CHAPTER 3
CHANNEL ROUTING TECHNIQUES

3.1 INTRODUCTION

Channel routing which refers to routing floods through channels, streams and rivers considers that the storage in the channel reach is a function of outflow as well as of inflow. If the storage in the channel reach is plotted against outflow, the result will be a wide loop, as in Figure 3.1, indicating greater storage for a given outflow during rising stages than during falling stages. When the flood is in rising stages a considerable volume of wedge storage may exist in the channel reach before any large increase occurs in the outflow. Similarly, during the falling stages of the flow, outflow does not drop down as fast as inflow, resulting in a negative wedge as seen in Figure 3.2. Hence, routing a streamflow requires a storage relationship which adequately represents the wedge storage. This is usually done by including the inflow as a parameter in the storage equation.

The inflow and outflow hydrographs in the channel reach would follow the pattern described for reservoir routing, as in Figure 2.3 with the exception that the peak outflow would occur at a time somewhat later than the time when the two hydrographs cross each other, see Figure 3.3. The reason is that unlike in reservoir routing, the discharge corresponding to given volume of storage varies due to the difference in water surface profile in the reach.

3.1.1 Classification of flood routing methods

Based on the wave models presented in Section 2.4, the flood routing methods can be classified under four groups. They are as follows.
Figure 3.1 Storage Vs flow for a channel reach

Figure 3.2 Definition sketch for wedge storage

Figure 3.3 Inflow and outflow hydrographs in a channel reach
a. Methods based on hydrological principles.
b. Methods based on kinematic wave model.
c. Methods based on diffusion wave model.
d. Methods based on dynamic wave model.

The latter three methods include the dynamic equation in full or in approximate form and are called hydraulic routing methods. Methods in the first mentioned group ignore the momentum equation and solve only the continuity equation, employing some empirical relationship for storage. Accordingly they are called storage routing methods and are also known as hydrological routing methods.

To route a flood any of the methods can be chosen, basing the selection on the nature of the problem on hand, the data available and the results that are expected of it. Routing a flood which comes through the junction with a tributary, is influenced by backwater effect. Also routing a flood regulated by a dam may be influenced by the effect of surges. In these cases, the depth, discharge, rate of change of depth and rate of change of discharge are all important and they can be evaluated only by a method which solves the complete dynamic equation. On the other hand if the solution requires only discharge at the downstream section, a simple storage routing model would suffice.

3.2 HYDROLOGICAL METHODS

3.2.1 Muskingum methods

In hydrological methods the continuity is expressed by equating the difference between inflow and outflow to the rate of change of storage in the reach as

\[ Q_1 - Q_0 = \frac{dS}{dt} \quad \ldots \ (3.1) \]

in which \( Q_1 \) is the inflow at the upstream section, \( Q_0 \) is the outflow in the downstream section and \( S \) is the storage in the reach. There are two unknowns in \( Q_0 \) and \( S \) and so Equation (3.1) is solved in association with a second equation for \( S \) involving \( Q_1 \) and \( Q_0 \).
McCarthy's popular Muskingum method [22] uses a storage equation which is linear in its form and is given by

\[ S = K \left[ \varepsilon Q_i + (1-\varepsilon)Q_o \right] \quad \ldots (3.2) \]

where \( K \) is the storage coefficient having the unit of time and \( \varepsilon \) is a dimensionless factor that defines the relative weights of inflow and outflow.

Substituting the Equation (3.2) in equation (3.1), one obtains

\[ \frac{1}{2}(Q_{i1} + Q_{i2}) \Delta t - \frac{1}{2}(Q_{o1} + Q_{o2}) \Delta t = K \varepsilon (Q_{i2} - Q_{i1}) + K (1-\varepsilon) (Q_{o2} - Q_{o1}) \ldots (3.3) \]

where subscripts 1 and 2 refer to the beginning and ending respectively of the routing time \( \Delta t \). Collecting like terms and rearranging

\[ Q_{o2} = C_1 Q_{i2} + C_2 Q_{i1} + C_3 Q_{o1} \quad \ldots (3.4) \]

where

\[ C_1 = \frac{-(K\varepsilon - 0.5 \Delta t)}{(K-K\varepsilon + 0.5 \Delta t)} \]

\[ C_2 = \frac{(K\varepsilon + 0.5 \Delta t)}{(K - K\varepsilon + 0.5 \Delta t)} \]

and

\[ C_3 = \frac{(K - K\varepsilon - 0.5 \Delta t)}{(K - K\varepsilon + 0.5 \Delta t)} \quad \ldots (3.5) \]

If \( q_i \) is the lateral inflow per unit length of the reach, an additional term

\[ C_4 = \frac{q_i L \Delta t}{(K - K\varepsilon + 0.5 \Delta t)} \quad \ldots (3.6) \]

is added to the right hand side of the Equation (3.4). With \( K, \varepsilon, L \) (the reach length) and \( \Delta t \) established the routing operation is simply a solution of Equation (3.4).
Conventional Muskingum method

The Muskingum parameters $K$ and $e$ are determined conventionally by a trial process in a graphical method. Such a process is termed Conventional Muskingum (CM) method in this study. Here the flood data of historic floods are used to find the storage and the weighted flow $\varepsilon Q_1 + (1 - \varepsilon) Q_0$, for assumed values of $\varepsilon$ which vary from 0.0 when the influence of inflow is not considered to 0.5 when both inflow and outflow have equal importance. The storage in the reach can be found from

$$S_2 = S_1 + \frac{Q_{11} + Q_{12}}{2} - \frac{Q_{01} + Q_{02}}{2} \Delta t \quad (3.7)$$

where $S_1$ can be conveniently assumed. The accumulated storage at every time level can be calculated and these values are plotted against weighted flow $\varepsilon Q_1 + (1 - \varepsilon) Q_0$ for various parametric values of $\varepsilon$ ranging from 0.0 to 0.5.

These values are then plotted for each of the assumed $\varepsilon$ values, producing curves in the form of loops as presented in Figure 3.4.

![Figure 3.4 Conventional determination of Muskingum parameters](image)

The inverse of the slope of the mean line passing through the narrowest loop provides the value of $K$, the storage coefficient and the corresponding assumed value of $\varepsilon$ is taken to be the correct weighting factor.
Using a few historic flood flow data, an average value for $K$ and $\varepsilon$ can be arrived at and used for routing flows of future floods.

Gill [23] presented a mathematical method of determining Muskingum parameters. The storage $S$ in the Muskingum equation is the absolute storage. The practical values of the storage that are normally available are relative values. According to this equation, if absolute values of storage are plotted against the corresponding weighted flow $\varepsilon Q_i + (1 - \varepsilon) Q_o$ values, a straight line passing through the origin can be drawn through the mean position of the plotted points. If the storage values are relative, the straight line will not pass through the origin. In view of this observation, Equation (3.2) is modified as

$$S = K \left\{ \varepsilon Q_i - (1 - \varepsilon) Q_o \right\} + \sigma \quad \ldots \ (3.8)$$

where $S$ is relative storage and $\sigma$ is the difference between the relative and absolute storages. If $K \varepsilon = A$ and $K (1-\varepsilon) = B$, then

$$S = AQ_i + BQ_o + \sigma \quad \ldots \ (3.9)$$

Gill used the least squares method to establish the values of $A$, $B$ and $\sigma$ which will fix the position of the straight line such that the width of the loop is minimized.

3.2.1.2 Constant parameter Muskingum-Cunge method

The conventional Muskingum method assumes basically that the stage and discharge have one to one relationship. Such an assumption allows the flood to pass the reach as it is and there should not be any attenuation of the peak flow. But the Muskingum method does give an attenuation of the wave. Cunge [24] explained this contradiction by substantiating that the attenuation in the CM method is not any real attenuation of the flood wave, but is the result of the truncation error in the numerical scheme employed.

Storage routing is the application of the continuity equation
\[ \frac{dS}{dt} = Q_i - Q_o + q_L \] ... (3.10)

Substituting Equation (3.2) in Equation (3.10),

\[ K \varepsilon \frac{dQ_i}{dt} + K(1 - \varepsilon) \frac{dQ_i}{dt} = Q_i - Q_o + q_L \] ... (3.11)

Rewriting Equation (3.11) in finite difference form in an x-t plane, vide Figure 3.5, it is obtained that

\[ K \varepsilon \frac{1}{\Delta t} (Q_{i+1}^j - Q_i^j) + K(1 - \varepsilon) \frac{1}{\Delta t} (Q_{i+1}^{j+1} - Q_{i+1}^j) \]

\[ = \frac{1}{2} (Q_{i+1}^j + Q_i^j - Q_{i+1}^{j+1} - Q_{i+1}^j) + q_L \Delta x \] ... (3.12)

Neglecting \( q_L \) and defining \( K \) as

\[ K = \frac{\Delta x}{C_w} \] ... (3.13)

where \( C_w \) is the average speed of the flood wave, Cunge showed that Equation (3.12) is the finite difference representation of the kinematic wave equation

\[ \frac{\partial Q}{\partial t} + \omega \frac{\partial Q}{\partial x} = 0 \] ... (3.14)

where \( \omega \) is the speed of the wave.

Considering Equation (3.12) as the second order approximation of the linear convection diffusion equation

\[ \frac{\partial Q}{\partial t} + \omega \frac{\partial Q}{\partial x} = \nu \frac{\partial^2 Q}{\partial x^2} \] ... (3.15)

the weighting parameter \( \varepsilon \) can be obtained as

\[ \varepsilon = \frac{1}{2} \left( 1 - \frac{Q}{\Delta x C_w B S_o} \right) \] ... (3.16)
where \( B \) is the top width of flow and \( \mu \) is the diffusion parameter, see also Ponce and Yevjevich [25]. Muskingum parameters thus obtained should correspond nearer to real damping effect, for, by virtue of the fact that the diffusion coefficients are brought into play, the parameters \( K \) and \( \varepsilon \) relate to the physical characteristics. This method of determining \( K \) and \( \varepsilon \) can be stated as Muskingum-Cunge method. If the parameters are deemed constant for any discharge and throughout the length of the reach, the method is named the constant parameter Muskingum-Cunge (CPMC) method in this study.

The parameters are determined for a reach from historic flood flow records and also records of cross-section at upstream and downstream sections of the reach. The plots, \( Q \) Vs \( A \) and \( Q \) Vs \( B \) are made and the steepest tangent of the \( Q \) Vs \( A \) plot would furnish \( \left( \frac{dQ}{dA} \right)_{\text{max}} \) which is the maximum flood wave velocity at that section. The average of these values at the inflow and outflow sections provides the value of \( V_{w} \), the average flood wave velocity along the reach. Similarly \( Q \) Vs \( B \) plots at the inflow and outflow sections can be used for finding the average top width of flow along the reach. Once these values are found the Muskingum parameters are readily calculated from Equations (3.13) and (3.16).
3.2.1.3 Variable parameter Muskingum-Cunge Method

The Muskingum parameters may also be considered to vary with discharge and distance. Such an approach should be more physically realistic. The parameter $K$ is essentially discharge dependent but routing computations may be relatively insensitive to the variations in the weighting parameter $\varepsilon$. Ponce and Yevjevich [25], in their variable parameter Muskingum-Cunge (VPMC) method have allowed both the parameters $K$ and $\varepsilon$ to vary with discharge. The values of $V_w$ and $q$ at grid point $(i,j)$ in Figures 3.5 are defined by

$$V_w = \frac{dQ}{dA} \bigg|_{i,j} \quad \ldots \quad (3.17)$$

and

$$q = \frac{Q}{B} \bigg|_{i,j} \quad \ldots \quad (3.18)$$

where $q$ is the discharge per unit top width of flow.

Determination of $V_w$ and $\varepsilon$ for various values of $Q$ was attempted by Ponce and Yevjevich in the following three ways.

i. Directly by using a two point average of the values at the grid points $(i,j)$ and $(i+1, j)$,

ii. Directly by using a three point average of the values at the grid points $(i, j)$ $(i+1, j)$ and $(i, j+1)$,

iii. By iteration, by using a four point average calculation for which the values at $(i+1, j+1)$ obtained by the three point average are used as first guess of iteration.

Their results showed that this attempt took into account the nonlinearity of the phenomenon. They also found that the four point average gave more satisfactory results.
3.2.2 General storage equation method

3.2.2.1 Kulandaiswamy general storage equation

A storage equation, much general in nature, was proposed by Kulandaiswamy [26] in his studies of rainfall-runoff relationship in a river basin. The equation has since been known as the Kulandaiswamy general storage equation (KGSE). The equation highlights the fact that the storage is not only dependent on the flow $Q$, but also on the rate of change of flow $dQ/dt$. Kulandaiswamy et al. [27] tested the equation for flood routing in some American rivers. Later Karmegam [14] and Karmegam and Wormleaton [28] applied the equation for routing a large number of floods in British rivers. For a detailed study of Kulandaiswamy general storage equation reference can be made to Babu Rao and Sakthivelv [29].

The KGSE is stated as

$$S = \sum_{m=1}^{M} a_{m-1} \frac{d^{m-1}Q_o}{dt^{m-1}} + \sum_{n=1}^{N} b_{n-1} \frac{d^{n-1}Q_i}{dt^{n-1}}$$

... (3.19)

In which $a$ and $b$ coefficients are either constants or functions of variables involving the outflow $Q_o$ and the inflow $Q_i$ respectively. If they are constants the equation is linear and if one or both of them are functions of the variables $Q_o$ or $Q_i$, it is nonlinear. $M$ and $N$ are the order of time derivatives of $Q_o$ and $Q_i$ that need to be taken into consideration.

It may be noted that if the equation is linear and if $M = N = 1$, it reduces to

$$S = a_0 Q_o + b_0 Q_i$$

... (3.20)
which is the same as a Muskingum equation with

\[ a_0 = K(1 - \varepsilon) \quad \text{and} \quad b_0 = K\varepsilon \quad \ldots (3.21) \]

Also if values higher than 1 are accorded for M and/or N, the equation lends itself to the inclusion of the rate of change of outflow and/or the influence of the rate of change of inflow. Thus the KGSE can be a very generalized equation or can be treated as simply as the Muskingum equation.

Restricting \( M = N = 2 \) from practical point of view, four KGSE models can be formed. They are

i. For \( M = 1 \) and \( N = 1 \)

\[ S = a_0 Q_o + b_0 Q_1 \quad \ldots (3.22) \]

ii. For \( M = 2 \) and \( N = 1 \)

\[ S = a_0 Q_o + a_1 \frac{dQ_o}{dt} + b_0 Q_1 \quad \ldots (3.23) \]

iii. For \( M = 1 \) and \( N = 2 \)

\[ S = a_0 Q_o + b_0 Q_1 + b_1 \frac{dQ_1}{dt} \quad \ldots (3.24) \]

and iv. For \( M = 2 \) and \( N = 2 \)

\[ S = a_0 Q_o + a_1 \frac{dQ_o}{dt} + b_0 Q_1 + b_1 \frac{dQ_1}{dt} \quad \ldots (3.25) \]

Equation (3.22), which is equivalent to the Muskingum model, is called two coefficient (2C) model. A three coefficient (3C) model which considers the influence of rate of change of outflow alone is represented by Equation (3.23). In Equation (3.24) another three
coefficient (C3) model which considers the rate of change of inflow alone is presented. Equation (3.25) is the four coefficient (4C) model. It is also seen from these equations and \( a_0 \) and \( b_0 \) have dimensions of time and \( a_1 \) and \( b_1 \) dimensions of time squared. These equations will be of use only if the values of the coefficients can be ascertained.

3.2.2.2 Routing with KGSE models

The 4C model obtained from Equation 3.25 is the most general form of the four equations and the others are all particular cases where either \( a_1 \) or \( b_1 \) or both are made equal to zero. Routing a flood with 4C model is explained below and routing with other models 2C, 3C and C3 - can be proceeded on the same lines. Proceeding on the lines of Muskingum method

\[
\frac{dS}{dt} = Q_i - Q_0 + q_L
\]

and

\[
S = a_0 Q_0 + a_1 \frac{dQ_0}{dt} + b_0 Q_1 + b_1 \frac{dQ_1}{dt}
\]

Substituting the latter equation in the former

\[
a_0 \frac{dQ_0}{dt} + a_1 \frac{d^2Q_0}{dt^2} + b_0 \frac{dQ_0}{dt} + b_1 \frac{d^2Q_1}{dt^2} = Q_i - Q_0 + q_L
\]

Expanding numerically in an x-t plane as in Figure 3.6

\[\text{Figure 3.6 Rectangular net of points for the general storage equation method}\]
\[ \frac{a_o}{\Delta t} \left( Q_j^i - Q_{o_j}^{i-1} \right) + \frac{a_1}{\Delta t^2} \left( Q_j^i - 2Q_{o_j}^{i-1} + Q_{o_j}^{i-2} \right) + \frac{b_o}{\Delta t} \left( Q_i^j - Q_{i-1}^j \right) + \frac{b_1}{\Delta t^2} \left( Q_i^j - 2Q_i^{j-1} + Q_i^{j-2} \right) \]

\[ = \frac{1}{2} \left( Q_i^j + Q_{i-1}^j - Q_o^j - Q_{o_i}^{j-1} \right) + q_j L \]

... \( (3.29) \)

This equation is written on the basis that i) it is applied at a level halfway between \( j \) and \( j-1 \) i.e. \( j-\frac{1}{2} \), ii) central difference is used to evaluate the differential terms, both first and second order, and iii) the second order differential term is evaluated around \( j-1 \) instead of \( j-2 \) to avoid interpolation of values.

Multiplying Equation (3.29) by 2 and rearranging the terms

\[ \frac{2a_o}{\Delta t} + \frac{2a_1}{\Delta t^2} + 1) Q_j^i + (- \frac{2a_o}{\Delta t} - \frac{4a_1}{\Delta t^2}) Q_{o_j}^{i-1} \]

\[ + \left( \frac{-2b_o}{\Delta t^2} \right) Q_o^{i-2} + \left( \frac{-2b_1}{\Delta t^2} \right) - 1) Q_i^j \]

\[ + \left( \frac{-2b_o}{\Delta t} - \frac{4b_1}{\Delta t^2} \right) Q_{i-1}^j + \left( \frac{2b_1}{\Delta t^2} \right) Q_{i-2}^j - 2q_j L = 0 \]

... \( (3.30) \)

Substituting

\[ A_o = \frac{2a_o}{\Delta t} \quad A_1 = \frac{2a_1}{\Delta t^2} \]

\[ B_o = \frac{2b_o}{\Delta t} \quad B_1 = \frac{2b_1}{\Delta t^2} \]

... \( (3.31) \)
Equation (3.30) can be rewritten as

\[(A_0 + A_1 + 1)Q_o^j + (-A_0 - 2A_1 + 1)Q_o^{j-1} + AQ_o^{j-2}\]

\[+ (B_0 + B_1 - 1) Q_1^j + (-B_0 - 2B_1 - 1) Q_1^{j-1}\]

\[+ B_1Q_1^{j-2} - 2q_1L = 0\]

Equation (3.32)

Dividing by \((A_0 + A_1 + 1)\) and rearranging

\[Q_o^j = A_0Q_o^{j-1} + B_0Q_o^{j-2} + CQ_1^{j-1} + DQ_1^{j-1} + EQ_1^{j-2} + F\] (3.33)

in which

\[A = (A_0 + 2A_1 - 1)/(A_0 + A_1 + 1)\]

\[B = -A_1/(A_0 + A_1 + 1)\]

\[C = (1 - B_0 - B_1)/(A_0 + A_1 + 1)\]

\[D = (B_0 + 2B_1 + 1)/(A_0 + A_1 + 1)\]

\[E = -B_1/(A_0 + A_1 + 1), \text{ and}\]

\[F = 2q_1L/(A_0 + A_1 + 1)\]

Equation (3.34)

Here the initial condition needs to be taken as when \(t = 0\), \(Q_j = Q_1^{j-1}\) at both space sections. From Equation (3.31) and (3.34), the constants \(A, B, C, D, E\) and \(F\) can be worked for a known time increment \(\Delta t\) and Equation (3.33) is solved for the outflow at the time level \(j\).

The routing equation employs \(A_0, A_1, B_0\) and \(B_1\), instead of the original coefficients \(a_0, a_1, b_0\) and \(b_1\). The latter coefficients
have dimensions of time or time squared whereas the former are dimensionless. Also they are of same magnitude. These factors enable their easy use. However, it should be noted that they have to be always identified with routing period $\Delta t$.

### 3.2.2.3 Determination of parameters in the KGSE models

If Equation (3.29) is rewritten with Equation (3.31), it will lead to

$$
\begin{align*}
(Q_0^1 - Q_o^{1-1})A_0 + (Q_o^1 - 2Q_o^{1-1} + Q_o^{1-2})A_1 + (Q_o^1 - Q_o^{1-1})B_0 \\
- (Q_1^1 - 2Q_1^{1-1} + Q_1^{1-2})B_1 - (Q_1^1 + Q_1^{1-1} - Q_1^1 - Q_1^{1-1}) \\
- 2q_1L &= 0
\end{align*}
$$

For a historic flood, $Q$ and $Q_o$ values of the reach can be fitted in the above equation to obtain the best values of $A_0$, $A_1$, $B_0$ and $B_1$ by the use of some optimization techniques. These techniques may require an initial guess of the values of these parameters which can be obtained by resorting to the use of techniques like method of moments, method of maximum likelihood and method of least squares.

### 3.2.3 Limitations of hydrological methods

The Muskingum method assumes a unique relationship between the stage and the discharge along the reach. This assumption implies $S_f = S_o$ irrespective of whether the flood is rising or falling. Such an assumption is contrary to real flood situation. The momentum equation (Equation 2.2) can be simplified by dropping the acceleration terms when

$$
S_f = S_o - \frac{3y}{3x}
$$

... (3.36)
For a rising wave \( \frac{\partial y}{\partial x} \) is negative and for a falling wave it is positive. The instantaneous friction slope is greater than the normal slope for a rising stage (i.e., \( S_f > S_o \)) and smaller for a falling stage (i.e., \( S_f < S_o \)). Correspondingly the discharge for the rising and falling stages are respectively greater and less than the discharge for the uniform flow. This phenomenon is displayed graphically in the loop rating curve shown in Figure 3.7. Because of this assumption, the flow in the reach will not attenuate and any attenuation obtained in the Muskingum method is the result of the truncation error of the numerical scheme adopted. Any accuracy obtained in routing may be only coincidental. However, this limitation could be removed if the Muskingum parameters are calculated by the method due to Cunge, applying equations (3.13) and (3.16), as these parameters would then be related to convection diffusion equation.

Price [30] found that variable parameter Muskingum-Cunge method gives results similar to variable parameter diffusion method. He also mentioned the advantage of the former being a quicker method placing less demand on computer time and storage. But often times the parameters in Cunge version are arrived at with reference to flow data at just two sections, the upstream and downstream gauging stations. Values of \( V_w \) and \( B \) obtained at these two stations may not be truly representative of the whole reach.

Hydrological routing using the KGSE model is surely an improvement over the Muskingum routing model. However, there is a limitation in the sense that the methods of determining the parameters of the KGSE models from physical and hydraulic characteristics of the reach are absent. An elaborate method of optimization has to be resorted to. However, the KGSE models do not require cross-section data and that is a distinct advantage.

The primary disadvantage of the hydrological methods is that the methods would not compute the stages and as such the methods
Figure 3.7 Loop rating curve
are not indicated when one has to compute the depths of flood flow. Their use, therefore, is limited to problems where the quantity of flow is the only solution required.

Despite the limitations mentioned, the hydrological methods remain popular essentially because of their simplicity, the ease with which they can be applied and economy. They generally do not have the difficulties with tributaries which the diffusion methods have and so the storage routing methods are well suited when tributaries are not well gauged.

3.3 KINEMATIC WAVE MODEL

If the bed slope $S_0$ is the only significant term in the momentum equation, then Equation (2.2) can be approximated to

$$S_0 + S_f = 0$$

and so

$$V = C \sqrt{RS_0}$$

If Chezy's equation is used for the velocity of flow, where $R$ is the hydraulic radius. Equation (3.38) implies that velocity and hence discharge are functions of depth, $y$, alone.

Rewriting the continuity equation (Equation 2.1),

$$\frac{dQ}{dy} + \frac{2y}{B} \frac{\partial y}{\partial x} + \frac{2y}{\partial t} = 0$$

or

$$V_w \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0$$

where $V_w = 1/B \frac{dQ}{dy} = \frac{dx}{dt}$

From Equation (3.40), it follows that to an observer travelling with a velocity $V_w$, $y$ and therefore $Q$ will appear to be constant. Or a wave with constant $y$ will travel with a constant velocity $V_w$. 
Based on this, Lighthill and Whitham [31] showed that flood propagation can be described in terms of kinematic, rather than dynamic waves. They defined kinematic wave as one which has a constant amplitude and Q as a function of y alone. Thus the kinematic wave is based only on the continuity equation. Its form, more familiar than Equation (3.40), is

\[ \frac{\partial Q}{\partial t} + \omega \frac{\partial Q}{\partial x} = 0 \] \hspace{1cm} (3.42)

where \( \omega \) is the convective speed of the wave. Flood routing procedure based on Equation (3.42) is known as the kinematic method. These methods have limited application other than that of prismatic channel with waves of monoclinal type. They are seldom found useful in natural river routing.

### 3.4 DIFFUSION WAVE MODELS

When the acceleration terms in the momentum equation are dropped, the approximation leads to diffusion wave models. Ignoring lateral flow, the basic equations are, therefore,

\[ \frac{\partial Q}{\partial x} + B \frac{\partial y}{\partial t} = 0 \] \hspace{1cm} (3.43)

and

\[ \frac{\partial y}{\partial x} = S_o - S_f \] \hspace{1cm} (3.44)

These two equations can be combined together to form

\[ \frac{\partial y}{\partial t} + \omega \frac{\partial y}{\partial x} = \mu \frac{\partial^2 y}{\partial x^2} \] \hspace{1cm} (3.45)

where \( \omega \) and \( \mu \) are dispersion parameters, with \( \omega \) known as convection parameter and \( \mu \) known as diffusion parameter. Equation (3.45) is termed as the convection-diffusion equation, for whose derivation Hayami [32] and Henderson [8] can be referred to. The coefficients in this equation relate to prismoidal channel and require correction or modification.
to account for the non-uniformity of the channel geometry.

3.4.1 Linear diffusion method

The convection-diffusion equation, in its linear form has been studied by Hayami [32], Lighthill and Witham [31], Wormleaton [33] and Thomas and Wormleaton [34] [35]. Regarding the dispersion parameters as lumped measures of the convective and diffusive characteristics of the river reach, a few closed form solutions for a range of initial and boundary conditions were found. Hayami solved simple input hydrographs like step or sine functions. He suggested finding the value of \( \gamma \) by a trial and error comparison of results using his method with records of previous floods. The parameter \( \omega \) was defined as the speed of the flood peak. However, the value of \( \gamma \) was uncertain and complex forms of natural input hydrographs are not amenable for analytical representation. A finite difference form would be more appropriate.

Thomas and Wormleaton [35] developed a numerical flood routing technique based on convection-diffusion equation. They tried to determine \( \omega \) and \( \mu \) by fitting the peak value of the stage at the downstream section and the travel time of the peak through the reach. Their results showed marked variations in the values of \( \omega \) and \( \mu \), probably indicating the influence of the irregularities of the river reach. These difficulties led to the search for a convection-diffusion scheme with variable parameters.

3.4.2 Variable parameter diffusion method

3.4.2.1 Basic equation

Price [36] considered the dispersion parameters as functions of discharge and length of the reach, i.e., \( \omega (Q,x) \) and \( \mu (Q,x) \) and formulated the variable parameter diffusion (VFD) method. In his model
he demarcated the storage and flow over the flood plain by dividing the total discharge along the river as discharge in the channel and discharge over the flood plain. Applying the basic equations to channel and flood plain separately, and introducing the concepts of convection speed and attenuation parameter, he mathematically derived his variable parameter diffusion equation as

\[
\frac{\partial Q}{\partial t} + \bar{C} \frac{\partial Q}{\partial x} = \frac{a}{L} Q \frac{\partial^2 Q}{\partial x^2} + \bar{C} q_1 \quad \ldots \ (3.46)
\]

where \( \bar{C} \) is the average value of convection speed and \( a \) is the attenuation parameter. The details of derivation can be found in Flood Studies Report [13].

3.4.2.2 Attenuation parameter

Price developed the following equation which calculates the attenuation parameter from the flood plain geometry.

\[
a(Q) = \frac{1}{L} \left[ \frac{1}{L} \sum_{m=1}^{M} \frac{P_m}{S_m^3} \right]^{-3} \sum_{m=1}^{M} \frac{L_m^2}{S_m^3} \quad \ldots \ (3.47)
\]

where \( P_m \) is the area in plan of the inundated flood plain and channel in the \( m \)th subreach with \( L_m \) and \( S_m \) being the corresponding length and bed slope of the channel. For a small in bank flood \( a \) can be found from,

\[
a = \frac{1}{2W_c^2} \left[ \frac{1}{L} \sum_{m=1}^{M} \frac{L_m^3}{S_m^{3/3}} \right]^{-3} \sum_{m=1}^{M} \frac{L_m^2}{S_m^3} \quad \ldots \ (3.48)
\]

Using Equation (3.47) the attenuation parameter for the largest recorded flood can be calculated, while Equation (3.48) provides the same value for an inbank flood. Intermediate values of \( a(Q) \) can be obtained only if data for different peaks of overbank floods are available.
Otherwise the curve of $\alpha(Q)$ will have to be estimated for intermediate values. A judicious approach is necessary in plotting the curve $\alpha(Q)$.

### 3.4.2.3 Convection speed

The value of the average convection speed in the reach $\bar{C}(Q)$ is found as follows. Hayami [32] had noted that flood waves with short periods have a propagation speed greater than $\omega$ and had also derived a correction factor. Accordingly,

$$\omega = \frac{L}{T_p} - \frac{2a}{L} Q^a$$ \ldots (3.49)

where $T_p$ is the time taken by peak flow to travel along the reach length $L$ and $Q^a$ is the attenuation. Price found that $\bar{C}$ strongly depends on $\frac{d}{dQ} \left( \frac{L}{T_p} \right)$ and on $Q^a$. He suggested

$$\bar{C} = \omega + Q^a \frac{d}{dQ} \left( \frac{L}{T_p} \right)$$ \ldots (3.50)

From the records of floods in the reach under study, $\bar{C}(Q)$ can be calculated.

### 3.4.2.4 Numerical procedure for diffusion method

The routing of flood is effected by solving Equation (3.46) using a numerical procedure. The equation is written in its finite difference form (See Figure 3.8).

![Figure 3.8 Finite difference net for variable parameter diffusion method](image-url)
\[ Q_{i+1}^{j+1} - Q_{i}^{j} + \frac{\Delta t}{4\Delta x} C_a(Q_a) [ Q_{i+1}^{j+1} - Q_{i-1}^{j-1} + Q_{i+1}^{j} - Q_{i-1}^{j}] \]

\[- \frac{C_a(Q_a)}{2L\Delta x} a(Q_a) [Q_{i+1}^{j+1} - 2Q_{i}^{j} + Q_{i-1}^{j}] = 0 \quad \cdots (3.51)\]

for all \(0 < i \leq N-1\), where \(Q_a = \frac{1}{2}(Q_{i+1}^j + Q_i^j)\), and \(N\) is the label for downstream boundary. The subscript \(i\) represents the distance \((i \Delta x)\) from upstream boundary. The superscript \(j\) refers to the time \((j \Delta t)\) after the beginning of the calculation.

The boundary conditions are taken as follows:

i. Initial condition is assumed to be steady flow expressed by

\[ Q_0^i = Q_{\text{Initial}} = \text{constant} \]

for \(0 \leq i \leq N \quad \cdots (3.52)\)

ii. The upstream boundary condition is the input hydrograph as a function of time written as

\[ Q_0^j = F(\Delta t) \]

for \(0 \leq j \leq M \quad \cdots (3.53)\)

where \(M\) is the number of time levels to which the routing is carried.

iii. A free boundary condition based on the characteristics form of the Equation (3.46) is used at the downstream boundary. Thus
\[
\frac{dQ}{dt} = \frac{a}{L} Q + \frac{2Q}{\frac{d}{dx}^2} + \bar{C} q_1
\]
with the characteristics curve given by

\[
\frac{dx}{dt} = \bar{C}
\]
Hence, the downstream condition is calculated by

\[
Q_{j+1}^N = Q_j^N + \left[ \bar{C}_{N'} \left( q_1^j N' + \left( \frac{Q_j^j}{N'} \frac{2Q}{\frac{d}{dx}^2 N'} \right) \Delta t \right) \right] \Delta t
\]
where \( N' \) refers to the point at time \((j \Delta t)\) on the characteristic curve through the point \((N \Delta x, (j+1) \Delta t)\).

Then \( Q_j^N \), is calculated applying the characteristic curve equation. The relevant values of the other terms in Equation (3.56) are then evaluated and substituted to find \( Q_{j+1}^N \). This leads to a set of nonlinear simultaneous equations in \( \{Q_i^{j+1}\} \). It is solved by using the generalized Newton iteration procedure and setting up a Gaussian elimination process.

To obtain maximum accuracy of the above procedure \( \Delta x \) and \( \Delta t \) are to be so chosen such that

\[
\frac{\Delta x}{\Delta t} \geq \bar{C}_{ave}
\]
where \( \bar{C}_{ave} \) is an average of \( \bar{C} \) over a range of \( Q \).

### 3.4.2.5 Limitations of diffusion methods

An accurate determination of \( \omega \) and \( \mu \) as constants for any flood is difficult. Hence linear diffusion methods are seldom useful. The accuracy of the variable parameter diffusion method depends on
the accuracy with which $Q$ vs $a$ and $Q$ vs $C$ curves are drawn. Flood plain geometry for floods of various peak flows are often not available and hence the curves have to be estimated, which may affect the accuracy of routing. The effect of discrete lateral inflow may upset calculations by the diffusion methods, often necessitating routing from tributary to tributary. Even then, the tributaries will have to be well gauged.

Also, a diffusion model is indicated only where the acceleration terms in the momentum equation can be ignored. This can be decided only by an order of magnitude study. Where such studies show contrary effect, the diffusion methods are not suitable.

### 3.5 Dynamic Wave Models

#### 3.5.1 Introduction

A complete solution to the full Saint-Venant equation is obtained in dynamic wave models. With $Q$ and $y$ as dependent variables, the equations are written as

$$
\frac{\partial Q}{\partial x} + B \frac{\partial y}{\partial t} = q_1 \quad \ldots \ (3.58)
$$

$$
\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \left( \frac{\partial y}{\partial x} \right) = gA \left( S_0 - S_f \right) + q_1 \frac{Q}{A} \quad \ldots \ (3.59)
$$

where the friction slope $S_f$ can be evaluated using either one of the uniform flow equations. If Manning's equation is used

$$
S_f = \left( \frac{nQ}{A} \right)^2 R^{-4/3} \quad \ldots \ (3.60)
$$

where $n$ is Manning's friction factor and $R$ is the hydraulic radius.

If $P$ is the wetted perimeter of flow, then
Equation (3.58) and (3.59) are nonlinear first order partial differential equations of hyperbolic type. They are mathematically intractable. Saint-Venant approached the solution by making simplifying assumptions like i) the frictional resistance is negligible, ii) the channel is horizontal, iii) the cross-section of the channel is rectangular etc. Obviously such assumptions cannot stand a practical flood routing problem. Hence in natural river routing the closed form integration of these equations is not possible.

Numerical solutions, adopting finite difference techniques, are possible. This development can be traced from Junius Massau [37] who proposed a graphical solution, to Thomas [4], Putman [38], Stoker [39], Gilcrest [5], Lin [40], Isaacson et al. [41], Fox [42], Abbot [43] and Amein [44] [45]. Thomas who proposed a numerical solution for the case of a long rectangular channel with an ideal hydrograph as input did not complete the solution for lack of computing facilities. The advent of digital computers has enabled the development of many finite difference schemes. These methods can be classified as follows:

a. Characteristics methods
b. Explicit methods
and, c. Implicit methods.

3.5.2 Characteristics methods

Characteristics method is perhaps the earliest finite difference technique, first developed by Stoker [39]. Lin [40] proposed a technique with constant time, and distance intervals. Isaacson et al. [42], Stoker [39] [6] and Mahmood and Yevjevich [46] present detailed analysis of the relevant theory as applied to open channel flows. Later many investigators have done extensive work on the method. See for example
The Saint-Venant equations with $V$ and $y$ as dependent variables are

$$\frac{A}{B} \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = \frac{q_1}{B} \quad \ldots \ (3.62)$$

$$V \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} + g \frac{\partial y}{\partial x} = g(S_o - S_f) + q_1 \frac{V}{A} \quad \ldots \ (3.63)$$

Since $V$ and $y$ are functions of $x$ and $t$, the total differentials can be written as

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial t} dt \quad \ldots \ (3.64)$$

$$dy = \frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial t} dt \quad \ldots \ (3.65)$$

Equations (3.62) to (3.65) form a system of four nonlinear simultaneous algebraic equations which can be represented in a matrix form

$$\begin{bmatrix}
\frac{A}{B} & 0 & V & 1 \\
V & 1 & g & 0 \\
dx & dt & 0 & 0 \\
0 & 0 & dx & dt
\end{bmatrix}
\begin{bmatrix}
\frac{\partial V}{\partial x} \\
\frac{\partial V}{\partial t} \\
\frac{\partial y}{\partial x} \\
\frac{\partial y}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
\frac{q_1}{B} \\
g(S_o - S_f) + q_1 \frac{V}{A} \\
V \\
dy
\end{bmatrix} \quad (3.66)$$

The propagation of flood wave may be treated as a large number of infinitesimal surges. These surges are formed due to the
disturbances caused by the flood and each surge has a discontinuous surface profile. The water surface breaks at the point of discontinuity and its slope has two values. However, these two surface slopes bear no definite relationship to each other. Therefore such slopes must be indeterminate.

If the determinant of the coefficient matrix and the determinant obtained by substituting the right hand side of Equation (3.66) for one of the columns in the coefficient matrix both vanish, the system becomes indeterminate, implying there are infinite number of solutions for the system. The variables themselves may be indeterminate individually but they may be determinate in their relationship to each other. To satisfy this proposition, it is necessary and sufficient that all determinants of the type shown in Equation (3.66) be zero. See, for example, Abbot [50] and Chow [7]. Hence

$$\begin{vmatrix} A & V & 1 \\ B & 0 & 0 \\ V & 1 & g \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{vmatrix} = 0 \quad \cdots (3.67)$$

and

$$\begin{vmatrix} q_1 \\ \frac{1}{B} \\ g(S_o - S_r) + \frac{q_1}{A} \\ dv & dt & 0 & 0 \\ dy & 0 & dx & dt \end{vmatrix} = 0 \quad \cdots (3.68)$$
Expanding and simplifying Equation (3.67)

\[
\frac{dx}{dt} = V' + \sqrt{\frac{g}{A}} \quad \text{(3.69)}
\]

\[
\frac{dx}{dt} = V - \sqrt{\frac{g}{B}} 
\]

Also, from Equation (3.68),

\[
dV + \sqrt{\frac{g_B}{A}} \frac{dy}{dt} = \left[ q_1 \sqrt{\frac{g}{AB}} + g(S_o - S_f) + q_1 \frac{V}{A} \right] \quad \text{(3.71)}
\]

\[
dV - \sqrt{\frac{g_B}{A}} \frac{dy}{dt} = \left[ -q_1 \sqrt{\frac{g}{AB}} + g(S_o - S_f) + q_1 \frac{V}{A} \right] \quad \text{(3.72)}
\]

Equations (3.69) to (3.72) describe a system of characteristics curves intersecting each other in the x-t plane. Equations (3.69) and (3.70) give the slopes of the advancing (positive) and receding (negative) characteristics respectively when the flow is subcritical as is the case in most flood flow problems. If standing waves do not form they can be used for supercritical flow also.

### 3.5.2.1 Routing with characteristics curves

The routing procedure starts with the plotting of positive and negative characteristics in the x-t plane. The slopes of these characteristics are obtained from Equations (3.69) and (3.70) respectively with Equation (3.71) valid on advancing characteristics and Equation (3.72) useful in receding characteristics. The intersection of the characteristics gives the solution to the basic equations.
The procedure for constructing this network of characteristics can be referred to in Amein [44]. To start with, the initial and upstream boundary conditions are known. Referring to Figure 3.9, the initial characteristic $C_0$ is drawn first. On this advancing characteristic the values of $V$ and $y$ are known as functions of $x$. With $x$, $V$ and $y$ known at the point $P$, the application of Equation 3.71, in its finite difference form enables one to evaluate the value of $t$ at that point if $S$ is evaluated by a uniform flow formula. Thus $P_0$, $P_1$, $P_2$, etc. can be fixed. Then the solution along the interior characteristics are obtained as follows. Suppose the solution has progressed up to the line $P_1$, $Q_1$, $R_1$, ..., $J_1$; then values of $V$, $y$, $x$ and $t$ will be available at all characteristics to the left of this line, and also at $P_2$, it being on the initial characteristic. If a suitable time interval $\Delta t$ is chosen, $t(P_2)$ can be calculated from $t(P_2) = t(P_1) + \Delta t$. Now the application of Equations (3.69) to (3.72) will solve for the values of $V$, $y$, $x$ and $t$ at $Q_2$. The solution will progress to $R_2$, $S_2$, etc.
3.5.2.2 Fixed grid characteristics method

The irregular grid with different values of $\Delta x$ and $\Delta t$ in the characteristics method involves elaborate book-keeping during the course of solution. This difficulty may be somewhat lessened if a regular grid is considered. The method due to Hartee developed by Fox [51] is briefly outlined here with reference to Figure 3.10. The values of $V$ and $y$ at all points of time row $j$ are known. The computation is to be advanced to time row $(j+1)$. The aim is to find the characteristics passing through any point in row $(j+1)$, for example $M(i, j+1)$, and to project them to row $j$. These characteristics will intersect row $j$, not at the nodal points but at some intermediate points like $L$ and $R$ shown in the Figure 3.10. The variables at $L$ and $R$ have to be found by interpolation and then the calculation is advanced to the point $M$ in the same lines as mentioned in Section 3.5.2.2.

3.5.2.3 Limitations of characteristics method

In proceeding with the irregular characteristics grid method a considerable amount of book-keeping of values is necessary, which makes the solution tedious, slow and inefficient. Even in the fixed grid method, the book-keeping will be only slightly less as interpolation of variables at intersections of characteristics points is required. The time step restriction imposed by the Courant condition for stability given by

![Figure 3.10 Fixed rectangular grid for method of characteristics](image-url)
\[
\frac{\Delta t}{\Delta x} \leq \frac{1}{V_0 \pm \sqrt{gy_0}} \tag{3.73}
\]

with \( V_0 \) and \( y_0 \) referring to steady uniform flow situation, places a severe limitation on this method. This limitation will reduce the values of \( \Delta x \) and \( \Delta t \) to make the computations tedious and cumbersome.

### 3.5.3 Explicit method

Techniques based on rectangular grids directly employing the finite difference formulations of the Saint-Venant equations followed the characteristics methods. Explicit and implicit methods fall under this category. The development and the application in unsteady flow of this method can be traced to Isaacson et al. [41] [52]. Since then many explicit difference schemes have come into vogue, see Gunaratnam and Perkins [20]. They differ from each other in the formation of x-t grid or in the method of calculation of numerical derivatives. A simple explicit difference scheme, known as diffusing scheme, is presented briefly with reference to Figure 3.11.

Applying the Saint-Venant equations, Equations (3.58) and (3.59) at a point M shown in Figure 3.11, one solves for interior

![Figure 3.11 Rectangular net of points for the explicit method](image-url)
grid points. Taking central difference in x-direction and forward difference in \( t \)-direction.

\[
\frac{1}{2 \Delta x} (Q_{i+1}^j - Q_{i-1}^j) + \frac{b^j}{\Delta t} (y_{i+1}^{j+1} - y_{i}^j) - (q_1)_i^j = 0
\]  

... (3.74)

\[
\frac{1}{\Delta t} (Q_{i+1}^{j+1} - Q_i^j) + \frac{1}{2 \Delta x} \left[ \left( \frac{Q_1^2}{A_{i+1}} - \frac{Q_1^2}{A_{i-1}} \right) \right]
\]

\[
gA \left[ \frac{1}{2 \Delta x} (y_{i+1}^{j+1} - y_{i-1}^j) - S_f + (S_f)_i^j \right]
\]

\[- (q_1)_i^j \left( \frac{Q_1^2}{A_{i+1}} \right) = 0
\]  

... (3.75)

Friction slope \( S_f \) can be evaluated from Manning's formula and then the two Equations (3.74) and (3.75) explicitly compute \( y_{i+1}^{j+1} \) and \( Q_{i+1}^{j+1} \).

The unknowns at upstream and downstream boundaries are solved using the equations in the form of characteristics curves. Multiplying Equation (3.59) by \( 1/\sqrt{gL} \) for dimensional compatibility and subtracting it from Equation (3.58), the backward characteristic curve is obtained and their addition gives the forward characteristic curve. Thus

\[
\frac{\partial Q}{\partial x} + \left( \frac{\alpha}{gB} \right) \frac{\partial Y}{\partial t} - q_1 + \frac{1}{\sqrt{gL}} \left[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{2A} \right) \right]
\]

\[- gA (S_0 - S_f - \frac{\partial Y}{\partial x}) - q_1 \left( \frac{Q^2}{A} \right) = 0
\]  

... (3.76)

in which the negative sign is for backward characteristic and the positive sign is for forward characteristic. Applying the backward characteristic from Equation (3.76) in its finite difference form at the upstream
section an equation containing two unknowns, $y_{j}^{i+1}$ and $u_{j+1}^{i+1}$ is formed. This is solved easily as either one of them would be known as the upstream boundary condition.

Similarly the forward characteristic would provide an equation involving two unknowns, $y_{N}^{j+1}$ and $Q_{N}^{j+1}$, for the downstream boundary. This equation together with the rating curve at the downstream boundary can be simultaneously solved for the unknowns.

### 3.5.3.1 Limitations of the explicit method

The calculations are straightforward and faster in the explicit method than in the characteristics method because of the rectangular grid of points. In this scheme, the value of the space interval should also be small. The finite difference form expands the space derivatives in a central difference scheme with two space steps (i.e. $2 \Delta x$), as seen in Equations (3.74) and (3.75). Taylor's series expands a function $f(x)$, with $\Delta x$ as arbitrary variation, as

$$ f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + \ldots \quad (3.77) $$

and

$$ f(x - \Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) - \frac{\Delta x^3}{3!} f'''(x) + \ldots \quad (3.78) $$

which gives the first order differential term

$$ f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2 \Delta x} \quad \ldots (3.79) $$
The truncation error in the Equation (3.79) is of $O(\Delta x^3)$. Hence, for accuracy the value of $\Delta x$ must be very small.

The more severe restriction in explicit method is related to the Courant condition, as in characteristics method. The stability of the explicit scheme is conditional in regard to the satisfaction of Courant condition stated by Equation (3.73). Thus for small values of $\Delta x$, $\Delta t$ will also be correspondingly small, thereby increasing the number of computations.

Different types of grids and finite difference formulations also influence the solution. Often it is found, the scheme is more reliably stable when applied to a staggered net of points known as leap-frog scheme, as stated by Strelkoff [48]. For a comparison of the performance of some explicit schemes reference can be made to Price [12].

Whatever be the scheme used, the Courant condition would still place restrictions on $\Delta t$ values for the stability of the scheme. Also Equation (3.79) may be very approximate for a flood wave, especially at the foot of the rising limb and also around the peak of the hydrograph, where the variation of flow is expected to be sharp.

3.5.4 Implicit method

Though the implicit method of flood routing is of comparatively recent popularity, it has its origin as early as in 1934. Thomas [4] indicated the lines in which the implicit solution should proceed though he did not complete the solution for lack of computing facilities. His works and, later, those of Isaacsen et al. [41] led to fast solution techniques that could be obtained from implicit schemes. Many algorithms employing the unsteady flow equations have been devised and the relevant references include Stoker [6] [53], Forsythe and Wasow [54], Brakensick et al. [55], Vasiliev et al. [56], Abbot and Ionescu [57], Lai [58], Ligget and Woolhiser [47], Amein [45], Amein [45], Baltzer and Lai [59], Amein and Fang [60], Amein and Chu [61] and Greco and Panattoni [62]. These techniques mainly differ in the method of transforming the basic partial differential equations into finite difference forms and in the method of solving the resulting finite difference equations.
In the implicit scheme, the unknown values at any time level occur implicitly in the difference equations. The solution for an implicit scheme at a given time step depends not only on the solution of the previous time step but also on the present time step. The technique transforms a set of nonlinear partial differential equations to a set of finite difference expressions which are nonlinear algebraic equations. The latter have to be solved simultaneously by iteration.

Earlier implicit schemes were generally six point schemes. A scheme used by Vasiliev et al. [56] could be referred to in Strelkoff [48]. A more versatile scheme is Amein's four point implicit scheme as found in Amein and Fang [60] and Price [12]. It is also known as the box scheme as mentioned in Isaacson [63]. This scheme is explained in the succeeding section.

### 3.5.4.1 Finite difference scheme for implicit method

Referring to Figure 3.12, it is required to find the variables at all points in the time level \((j+1)\) with variables at all points in

![Figure 3.12 Rectangular net of points for the implicit method](image-url)
time row j already known. If M is a point centered within the nodes 
(i, j), (i+1, j), (i+1, j+1) and (i, j+1), the average values and the 
partial derivatives of any function a at that point are

\[
\alpha(M) = \frac{1}{4} (a_i^j + a_{i+1}^j + a_{i+1}^{j+1} + a_i^{j+1}) \quad \ldots (3.80)
\]

\[
\frac{\partial \alpha(M)}{\partial x} = \frac{1}{2 \Delta x} \left[ (a_{i+1}^j - a_i^j) + (a_{i+1}^{j+1} - a_i^{j+1}) \right] 
\quad \ldots (3.81)
\]

\[
\frac{\partial \alpha(M)}{\partial t} = \frac{1}{2 \Delta t} \left[ (a_i^{j+1} - a_i^j) + (a_{i+1}^{j+1} - a_i^{j+1}) \right] 
\quad \ldots (3.82)
\]

where \( \alpha \) may be any variable such as Q, y, A, B, etc. so that Q, 
y, \( \partial Q/\partial x \), \( \partial Q/\partial t \), \( \partial y/\partial x \), \( \partial y/\partial t \) etc. at the point M are evaluated 
with reference to the variables at the four corner grid points around 
M. Applying the above scheme to Equations (3.58) and (3.59)

\[
\frac{1}{2 \Delta x} \left( Q_i^{j+1} - Q_i^j + Q_{i+1}^{j+1} - Q_{i+1}^j \right) + 
\]

\[
\frac{1}{2 \Delta t} \left( B_i^{j+1} + B_{i+1}^j - B_i^j + B_{i+1}^{j+1} \right) \frac{1}{2 \Delta t}
\]

\[
(y_i^{j+1} - y_i^j + y_{i+1}^{j+1} - y_{i+1}^j) - \frac{\partial y}{\partial i} = 0 \quad \ldots (3.83)
\]
Either Manning's or Chezy's formula can be used for evaluating $S_f$ at the respective nodal points. Also $\Delta x$, $\Delta t$, $q_1$, $S_0$ and the friction factor used in evaluating $S_f$ are all considered constants but some or all of them may be allowed to vary where necessary.

The above two nonlinear algebraic equations possess four unknowns in $Q$ and $y$ at time level $j+1$. Hence the equations by themselves are inadequate to solve for the unknowns. But, when two adjoining grids are considered, two of the unknown will be common to both grids and hence there would be four equations in six unknowns. Thus, in the total reach of $N$ nodes having $N-1$ grids there will be $2N-2$ equations for $2N$ unknowns. The remaining two equations necessary to solve the system will be provided by the upstream and the downstream boundary conditions.
In flood routing problems, the stage or discharge hydrograph at the upstream boundary is known. Thus, either

\[ y_{1}^{j+1} = f_{1}(t_{1}^{j+1}) \quad \ldots (3.85) \]

or

\[ Q_{1}^{j+1} = f_{2}(t_{1}^{j+1}) \quad \ldots (3.86) \]

At the downstream boundary either a depth-discharge relationship or a stage hydrograph (in tidal reaches) is to be known. The former is more often the case and is described by

\[ Q_{N}^{j+1} = f_{N}(y_{N}^{j+1}) \quad \ldots (3.87) \]

With Equations (3.85) or (3.86) and (3.87) available, the system of 2N equations can be solved for 2N unknowns.

The functional form of the system of Equations (3.83), (3.84), (3.86) and (3.87) can now be written as follows.

\[
\begin{align*}
F_{1} (y_{1}, Q_{2}, y_{2}) &= 0 \\
G_{1} (y_{1}, Q_{2}, y_{2}) &= 0 \\
F_{i} (Q_{i}, y_{i}, Q_{i+1}, y_{i+1}) &= 0 \\
G_{i} (Q_{i}, y_{i}, Q_{i+1}, y_{i+1}) &= 0 \quad \text{for } i=2 \text{ to } N-1 \\
G_{N} (y_{N}, Q_{N}) &= 0
\end{align*}
\]

\ldots (3.88)
in which the functions $F$ and $G$ are respectively due to continuity and momentum equations except that $Q_N$ is a function due to the depth-discharge relationship describing the downstream boundary.

The system of Equations (3.88) consists of $2N-1$ unknowns in $(2N-1)$ nonlinear simultaneous equations. For different methods of solving these equations, reference can be made to Fang [9], Gunaratnam and Perkins [20] and Strelkoff [48]. The Newton-Raphson method is recommended by Amein and Fang [60] and has been widely adopted by Price [12], Karmegam [14] and Wormleaton and Karmegam [64].

The Newton-Raphson method, being iterative by nature, requires initial estimates of the unknowns. In flood routing problem the initial estimates of the variables at time level $j+1$ may be assumed to be those values obtained at the $j$th time level. In every iteration these initial estimates are successively improved till the variables fail to improve or the difference in the values of the variables in the last two iterations is less than a specified tolerance. In this technique the system of nonlinear equations is reduced to a system of linear equations in every iteration. The linear equations can be solved by any of the standard methods like matrix inversion or Gauss elimination in every iteration. This technique is explained in detail in Appendix 1.

The coefficient matrix in the system of Equations (3.88) is a banded matrix having only a maximum of four elements centred around the main diagonal. The rest of the elements are all zeroes. This property may be made use of to avoid unnecessary computer storage requirements. Fread [65] suggested Gauss elimination technique and has used the same in his DWOPER-NETWORK models; see Fread [66]. A Gauss elimination technique condensing the $2N-1$ columns of the matrix to just 5 columns effecting a considerable savings of computer storage was used by Karmegam [14].
3.5.4.2 Improved implicit scheme

Amein and Chu [61] developed an improved implicit scheme to incorporate large changes in channel geometry, discharge and boundary conditions. Referring to Figure 3.13, the scheme with reference to any function \( a \) is given by

\[
a(M) = \frac{1}{2} \left[ \theta (a_{i+1,j+1}^j) + (1-\theta) a_{i+1,j+1}^j \right] \quad \cdots (3.89)
\]

\[
\frac{\partial a(M)}{\partial x} = \frac{1}{\Delta x} \left[ \theta (a_{i+1,j+1}^j - a_{i+1,j+1}^j) + (1-\theta) (a_{i+1,j+1}^j - a_{i,j+1}^j) \right] \quad \cdots (3.90)
\]

\[
\frac{\partial a(M)}{\partial t} = \frac{1}{2\Delta t'} \left( a_{i+1,j+1}^j - a_{i,j+1}^j + a_{i+1,j+1}^j - a_{i,j+1}^j \right) \quad \cdots (3.91)
\]

where \( a_{i+1,j+1}^j = \frac{1}{x_{j+1} - x_{j}} \int_{x_{j}}^{x_{j+1}} a(x) \, dx \)

\[
= \frac{1}{2} (a_{i+1,j+1}^j + a_{i,j+1}^j) \quad \cdots (3.92)
\]

and \( \theta = \Delta t'/\Delta t \), a weighting factor whose value ranges from 0 to 1. \( \Delta t' \) is the time difference from point M to the time level \( j \). Equations (3.89) to (3.91) reduce to Equations (3.80) to (3.82) in the original scheme if \( \theta = 0.5 \).
3.5.4.3 Limitations of implicit scheme

The implicit scheme is unconditionally stable and is not restricted by the Courant condition unlike the characteristics and explicit methods. So a flexibility is allowed in choosing the values of $\Delta x$ and $\Delta t$. Nevertheless the values of $\Delta x$ and $\Delta t$ have to be restricted from accuracy point of view. Price [12] and Karmegam [14] tested different combinations of $\Delta x$ and $\Delta t$ to conclude that for the implicit routing to be accurate, $(\Delta x/\Delta t) \leq V_x$. Thus, the accuracy of the results again depends on $\Delta x$ and $\Delta t$ values.

The downstream boundary condition has to be specified by a stage discharge relationship or a rating curve. These relationships are empirical and are valid only for specified sections. When using such a relationship, the river section for which it is applicable should be accurately modelled. This may present a problem due to insufficient data particularly in developing countries.

Another limitation involves the determination of friction slope $S_f$, which is generally done using a uniform flow formula. However, this limitation is not peculiar to implicit method alone. All dynamic wave models are beset with this limitation.

The implicit scheme involves complex programming and iterative procedures to solve a large number of nonlinear partial differential equations simultaneously. In many field problems this factor may enormously increase the computing costs; that perhaps would be the most serious limitation of the implicit method.