mu_1 e^{-\mu_1 x}$, $h(x) = \mu_2 e^{-\mu_2 x}$ and $a(x) = \eta e^{-\eta x}$ respectively. The effect of varying retrial rate $\eta$ and push-out probability $\alpha$ on the mean system size $L_s$ is shown as the surface in Figure 5.1 for the set of parameters $(\lambda, \mu_1, \mu_2, \eta) = (2, 18, 10, 0.5)$. The surface displays a downward trend for $L_s$ against increasing $\alpha$ and $\eta$ values as expected.

In Figure 5.2, the variation of $L_s$ with respect to $\alpha$ and $\eta$ is displayed for the set of parameters $(\lambda, \mu_1, \mu_2, \eta) = (3, 20, 18, 8)$. Here also the surface displays a downward trend for $L_s$ against increasing $\alpha$ and $\eta$. In Figure 5.3, the surface trend for the set of parameters $(\lambda, \mu_1, \mu_2, \eta) = (2, 20, 12, 0.9)$ is displayed, indicating sharp fall in the beginning following a slow decrease in mean system size $L_s$. 

Figure 5.1 $L_\nu$ versus $(\eta, \alpha)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, p) = (2, 18, 10, 0.5)$.

Figure 5.2 $L_\nu$ versus $(\alpha, q)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \eta) = (3, 20, 18, 8)$.
Figure 5.3 $L_s$ versus $(\eta, q)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \alpha) = (2, 20, 12, 0.9)$.

Figure 5.4 $R_0$ versus $(\eta, \alpha)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, p) = (0.4, 7, 1.5, 0.5)$.
Figure 5.5 $R_0$ versus $(\alpha, q)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \eta) = (0.4, 7, 1.5, 4)$.

Figure 5.6 $R_0$ versus $(\eta, q)$ for a queue with Exponential distribution $(\lambda, \mu_1, \mu_2, \alpha) = (0.4, 7, 1.5, 0.5)$.
Figure 5.7 \( L_s \) versus \((\eta, \alpha)\) for a queue with Erlangian distribution 
\((\lambda, \mu_1, \mu_2, p) = (1, 20, 15, 0.5)\).

Figure 5.8 \( L_s \) versus \((\alpha, q)\) for a queue with Erlangian distribution 
\((\lambda, \mu_1, \mu_2, \eta) = (2.5, 20, 18, 8)\).
Figure 5.9 $L_0$ versus $(\eta, q)$ for a queue with Erlangian distribution $(\lambda, \mu_1, \mu_2, \alpha) = (1, 25, 22, 0.5)$.

Figure 5.10 $R_0$ versus $(\eta, \alpha)$ for a queue with Erlangian distribution $(\lambda, \mu_1, \mu_2, p) = (1, 30, 20, 0.5)$.
Figure 5.11 $R_0$ versus $(\alpha, q)$ for a queue with Erlangian distribution $(\lambda, \mu_1, \mu_2, \eta) = (1, 30, 20, 5)$.

Figure 5.12 $R_0$ versus $(\eta, q)$ for a queue with Erlangian distribution $(\lambda, \mu_1, \mu_2, \alpha) = (1, 30, 25, 0.5)$. 
Figure 5.13 Mean number of customers in the system versus $p$ with $(\lambda, \mu_1, \mu_2, \eta, \alpha, u) = (2, 18, 13, 9, 0.5, 0.6)$.

Figure 5.14 Mean number of customers in the system versus $\alpha$ with $(\lambda, \mu_1, \mu_2, \eta, p, u) = (2, 15, 13, 9, 0.4, 0.6)$.
Figure 5.15 Mean number of customers in the system versus $\eta$ with $(\lambda, \mu_1, \mu_2, \alpha, p, u) = (1, 20, 13, 0.5, 0.4, 0.6)$.

Figure 5.16 Probability of no customer in the system versus $p$ with $(\lambda, \mu_1, \mu_2, \eta, \alpha, u) = (2, 16, 12, 9, 0.5, 0.5)$. 
Figure 5.17 Probability of no customer in the system versus $\alpha$ with $(\lambda, \mu_1, \mu_2, \eta, p, u) = (1, 16, 12, 0.2, 0.4)$.

Figure 5.18 Probability of no customer in the system versus $\eta$ with $(\lambda, \mu_1, \mu_2, \alpha, p, u) = (1, 16, 12, 0.4, 0.6, 0.5)$.
The Figures 5.4, 5.5 & 5.6 display the surface for the variation of the probability of an empty system under steady-state conditions, for the chosen parametric values:

Figure 5.4 : \((\lambda, \mu_1, \mu_2, p) = (0.4, 7, 1.5, 0.5)\),

Figure 5.5 : \((\lambda, \mu_1, \mu_2, \eta) = (0.4, 7, 1.5, 4)\),

Figure 5.6 : \((\lambda, \mu_1, \mu_2, \alpha) = (0.4, 7, 1.5, 0.5)\).

In all the cases the probability of the system size being zero increases gradually as plotted in the figures. Similar trends are displayed (in Figures 5.7, 5.8, 5.9 and 5.10, 5.11, 5.12) for the case of preliminary service, primary service and retrial distributions following Erlangian distribution of order 2 as 

\[ b(x) = \mu_1^2 x e^{-\mu_1 x}, \quad h(x) = \mu_2^2 x e^{-\mu_2 x} \]

and 

\[ a(x) = \eta^2 x e^{-\eta x} \]

respectively for the mean system size \(L_s\) and \(R_0\), steady-state probability of system size zero.

The graphs illustrated in Figures 5.13, 5.14 & 5.15 compare the behaviour of \(L_s\) against the parameters (i) \(p\), the probability of selecting the primary service, (ii) increasing values of push-out probability \(\alpha\) and (iii) increasing retrial rate \(\eta\) for the exponential, Erlangian of order 2 as given earlier and hyper-exponential distributions for preliminary service, primary service and retrial distributions as

\[ b(x) = u \mu_1 e^{-\mu_1 x} + (1 - u) \mu_2 e^{-\mu_2 x}, \quad h(x) = u \mu_2 e^{-\mu_2 x} + (1 - u) \mu_2^2 e^{-\mu_2 x} \]

and 

\[ a(x) = u \eta e^{-\eta x} + (1 - u) \eta^2 e^{-\eta^2 x} \]

respectively. Similarly the graphs 5.16, 5.17 & 5.18 compare the trend for \(R_0\), for the chosen parametric values and distributions.

### 5.6 STOCHASTIC DECOMPOSITION

In this section, the stochastic decomposition property of the system size distribution is analyzed. The literature on vacation models recognizes this problem as one of the most interesting features (see Cooper (1970), Doshi (1986), Fuhrmann and Cooper (1985)). The classical interpretation of stochastic decomposition property shows that
the steady-state number of customers present in the system at an arbitrary point is distributed as the sum of two independent random variables — the steady-state number of customers present at an arbitrary point in time in the corresponding queueing model without server vacations and the number of customers at an arbitrary point in time given that the server is on vacation (see Doshi (1986), Takagi (1991) for more details).

Stochastic decomposition has also been obtained to hold for some $M/G/1$ retrial queues. Retrial queue, under consideration, can be thought of as an $M/G/1$ queue with generalized vacations (see, Fuhrmann and Cooper (1985)) in which the vacation begins at the end of either preliminary service or primary service time. Let $\pi(z)$ be the probability generating function of the number of customers in the $M/G/1$ queue with preliminary and primary service facility in the steady-state at a random point in time, $\chi(z)$ be the probability generating function of the number of customers in the generalized vacation system at a random point in time given that the server is idle due to retrials and $K(z)$ be the probability generating function of the random variable being decomposed. Then the mathematical version of the stochastic decomposition law is

$$K(z) = \pi(z)\chi(z).$$

Observe that the decomposition law is applicable to the present model of retrial queue with preliminary and primary services. For the $M/G/1$ queue with preliminary and primary services,

$$\pi(z) = \begin{cases} 
(1-z)(1-z+\alpha z)\beta^*(\lambda(1-z) + \alpha \lambda z)[q + \nu^*(\lambda(1-z))] \\
\frac{\beta^*(\alpha \lambda)(1 + \alpha - \rho \alpha \nu_1) - 1}{\beta^*(\alpha \lambda)(1 - \alpha z)[q + \nu^*(\lambda(1-z))] - \alpha z} - z(1-z)\alpha \beta^*(\alpha \lambda) 
\end{cases}. $$

To obtain an expression for $\chi(z)$, the vacation for this context, it is to be defined first. It is observed that the server is on vacation if the server is idle (There may be customers in the system even when the server is idle in the retrial queueing context). Under this
definition, \( x(z) \) is given by

\[
x(z) = \frac{R_0 + R(z)}{R_0 + R(1)}.
\]

From (5.36) and (5.37), it can be seen that \( K(z) = \pi(z)x(z) \), which confirms that the decomposition law of Fuhrmann and Cooper (1985) is also valid for this special vacation system. However, it must pointed out that if the idle periods were not considered as vacations, the decomposition law would not apply here due to interference between customer retrials and server vacation.

For appropriate choice of parametric values and distributions, the decomposition property for the models considered by Madan (2000) and Gomez-Corral (1999) can be deduced as special cases.

5.7 CONCLUSION

In the foregoing analysis, an \( M/G/1 \) retrial queueing system with preliminary and primary service under possible preemptive resume service discipline is considered to obtain analytical expressions for various performance measures of interest. For an arbitrarily distributed retrial time distribution, the necessary and sufficient condition for the system stability is discussed. This model unifies the FCFS and LCFS preemptive resume disciplines (for \( \alpha = 0 \) and \( \alpha = 1 \) respectively). Numerical works have been carried out to observe the trend for (1) the probability of no customer in the system, (2) mean number of customers in the system for varying parametric values. The general decomposition law for this system has also been established.

So far, some retrial queueing models are analyzed using supplementary variable technique. The rest of the thesis analyzes queueing systems with \( N \)-policy in different settings.