CHAPTER 0

Introduction
In this thesis I present my work related to the study of some aspects of Fibonacci numbers. The aim of the thesis is to study various properties of the family of generalized (a,b)-Fibonacci numbers. The thesis consists of six chapters. In this chapter we describe briefly the contents of the chapters of this thesis.

**CHAPTER 1** : The **First Chapter** consists of introduction of the sequence \( \{F_n^G\} \), the sequence of *generalized Fibonacci numbers*. The terms of this sequence are defined by the recurrence relation

\[
F_n^G = p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^G + r^{\chi(n)} s^{1-\chi(n)} F_{n-2}^G,
\]

where \( p, q, r, s \) are any fixed real numbers and \( \chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases} \).

In this thesis, we develop extended Binet’s formula for the various sub-sequences of \( \{F_n^G\} \), which is obtained by considering some fixed values of \( p, q, r \) and \( s \). We also derive some interesting identities and results for these sub-sequences.

We first consider the subsequence \( \{F_n^{L(a,b)}\} \) of \( \{F_n^G\} \) by considering \( p = a, q = b, r = s = 1 \) in \((*)\). The terms of this subsequence are defined by the following recurrence relation:
\[ F_n^{L(a, b)} = \begin{cases} \alpha F_{n-1}^{L(a, b)} + F_{n-2}^{L(a, b)} & \text{when } n \text{ is odd} \\ \beta F_{n-1}^{L(a, b)} + F_{n-2}^{L(a, b)} & \text{when } n \text{ is even} \end{cases}; \quad (n \geq 2) \]

with the initial conditions \( F_0^{L(a, b)} = 0, F_1^{L(a, b)} = 1. \)

For convenience, we write \( F_n^L \) for \( F_n^{L(a, b)} \).

In this chapter, we first show that \( \gcd(F_n^L, F_{n+1}^L) = 1; \) for all \( n \).

We then derive some simple identities for \( \sum_{i=1}^{n} F_i^L \), \( \sum_{i=1}^{n} F_{2i}^L \) and \( \sum_{i=1}^{n} F_{2i}^L \).

Using the techniques of generating functions, we next derive the explicit formula, known as extended Binet formula for \( F_n^L \). In fact we prove that

\[
F_n^L = \frac{b^{1-x(n)}}{(ab)^{\lfloor n/2 \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right),
\]

where \( \alpha, \beta \) are the roots of the characteristic equation \( x^2 - abx - ab = 0 \) and \( \lfloor x \rfloor \) denotes the integer part of \( x \).

We then define \( P_n = \alpha^n, Q_n = -\beta^n \) and derive the second order recurrence relation for \( P_n \) and \( Q_n \). We also derive some simple and interesting properties involving \( \alpha \) and \( \beta \).

We then develop some very interesting results related to \( F_n^L \).

First we derive the following simple formula which gives the value of \( F_n^L \) for any positive integer \( n \):

\[
F_n^L = \left\lfloor \gamma a^n + \frac{1}{2} \right\rfloor,
\]

where \( \gamma = \frac{b^{1-x(n)}}{(ab)^{\lfloor n/2 \rfloor}} \frac{1}{(\alpha - \beta)} \) is a fixed constant.
We further derive recursive formula to compute the successor of any given generalized Fibonacci number. The formula reads as

\[ F_n^L = \left[ \frac{a}{a^{1-x(n)}b^{x(n)}} F_{n-1}^L + \frac{1}{2} \right] ; n \geq 2. \]

Finally we derive the reverse formula to compute the predecessor of a given generalized Fibonacci number as shown below:

\[ F_n^L = \left[ \frac{a^{x(n)}b^{1-x(n)}}{a} (F_{n+1}^L + \frac{1}{2}) \right] ; n \geq 2. \]

**Chapter 2**: In Second Chapter, we continue our investigation and derive some more interesting properties related to \( F_n^L \).

We first derive *extended Catalen’s identity* connecting three consecutive \( F_n^L \)'s in arithmetic progressions and prove that

\[ a^{1-x(n-r)}b^{x(n-r)} F_{n-r}^L F_{n+r}^L - a^{1-x(n)}b^{x(n)} (F_n^L)^2 \]

\[ = b^{x(r)} a^{1-x(r)} (-1)^{n+1-r} (F_r^L)^2, \]

for any two non-negative integers \( n \) and \( r \) with \( n \geq r \).

We next prove the *extended d’Ocagne’s identity* and show that

\[ a^{x(mn+n)}b^{x(mn+m)} F_m^L F_{n+1}^L - a^{x(mn+m)}b^{x(mn+n)} F_{m+1}^L F_{n}^L \]

\[ = (-1)^n b^{x(m-n)} F_{m-n}^L, \]

where \( m, n \) are non-negative integers with \( m \geq n \).

We next use the above extended Binet formula to derive an interesting combinatorial identity for \( F_n^L \) of the type
\[ F_n^L = \frac{b^n (n-1)(ab)^{\lfloor n/2 \rfloor} - x(n-1)}{2^{n-1}} \left( \sum_{r=1}^{\infty} \left( \frac{n}{2r-1} \right) \left( 1 + \frac{4}{ab} \right)^{r-1} \right). \]

We again use extended Binet’s formula and find two different expressions for \( F_n^L \) in terms of \( F_{n-1}^L \) along with some powers of \( \alpha \) or \( \beta \).

For any non-negative integer \( n \), we also derive the following two different sums involving binomial coefficients:

a) \( F_{2n}^L = \sum_{k=0}^{n} \binom{n}{k} b^{x(k)} (ab)^{\lfloor k/2 \rfloor} F_k^L \)

b) \( bF_{2n+1}^L = \sum_{k=0}^{n} \binom{n}{k} b^{x(k+1)} (ab)^{\lfloor k+1/2 \rfloor} F_{k+1}^L \).

We next derive exponential generating function for \( \frac{F_{nk}^L}{n!} \) and \( 2^n \frac{F_n^L}{n!} \).

Using the similar techniques of generating functions, we also prove the following identity:

\[ F_{2n}^L = (ab)^n \sum_{k=0}^{n} \binom{n}{k} F_k^L. \]

Before the conclusion of this chapter, we derive the following two interesting results for \( F_n^L \):

a) The number of digits of \( F_n^L \) is given by

\[ \# F_n^L = \lfloor n \log \alpha + \log \mu \rfloor + 1, \text{ where } = \frac{b^{1-x(n)}}{(ab)^{\lfloor n/2 \rfloor} (\alpha - \beta)}. \]

Here \( \log \) represents the logarithm with base 10.

b) \( \sum_{i=1}^{\infty} \frac{F_i^L}{m(i+1)^n} = \frac{m^{2n} + bm^{n-1}}{m^{4n} - (ab + 2)m^{n+1}} \); for any positive real number \( m \geq 2 \).
Chapter 3: The third chapter deals with the study of generalized Lucas sequence \( \{L_n^G(p, q, r, s)\} (= \{L_n^G\}) \) defined by the recurrence relation

\[
L_n^G(p, q, r, s) = p \chi(n) q^{1-\chi(n)} L_{n-1}^G(p, q, r, s) + r \chi(n) s^{1-\chi(n)} L_{n-2}^G(p, q, r, s)
\]

where \( p, q, r, s \) are any fixed integers and \( \chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases} \).

We define \( L_0^G(p, q, r, s) = 2 \) and \( L_1^G(p, q, r, s) = 1 \). This chapter deals with the study of generalized Lucas sequence \( \{L_n^G(a, b, 1, 1)\} = \{L_n^L\} \) defined by the recurrence relation

\[
L_n^L = a \chi(n) b^{1-\chi(n)} L_{n-1}^L + L_{n-2}^L,
\]

where \( L_0^L = 2, L_1^L = 1 \) and \( a, b \) are any two positive integers.

This can be equivalently expressed as

\[
L_n^L = \begin{cases} a L_{n-1}^L + L_{n-2}^L; & \text{if } n \text{ is odd} \quad (n \geq 2) \text{ with } L_0^L = 2, L_1^L = 1. \\ b L_{n-1}^L + L_{n-2}^L; & \text{if } n \text{ is even} \end{cases}
\]

We first derive simple identities for \( \sum_{i=1}^{n} L_i^L \), \( \sum_{i=1}^{n} L_{2i}^L \) and \( \sum_{i=1}^{n} L_{2i+1}^L \). Next we derive the corresponding extended Binet’s formula for \( L_n^L \) as

\[
L_n^L = \frac{\gamma a^n - \delta b^n}{(ab)^{n/2} b \chi(n)(\alpha - \beta)},
\]

where \( \gamma = 2\alpha - 2ab + b \) and \( \delta = 2\beta - 2ab + b \).

We further derive two relations connecting \( F_n^L \) and \( L_n^L \). One of them reads as
We next derive the extended Catalen’s identity for the three consecutive $L_n^L$’s in arithmetic progression. We also derive extended d’Ocagne identity for $L_n^L$. We then find the two different expressions for $L_n^L$ in terms of $L_{n-1}^L$ along with the powers of $\alpha$ or $\beta$.

We also derive two sums involving binomial coefficients in the form

$$L_{2n}^L = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\beta}{(ab)^{[n/2]}b\chi(n)}\right)^k L_k^L$$ and

$$L_{2n+1}^L = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\alpha}{(ab)^{[n/2]}b\chi(n)}\right)^{k+1} L_{k+1}^L.$$

We conclude the chapter by deriving the generating function for $\frac{L_n^L}{n!}$ and also an identity for $L_{2n}^L$.

**Chapter 4**: In the Fourth Chapter we consider two sub-sequences

\[ \{f_n^{L(a,b)}\} \text{ and } \{l_n^{L(a,b)}\} \] of $\{F_n^G\}$ by taking $p = a, q = b, r = s = -1$ in (*).

In the first part of the chapter, we define terms of the sub-sequence

\[ \{f_n^{L(a,b)}\} \] by the recurrence relation

\[ f_n^{L(a,b)} = \begin{cases} a f_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)} & \text{if } n \text{ is odd} \\ b f_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)} & \text{if } n \text{ is even} \end{cases} \quad (n \geq 2) \]

where $f_0^{L(a,b)} = 0, f_1^{L(a,b)} = 1$. For convenience, we write $f_n^L$ for $f_n^{L(a,b)}$. 

\[ (i) \quad L_n^L = \frac{\gamma R_n^L}{b} + \frac{2\beta^n}{(ab)^{[n/2]}b\chi(n)} \quad (ii) \quad L_n^L = \frac{\delta R_n^L}{b} + \frac{2\alpha^n}{(ab)^{[n/2]}b\chi(n)}. \]
Using the techniques of generating functions, we derive the extended Binet formula for \( f_n^L \) as

\[
f_n^L = \frac{b^{1-x(n)}(a^n - \beta^n)}{(ab)^{[n/2]}(\alpha - \beta)},
\]

where \( \alpha, \beta \) are the roots of the characteristic equation \( x^2 - abx + ab = 0 \). We next derive some simple and interesting properties connecting \( \alpha \) and \( \beta \).

We then use the above extended Binet formula to derive an interesting combinatorial identity for \( f_n^L \) of the type

\[
f_n^L = \frac{b^{x(n-1)}(ab)^{[n/2]} - x(n-1)}{2^{n-1}} \left( \sum_{r=1}^{\infty} \left( \begin{array}{c} n \\ 2r-1 \end{array} \right) \left( 1 + \frac{4}{ab} \right)^{r-1} \right).
\]

We again use extended Binet’s formula and find two different expressions for \( f_n^L \) in terms of \( f_{n-1}^L \) along with the powers of \( \alpha \) and \( \beta \). We also mention a number of results which can be proved easily by the techniques used earlier in the thesis.

In the second part of the chapter, we study the sub-sequence \( \{l_n^{L(a,b)}\} (= \{l_n^L\}) \) which is defined by the recurrence relation

\[
l_n^L = \alpha^{x(n)} b^{1-x(n)} l_{n-1}^L - l_{n-2}^L, \text{ with } l_0^L = 2, l_1^L = 1.
\]

This can be equivalently expressed as

\[
l_n^L = \begin{cases} \alpha l_{n-1}^L - l_{n-2}^L & \text{if } n \text{ is odd } (n \geq 2) \text{ with } l_0^L = 2, l_1^L = 1. \\
\beta l_{n-1}^L - l_{n-2}^L & \text{if } n \text{ is even } (n \geq 2) \end{cases}
\]

We derive the corresponding extended Binet’s formula for \( \{l_n^L\} \) as

\[
l_n^L = \frac{\gamma a^n - \beta^n}{(ab)^{[n/2]}} b^{x(n)}(\alpha - \beta),
\]
where \( \gamma = 2\alpha - 2ab + b \) and \( \delta = 2\beta - 2ab + b \). We conclude the chapter by mentioning some interesting results which can be proved easily by the techniques used earlier.

**Chapter 5**: In the **Fifth Chapter** we derive extended Binet’s formula for \( n^{th} \) term of some more sub-sequences of \( \{F_n^G\} \). In this chapter we first consider the sub-sequence \( \{F_n^{R(a,b)}\} (= \{F_n^R\}) \) of \( \{F_n^G\} \) by taking \( p = q = 1, r = a, s = b \) in (1). The terms of this subsequence are defined by the recurrence relation

\[
F_n^{R(a,b)} = \begin{cases} 
F_{n-1}^{R(a,b)} + aF_{n-2}^{R(a,b)}; & \text{if } n \text{ is odd} \\
F_{n-1}^{R(a,b)} + bF_{n-2}^{R(a,b)}; & \text{if } n \text{ is even}
\end{cases} \quad (n \geq 2)
\]

with \( F_0^{R(a,b)} = 0, F_1^{R(a,b)} = 1 \).

Using the techniques of generating functions, we derive the extended Binet formula for \( F_n^R \) as

\[
F_n^R = \left( \frac{\gamma^{\chi(n)} \alpha^\frac{n}{2} - \delta^{\chi(n)} \beta^\frac{n}{2}}{\alpha - \beta} \right); \text{ where } \gamma = \alpha - b, \delta = \beta - b.
\]

Here \( \alpha, \beta \) are the roots of equation \( x^2 - (1 + a + b)x + ab = 0 \).

We next define

\[
f_n^{R(a,b)} = \begin{cases} 
f_{n-1}^{R(a,b)} - af_{n-2}^{R(a,b)}; & \text{if } n \text{ is odd} \\
f_{n-1}^{R(a,b)} - bf_{n-2}^{R(a,b)}; & \text{if } n \text{ is even}
\end{cases} \quad (n \geq 2)
\]

with \( f_0^{R(a,b)} = 0, f_1^{R(a,b)} = 1 \). We note that \( f_n^{R(a,b)} = F_n^{R(-a,-b)} \).
This observation immediately helps to write the extended Binet’s formula for \( f_n^{R(a,b)} \).

Next we consider the sub-sequence \( \{L_n^{R(a,b)}\} \) of \( \{L_n^G\} \) defined by the recurrence relation

\[
L_n^{R(a,b)} = \begin{cases} 
L_{n-1}^{R(a,b)} + aL_{n-2}^{R(a,b)} & \text{if } n \text{ is odd} \\
L_{n-1}^{R(a,b)} + bL_{n-2}^{R(a,b)} & \text{if } n \text{ is even}
\end{cases} \quad (n \geq 2);
\]

with \( L_0^{R(a,b)} = 2, L_1^{R(a,b)} = 1 \). We derive extended Binet’s formula for this sequence.

We further define

\[
i_n^{R(a,b)} = \begin{cases} 
i_{n-1}^{R(a,b)} - aL_{n-2}^{R(a,b)} & \text{if } n \text{ is odd} \\
i_{n-1}^{R(a,b)} - bL_{n-2}^{R(a,b)} & \text{if } n \text{ is even}
\end{cases} \quad (n \geq 2);
\]

with \( i_0^{R(a,b)} = 2, i_1^{R(a,b)} = 1 \). We note that \( i_n^{R(a,b)} = L_n^{R(-a,-b)} \). Thus we immediately get the extended Binet’s formula for \( i_n^{R(a,b)} \).

We then reconsider the sequences \( \{F_n^L\} \) and \( \{f_n^L\} \). In both the cases we consider the characteristic equation different from that considered in earlier chapters. By using the techniques of generating functions, we derive the different form of extended Binet formula for \( F_n^L \) and \( f_n^L \) as follows:

\[
F_n^L = b^{1-x(n)} \left( \frac{\gamma x(n) a^{\frac{n}{2}} - \delta x(n) b^{\frac{n}{2}}}{a-b} \right); \quad \text{where } \gamma = \alpha - 1, \delta = \beta - 1
\]

and
In the first case, \( \alpha, \beta \) are the roots of the characteristic equation \( x^2 - (ab + 2)x + 1 = 0 \); where as, in the second case \( \alpha, \beta \) are the roots of equation \( x^2 - (ab - 2)x + 1 = 0 \).

\[
\begin{align*}
    f_n^L &= b^{1-\chi(n)} \left( \frac{\gamma^{\chi(n)} \alpha^{\left\lfloor \frac{n}{2} \right\rfloor} - \delta^{\chi(n)} \beta^{\left\lfloor \frac{n}{2} \right\rfloor}}{\alpha - \beta} \right); \text{ where } \gamma = 1 + \alpha, \delta = 1 + \beta.
\end{align*}
\]

\section*{Chapter 6 :}

In the final and \textbf{Sixth Chapter}, we consider the sub-sequences \( \{H_n^L(a,b)\} (= \{H_n^L\}) \) and \( \{H_n^R(a,b)\} (= \{H_n^R\}) \) of \( \{F_n^G\} \) by considering \( q = r = 1 \) and \( p = a, s = b \) in (*) for first case, and by considering \( p = s = 1 \) and \( q = b, r = a \) in (*) for second case. The terms of these sub-sequences are defined respectively by the following recurrence relations:

\textbf{a)} \( H_n^L(a,b) = \begin{cases} aH_{n-1}^L(a,b) + H_{n-2}^L(a,b); & \text{if } n \text{ is odd} \\ H_{n-1}^L(a,b) + bH_{n-2}^L(a,b); & \text{if } n \text{ is even} \end{cases} \quad (n \geq 2) \)

with the initial condition \( H_0^L(a,b) = 0, H_1^L(a,b) = 1. \)

\textbf{b)} \( H_n^R(a,b) = \begin{cases} H_{n-1}^R(a,b) + aH_{n-2}^R(a,b); & \text{if } n \text{ is odd} \\ bH_{n-1}^R(a,b) + H_{n-2}^R(a,b); & \text{if } n \text{ is even} \end{cases} \quad (n \geq 2) \)

with the initial condition \( H_0^R(a,b) = 0, H_1^R(a,b) = 1. \)

By using the techniques of generating functions, we derive the extended Binet formula for both \( H_n^L \) and \( H_n^R \) as follows:
a) $H_n^L = \frac{\gamma x(n) a_{\left\lfloor \frac{n}{2} \right\rfloor} - \delta x(n) b_{\left\lfloor \frac{n}{2} \right\rfloor}}{\alpha - \beta}$, where $\gamma = \alpha - b$, $\delta = \beta - b$,

and $\alpha, \beta$ are the roots of equation $x^2 - (a + b + 1)x + b = 0$;

b) $H_n^R = b^{1 - \chi(n)} \left( \frac{\gamma x(n) a_{\left\lfloor \frac{n}{2} \right\rfloor} - \delta x(n) b_{\left\lfloor \frac{n}{2} \right\rfloor}}{\alpha - \beta} \right)$, where $\gamma = \alpha - 1$, $\delta = \beta - 1$,

and $\alpha, \beta$ are the roots of the equation $x^2 - (a + b + 1)x + a = 0$.

* * * * *