Chapter IX

CSCC for multiattribute production process
9.1 INTRODUCTION

The Cumulative Sum Control Chart or CUSUM chart is used primarily to maintain current control of a process. Page (1954) introduced the idea of Cumulative Sum Control Chart (CSCC) to maintain current control by continuous inspection of a manufacturing process. Barnard (1959) proposed CSCC technique for continuous inspection scheme by treating the manufacturing process as a stochastic process. For this he used the theory of runs and suggested that a V-shaped mask superimposed on the CUSUM chart be used. Johnson (1961) showed that the Cumulative Sum Control Chart can be interpreted as a modified form of a pair of Sequential Probability Ratio (SPR) tests, operated simultaneously. The idea behind this was the sequential test for two alternative hypothesis. He also gave the procedure for determining the parameters of the V-mask viz. the lead distance d and angle $\theta$ between the arms of the V-mask and horizontal line. Later on, Johnson and Leone (1962 a) developed the mathematical principle for construction and selection of CSCC for detecting shifts in mean and standard deviation of a normal production process. Subsequently, Johnson and Leone (1962 c) developed CSCC for Poisson and binomial variables using the relationship between Wald’s (1947) Sequential Probability Ratio Test and CUSUM.

In this chapter, we have developed CSCC for a process monitored with respect to m independent attributes, each of which follows a binomial distribution. The procedure of construction of CSCC is the same as Johnson and
Leone (1962c) using the pair of sequential probability ratio tests. Johnson's (1961) technique has been utilized to determine the parameters of the mask. The expression for the Average Run Length (ARL) has also been derived. Numerical examples have been included to illustrate the mathematical findings.

9.2 DEVELOPMENT OF CUSUM CHART

Suppose a process is monitored with respect to m independent attributes each of which follow a binomial distribution, say, \( b(n, p_1), b(n, p_2), \ldots, b(n, p_m) \). The sum (or convolution) of the number of defective items \( D \) found in a sample of size \( n \) from the process with respect to all m attributes is well approximated by \( b(nm, \bar{p}) \), where \( \bar{p} \) is the mean of \( p_1, p_2, \ldots, p_m \).

Wald (1947) emphasized that for practical reasons, it may sometimes be preferable to take the observation in group rather than singly. The Group Sequential Probability Ratio test in the present context is carried out as follows: A group consisting of \( n \) successive units is drawn from the production line. If the number of defectives \( D_1 \) (with respect to all m attributes) in this group is less than or equal to the acceptance number \( a_1 \), inspection is terminated with the acceptance and continuation of the process. If \( D_1 \) is greater than or equal to rejection number \( r_1 \), inspection is terminated with rejection or termination of the process. If \( a_1 < D_1 < r_1 \), a second group of \( n \) successive units is drawn. Again, the process is accepted and continued, if the number of defectives \( (D_1 + D_2) \) in two groups is less than or equal to \( a_2 \), the process is rejected if \( (D_1 + D_2) \geq r_2 \).
and / otherwise another group of n unit is taken. This procedure is continued until either rejection or acceptance of the process is decided.

Given $D_1, D_2, \ldots, D_k$ as number of defectives in k samples respectively, our problem is to construct CSCC to maintain the process level at $\bar{p} = \bar{p}_0$. Now, following Johnson (1961), we consider the SPRT for the following hypothesis

$$H_0 : \bar{p} = \bar{p}_0 \quad ; \quad H_1 : \bar{p} = \bar{p}_1 (> \bar{p}_0) \quad (9.2.1)$$

The probability of obtaining $D$ defectives (with respect to all m attributes) in a group of n observations is written as

$$f(D) = \bar{p}^D (1-\bar{p})^{nm-D} \quad (9.2.2)$$

The likelihood function under the hypothesis $H_0$ and $H_1$ are respectively given by

$$P_{0k} = \prod_{i=1}^{k} \bar{p}_0^D (1-\bar{p}_0)^{nm-D_i} \quad (9.2.3)$$

$$P_{1k} = \prod_{i=1}^{k} \bar{p}_1^D (1-\bar{p}_1)^{nm-D_i} \quad (9.2.4)$$

The log likelihood function of the Sequential Probability Ratio Test (SPRT) is found to

$$\log \frac{P_{1k}}{P_{0k}} = \sum_{i=1}^{k} D_i \log \frac{\bar{p}_1}{\bar{p}_0} + \sum_{i=1}^{k} (nm - D_i) \log \frac{(1-\bar{p}_1)}{(1-\bar{p}_0)} \quad (9.2.5)$$
The SPRT discriminating between $H_0$ and $H_1$ had its continuation region

\[
\frac{\log \left( \frac{\alpha_i}{1-\alpha_0} \right)}{1-\alpha_0} + k \frac{\log \left( \frac{1-p_0}{p_0} \right)}{1-p_0} < \sum_{i=1}^{k} D_i < \frac{\log \left( \frac{\alpha_i}{1-\alpha_0} \right)}{1-\alpha_0} + k \frac{\log \left( \frac{1-p_0}{p_0} \right)}{1-p_0}
\]

(9.2.6)

where $\alpha_i = p_i \{\text{Accept } H_{1-i} / H_i \}$, $i = 0, 1$.

The inspection is terminated with the rejection of $H_0$, if $\sum_{i=1}^{k} D_i \geq r_k$

\[
r_k = \frac{\log \left( \frac{1-\alpha_1}{\alpha_0} \right)}{\alpha_0} + k \frac{\log \left( \frac{1-p_0}{p_0} \right)}{1-p_0}
\]

(9.2.7)

Similarly, the inspection process is terminated with the acceptance of $H_0$, if $\sum_{i=1}^{k} D_i \leq a_k$

\[
a_k = \frac{\log \left( \frac{\alpha_1}{1-\alpha_0} \right)}{1-\alpha_0} + k \frac{\log \left( \frac{1-p_0}{p_0} \right)}{1-p_0}
\]

(9.2.8)
Now, considering the CSCC as a reversed SPR test with a very small value of \( \alpha \) \( (\equiv 0) \) and using only the right hand side of the inequality in (9.2.5), we have

\[
\sum_{i=1}^{k} D_i < \frac{-\log \alpha_0 + k \log(\frac{1-\bar{P}_0}{1-\bar{p}_1})^{nm}}{\log \frac{\bar{p}_1}{\bar{p}_0} - \log(\frac{1-\bar{p}_1}{1-\bar{p}_0})} \tag{9.2.9}
\]

as an inequality for constructing the CSCC for detection of an increase in the form \( \bar{p}_0 \) to \( \bar{p}_1 \). The control limit \( \text{PQ} \), as shown in Fig.(9.1a), is obtained by plotting the points with co-ordinates \((k, \sum_{i=1}^{k} D_i)\) in a two dimensional space.

Any plotted point lying below the line \( \text{PQ} \) is an evidence of lack of control i.e. an indication of increase in \( p \). The lead distance \( \text{OP} \) where \( O \) is the last plotted point, and \( P \) is to the left of \( O \), and the slope of the line \( \text{PQ} \) with horizontal line are respectively given by

\[
\text{OP} = d = \frac{-\log \alpha_0}{\log(\frac{1-\bar{p}_1}{1-\bar{p}_0})^{nm}} \tag{9.2.10}
\]

\[
\text{and} \quad \tan \angle \theta = \tan \angle \text{OPQ} = \frac{\log(\frac{1-\bar{p}_0}{1-\bar{p}_1})^{nm}}{\log[\frac{\bar{p}_1(1-\bar{p}_0)}{\bar{p}_0(1-\bar{p}_1)}]} \tag{9.2.11}
\]
Similarly, for detecting a decrease in \( p \) from \( \bar{p}_0 \) to \( \bar{p}_2 \) \((<\bar{p}_0)\), we consider the following hypothesis:

\[
H_0 : \bar{p} = \bar{p}_0 \quad H_2 : \bar{p} = \bar{p}_2 \quad (<\bar{p}_0) \quad (9.2.12)
\]

The inequality to detect a change in the parameter is given by

\[
\sum_{i=1}^{k} D_i > \frac{\log \alpha_0 + k \log(\frac{1-\bar{p}_2}{1-\bar{p}_1})^{nm}}{\log[\frac{\bar{p}_0(1-\bar{p}_2)}{\bar{p}_2(1-\bar{p}_0)}]} \quad (9.2.13)
\]

The control limit \( P'Q' \) as shown in Fig.(9.1b) can be obtained by plotting the points with co-ordinates \((k, \sum_{i=1}^{k} D_i)\). The parameters of the control chart are

\[
OP' = d = \frac{-\log \alpha_0}{\log(\frac{1-\bar{p}_2}{1-\bar{p}_0})^{nm}} \quad (9.2.14)
\]

\[
\tan \angle \theta = \tan \angle OP'Q' = \frac{\log(\frac{1-\bar{p}_2}{1-\bar{p}_0})^{nm}}{\log[\frac{\bar{p}_0(1-\bar{p}_2)}{\bar{p}_2(1-\bar{p}_0)}]} \quad (9.2.15)
\]

Any plotted point lying below the line \( P'Q' \) will indicate a lack of control in the process level.

A two sided CSCC can be obtained by simultaneous operation of the pair of SPR tests given by (9.2.9) and (9.2.13). The control chart diagram is shown in Fig.(9.1c). Any plotted point with co-ordinate \((k, \sum_{i=1}^{k} D_i)\), lying outside the
two limits PQ and P'Q' will indicate a lack of control. The relevant parameters for two sided CSCC are still given by the equations (9.2.10), (9.2.11), (9.2.14) and (9.2.15) with first kind of error being $\alpha_0$ and second kind of error $(\alpha_1 = 0)$.

### 9.3 AVERAGE RUN LENGTH (ARL)

Following Johnson (1961), the approximate formula for the average run length (ARL) when $H_i$ is true is given by

$$ARL = (-\log \alpha_0) E^{-1} \tag{9.3.1}$$

where

$$E = E[\log \frac{f(D/\bar{p} = \bar{p}_i)}{f(D/\bar{p} = \bar{p}_0)} / H_i] \tag{9.3.2}$$

and $E[.] / H_i$ is the expectation under the condition that $H_i (i = 1, 2)$ is true.

Using the equation (9.3.1) and (9.3.2), the approximate ARL can be obtained as

$$ARL = \frac{1}{\nu} \left[ -\log \alpha_0 \frac{\log \frac{\bar{p}_i}{\bar{p}_0}}{\bar{p}_i} + (1-\bar{p}_i) \log \left( \frac{1-\bar{p}_i}{1-\bar{p}_0} \right)^{mn} \right] \tag{9.3.3}$$

where $\nu = (nm - 1)$.

For a two sided CSCC, the approximate ARL can be obtained by

$$ARL = (ARL)^{-1}_{H_1} + (\Lambda R L)^{-1}_{H_2}$$
where \((\text{ARL})_{H_i}\) is the ARL when \(H_i\) \((i = 1, 2)\) is true. It should be noted that here ARL gives the expected number of groups to detect a change in \(p_0\) to \(p_i \neq p_0\) \((i = 1, 2)\).

**AN ILLUSTRATIVE EXAMPLE**

In order to study the effect of \(m\) on parameters of the V-mask, the values of \(\theta\) and \(d\) are tabulated in Table (9.1) for \(n = 10, \alpha = .05\) and \(m = 2, 3, 4\). It is found from the table that increasing value of \(m\) causes an increase in \(\theta\) but a decrease in \(d\) values. In order to give a visual comparison of the ARL values, curves have been drawn in Fig.(9.2). It is observed from figure that an increase value of \(m\) decreases ARL values.
Fig. (9.1a): CSCC for detecting the increase in the process level.

Fig. (9.1b): CSCC for detecting the decrease in the process level.

Fig. (9.1c): two-sided CSCC for controlling the process level.
Fig. (9.2) : ARL Curve for different values of m

n = 10 and $\overline{p}_0 = .01$
Table (9.1): Parameters of V-mask for Multiattribute CSCC

Values of $\theta$ and $d$ for $n = 10$

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