Chapter 1

Introduction

Differential and integral equations arise frequently as mathematical models in various fields of applied mathematics, physics and engineering. Due to its importance in diverse fields, the theory integral equations have held a central place in attention of mathematicians. The theory of integral equations goes back to Abel who found the integral equation in 1812 starting from a problem in mechanics. Bois-Reymond is credited with the introduction of the term “Integral Equation”. It was Volterra V. who built up a theory of integral equations in 1896. Ivar Fredholm in 1900 made his famous contribution to the theory of integral equations containing a parameter. Excellent account on this subject may be found in monographs by C. Corduneanu [42], D. Guo, V. Lakshmikantham and Liu Xinzhi [68], T. A. Burton [31] and R. K. Miller [107].

By thinking in terms of operators, which takes the system from one state to another, we are led naturally to the concept of semigroup of operators,
For example, Giuseppe Peano [128], wrote the system of linear ordinary differential equations
\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + \cdots + a_{1n}x_n + f_1(t) \\
\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \cdots + a_{nn}x_n + f_n(t)
\end{align*}
\]
in matrix form as
\[
\frac{dx}{dt} = Ax + f
\]
and solved it using the explicit formula
\[
u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} f(s) ds,
\]
where \( e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \). That is, he transformed a complicated problem in one dimension to a formally simple one in higher dimensions and used the ideas of one-variable calculus to solve it.

The origin of the theory of abstract differential equations dates back from the pioneering work of E. Hille [80] and K. Yosida [150] on the Cauchy problem for the first order equation
\[
x'(t) = Ax(t); \quad a \leq t \leq b,
\]
where the operator \( A \) is infinitesimal generator of semigroup \( T(t) \) on a Banach space \( X \), in which results were formulated in terms of the theory of
semigroups of operators. The problems related with the semigroup theory, abstract cauchy problem and its variant, and their applications has been studied extensively in literature, see for example, E. Hille and R. Philips [81], A. Pazy [127], A. Friedman [60], Jerome A. Goldstein [65], G. E. Ladas and V. Lakshmikantham [90], V. Lakshmikantham, S. Leela [92] and V. Barbu [23].

Functional differential equations provide a mathematical model for a physical system in which the rate of change of the system may depend upon its past history; that is, the future state of the system depends not only on the present but also a part of its past history. A special case of such an equation is a differential-difference equation

\[ x'(t) = f(t, x(t), x(t - r)) \]

where \( r \) is a nonnegative constant. For \( r = 0 \), this is an ordinary differential equation. Jack Hale [70] has illustrated the importance and the frequency of occurrence of equations which depends upon past history, by giving number of examples in different fields. The equations and their variants have also been used as models in visco-elastic materials and in the theory of growth of a single species.
A more general equation, which we call a functional-differential equation, is one of the form

\[ x'(t) = f(t, x_t) \]

where \( x \) is an \( n \)-vector and the symbol \( x_t \) is defined as follows. If \( x \) is a function defined on \([-r, \infty)\), then for each fixed \( t \in [0, \infty) \), \( x_t \) is a function defined on the interval \([-r, 0] \), \( r \) is finite, whose values are given by \( x_t(\theta) = x(t + \theta), \ -r \leq \theta \leq 0 \). In other words, the graph of \( x_t \) is the graph of \( x \) on \([t - r, t]\) shifted to the interval \([-r, 0]\). To obtain solution of this equation for \( t \geq t_0 \), one specifies an initial function on the interval \([t_0 - r, t_0]\) and then extend the function to \( t \geq t_0 \).

Although the general theory and basic results for functional differential equations have by now been thoroughly investigated, in the past years there has been increasing interest because of the importance of such equations in variety of problems in divers field, such as, heat flow in materials with memory, visco-elasticity, reaction diffusion problem, wave equations, the vibrating beam equations, klein Gorden equations and many other physical phenomena. During the last few years number of papers and monograph have been appeared in the literature which deals with the problem of existence, uniqueness, continuation and other properties of various types of functional differential, functional integrodifferential equations and their special forms in Banach spaces using different approaches, see for example,
The study of functional differential equations with infinite delay has been studied in literature extensively. Most of the work devoted to this subject is concerned with the Cauchy problem. We mention here the works of Corduneanu and Lakshmikantham [43] and Hino, Murakami and Naito [83], and the references cited therein for the excellent guide to this subject. The interesting results concerning the existence and other properties of solutions of equations that can be described in the form

\[ x'(t) = Ax(t) + f(t, x_t), \quad t \geq 0 \]

\[ x_0 = \phi, \]

where \( A \) is the infinitesimal generator of strongly continuous semigroup of linear operators on a Banach space \( X \), may be found in literature, see for example, H. Henríquez [74, 75], J. Liang and T. Xiao [100], B. Yan [147], H.
Petzeltova [129]. The work in partial neutral functional differential equations with unbounded delay, was initiated by Hernandez and Henriquez [78, 79] and established results concerning existence and qualitative properties for neutral functional differential equations with unbounded delay.

Controllability problem of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators, see, K. Naito [108], S. Nakagri and R. Yamamoto [109] and E. N. Chukwu and S. M. Lenheart[41]. More details and results can be found in the monographs [28], [99], [151]. R. Triggiani [144] established sufficient conditions for controllability of linear and nonlinear systems in Banach space. Controllability for linear retarded systems has been considered by Manitius and Triggiani [105, 106]. Exact controllability of abstract semilinear equations has been studied by Lasiecka and Triggiani [94]. Quinn and Carmichael [131] have shown that the controllability problem can be converted into a fixed point problem. Balachandran and Dauer have considered various classes of first and second order semilinear ordinary, functional and neutral functional differential equations on Banach spaces in [6]. Fu [62] and Fu and Liu [63] have been studied the controllability result for non-densely defined functional differential systems.
After studying the available literature, the author strongly feels that the field of abstract functional integrodifferential equations of more general type, yet, awaits for its further development. This motivates the author to study functional integrodifferential equations of more general type. The present thesis “Studies in the Theory of Nonlinear Integral Equations” investigates the results pertaining to existence, uniqueness, continuous dependence on initial data and parameters, convergence properties, closeness, boundedness, controllability and other properties of the solutions of first and second order nonlinear functional integrodifferential equations of more general type. The main tools employed in our analysis are based on topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solution, the Banach’s and Schauder’s fixed point theorems, Sadovskii fixed point theorem, Krasnoselski-Schefer type fixed point theorem, nonlinear alternative of Leray-Schauder, existence of maximal solution of scalar initial value problem, fractional powers of operators, semigroup theory, the theory of cosine and sine family, Gronwall-Bellman inequality and Pachpatte inequality.

The problems investigated in this thesis are as follows:

In Chapter 2, we study the global existence of mild solutions of the nonlinear mixed Volterra-Fredholm functional integrodifferential equations
of the following types:

\[ x'(t) = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in J, \tag{1.0.1} \]

\[ x(t) = \phi(t), \quad -r \leq t \leq 0; \tag{1.0.2} \]

\[ x'(t) + Ax(t) \]

\[ = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in J, \tag{1.0.3} \]

\[ x(t) = \phi(t), \quad -r \leq t \leq 0, \tag{1.0.4} \]

\[ \frac{d}{dt} [x(t) - g(t, x_t)] + Ax(t) \]

\[ = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in J, \tag{1.0.5} \]

\[ x(t) = \phi(t), \quad -r \leq t \leq 0, \tag{1.0.6} \]

and

\[ (\varrho(t)x'(t))' = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in J, \tag{1.0.7} \]

\[ x(t) = \phi(t), \quad -r \leq t \leq 0, \quad x'(0) = \delta, \tag{1.0.8} \]

where \( J = [0, b], \quad -A \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \( T(t), t \geq 0 \) in a Banach space \( X \), \( f : J \times C \times X \times X \to X \), \( k, l : J \times J \to \mathbb{R} \), \( g, w, h : J \times C \to X \), are continuous functions, \( \varrho(t) \) is real valued positive sufficiently smooth
function on $[0, b]$ and $\phi$ is a given element of $C$. Here $C = C([-r, 0], X)$ is a Banach space with sup-norm. For any $x \in C([-r, b], X)$ and $t \in J$, $x_t$ denotes the element of $C$ defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

The main tools used in our analysis are based on the theory of strongly continuous semigroup, the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

In Chapter 3, we prove the existence results and establish the qualitative properties like uniqueness, continuous dependence on initial functions and parameters of mild solution, closeness and continuous dependence of mild solution on right hand side of the equations to semilinear mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces of the form

$$
x'(t) + Ax(t) = f(t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds), \quad t \in J, \quad (1.0.9)
$$

$$
x_0 = \phi, \quad (1.0.10)
$$

where $J = [0, b]$, $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on a Banach space $X$. 
\( f : J \times C \times X \times X \to X, \ k, l : J \times J \to \mathbb{R}, \ w, h : J \times C \to X, \) are continuous functions and \( \phi \) is a given element of a Banach space \( C = C([-r, 0], X). \)

Our analysis is based on the theory of strongly continuous semigroup, existence of maximal solution of scalar differential equations, the nonlinear alternative of Leray-Schauder and the integral inequality established by B. G. Pachpatte.

Chapter 4, deals with the basic problems such as the existence, uniqueness, continuous dependence and boundedness of mild solution of sobolev-type Volterra-Fredholm functional integrodifferential equations in Banach spaces of the following forms. As an application the controllability problem for such system is discussed. The equations considered in this chapter are

\[
(Bx(t))' + Ax(t) = f(t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds), \quad t \in [0, b], \quad (1.0.11)
\]

\[
x(t) = \phi(t), \quad t \in [-r, 0], \quad (1.0.12)
\]

and

\[
\frac{d}{dt} [Bx(t) - g(t, x_t)] + Ax(t) = f(t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds), \quad t \in [0, b], \quad (1.0.13)
\]

\[
x(t) = \phi(t), \quad t \in [-r, 0], \quad (1.0.14)
\]

where \( B \) and \( A \) are linear operators with the domains contained in a Banach space \( X \) and ranges contained in a Banach space \( Y. \) The nonlinear
functions \( f : [0, b] \times C \times Y \times Y \to Y, \ w, h, g : [0, b] \times C \to Y, \) the kernel functions \( k, l : [0, b] \times [0, b] \to \mathbb{R} \) are continuous and \( \phi \) is a given element of a Banach space \( C = C([-r, 0], X). \) The technique used in our analysis are based on Banach fixed point principle, semigroup theory, Gronwall-Bellman inequality and the Pachpatte’s inequality.

In Chapter 5, we establish the existence results for the mixed Volterra-Fredholm neutral functional integrodifferential equations with infinite delay of the form:

\[
\frac{d}{dt} \left[ x(t) - g \left( t, x_t, \int_0^t e(t, s, x_s)ds \right) \right] = Ax(t) + f \left( t, x_t, \int_0^t k(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds \right), t \in J, \quad (1.0.15)
\]

\[
x_0 = \phi \in \mathcal{B}_h, \quad (1.0.16)
\]

where \( J = [0, b], \) \( A \) is the infinitesimal generator of a compact analytic semigroup of bounded linear operators \( T(t), t \geq 0 \) in a Banach space \( X, \)

\( g : J \times \mathcal{B}_h \times X \to X, \) \( e, k, h : \Delta \times \mathcal{B}_h \to X \) and \( f : J \times \mathcal{B}_h \times X \times X \to X \)

are given functions, where \( \mathcal{B}_h \) is an abstract phase space and \( \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \). The histories \( x_t : (-\infty, 0] \to X, \) \( x_t(s) = x(t + s), s \leq 0, \)

belong to \( \mathcal{B}_h. \)

The main tools employed in our analysis are based on the theory of analytic semigroup, the theory of fractional powers of operators, Krasnoselski-Schaefer type fixed point theorem and the Pachpatte’s inequality.
In Chapter 6, we establish the controllability result for class of mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces, where the linear part is non-densely defined and satisfies the resolvent estimate of the Hille-Yosida condition. Further we investigate the controllability results for second order mixed Volterra-Fredholm functional integrodifferential equations. The equations considered in this chapter are of the following forms

\[
x'(t) = Ax(t) + Eu(t) \\
+ f \left( t, x_t, \int_0^t w(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds \right), \quad t \in J, \quad (1.0.17)
\]

\[
x(t) = \phi(t), \quad t \in [-r, 0]; \quad (1.0.18)
\]

\[
\frac{d}{dt}[x(t) - g(t, x_t)] = A[x(t) - g(t, x_t)] + Eu(t) \\
+ f \left( t, x_t, \int_0^t w(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds \right), \quad t \in J, \quad (1.0.19)
\]

\[
x(t) = \phi(t), \quad t \in [-r, 0], \quad (1.0.20)
\]

where \( J = [0, b] \), \( A : D(A) \subseteq X \to X \) is a non densely defined (i.e. \( \overline{D(A)} \neq X \)) closed linear operator which generates an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( X \), and

\[
x''(t) = Ax(t) + Eu(t) \\
+ f \left( t, x_t, \int_0^t w(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds \right), \quad t \in J \quad (1.0.21)
\]
\[ x_0 = \phi, \ x'(0) = \delta; \quad (1.0.22) \]

\[
\frac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + Eu(t) \\
+ f \left(t, x_t, \int_0^t w(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds\right), \ t \in J \quad (1.0.23)
\]

\[ x_0 = \phi, \ x'(0) = \delta, \quad (1.0.24) \]

where \( J = [0, b] \), \( A \) is the generator of a strongly continuous cosine family \( \{C(t) : t \in \mathbb{R}\} \) of bounded linear operators in \( X \).

In the above equations the state variable \( x(\cdot) \) takes values in Banach space \( X \) and the control function \( u(\cdot) \) is given in \( L^2([0, b], U) \)-the Banach space of admissible control functions with \( U \) as a Banach space, \( E \) is a bounded linear operator from \( U \) into \( X \) and \( \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \). The nonlinear functions \( g : J \times C \to X, \ w, h : \Delta \times C \to X \) and \( f : J \times C \times X \times X \to X \), are given continuous functions and \( \phi \) is a given element of Banach space \( C = C([-r, 0], X) \).

The results are obtained using the integrated semigroup theory, theory of strongly continuous cosine family of operators, Schauder fixed point theorem and the Sadoviskii fixed point theorem.

We note that, many interesting problems in the field of abstract functional differential and functional integrodifferential equations of more general types and of higher orders are not investigated in literature, for example, boundary value problems for abstract functional differential equations,
functional integrodifferential equations and inclusions of higher orders, fractional functional integrodifferential equations of more general types and impulsive functional integrodifferential equations of higher orders are not studied to a satisfactory level and there is a wide scope to study these problems.

We hope that these problems will be solved in near future and will open new fields of applications.