CHAPTER IV
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STOCHASTIC ANALYSIS OF A TWO UNIT PARALLEL
SYSTEM WITH MAXIMUM REPAIR AND ORDER TIME FOR
THE NEW UNIT

4.1 INTRODUCTION

Various authors including Goel and Gupta [1984], Gopalan et al [1982],
have studied two unit parallel /standby systems under different sets of assumptions,
using the theory of Semi-Markov process, regenerative Process and Markov Renewal
process. In all the models of two unit standby redundant systems considered so far, it
has been assumed that whenever operating unit fails standby unit operates
immediately. Practice reveals that to keep an unit in standby, increases the inventory
cost of the system. So, if there is no place for inventory, then it would be beneficial to
replace the failed unit by the new ordered one. Recently Nakagawa and Osaki [1976]
have analysed an one unit system under the assumption that as soon as an operating,
unit fails before prefixed time T an order is placed immediately for a new unit to
replace the failed one. Okumoto, Kazu [1981] has obtained the availability of a two
component repairable system using bivariate exponential failure and repair time
distributions with the very assumption that whenever both components fail
simultaneously, an order for a new units is placed to replace the failed ones. If the
new unit arrives before the completion of repair, the failed components are rejected
and replaced by the new ones, otherwise, the order is cancelled. However very few
studies have been made in this direction. This Chapter deals with the cost analysis of a
two identical unit parallel system, in which no unit is kept as standby. A single repair
facility is available in the system to replace the failed one by the new ordered one, if it
is not repaired up to a given prefixed time T. The failure, maximum repair, delivery
and replacement time distributions of an unit are taken to be negative exponential
while as repair time distribution is arbitrary. At last some particular cases are also
discussed.
4.2 Model:

The purpose of the present model is to study a two unit parallel system in which an order is placed to replace the failed unit if it is not repaired up to a fixed time $T$. A single repair facility is continuously available in the system which serves the dual role of repair and replacement of a failed unit by the new ordering one. Using regenerative point technique following measures of system effectiveness are obtained.

(i) Mean time to system failure (MTSF)
(ii) Point wise availability of the system in $(0, t]$ and in steady state.
(iii) Busy period of the repair facility in repair in $(0, t]$
(iv) Busy period of the repair facility in replacement of the failed unit with new order ones in replacement in $(0, t]$.
(v) Expected number of orders for the new unit in $(0, t]$.
(vi) Expected profit earned by the system in $(0, t]$ and in steady state.

4.3 SYSTEM DESCRIPTION

(i) The system comprises of two identical parallel units. Each unit has two modes normal (N) and total failure (F)
(ii) Whenever repair time of the failed unit exceeds some given maximum time, then that unit is rejected and an order is placed for a new unit to replace the failed one.
(iii) There is a single repair facility which serves the dual role of repair and replacement of the failed unit.
(iv) Priority is given to replacement over the repair of the failed unit.
(v) During the ordering time of a unit, if a unit fails and it is not repaired up to maximum repairing time, then this failed unit waits for order until the replacement of the first failed unit is not completed.
(vi) Failure, delivery, replacement and maximum repair time distributions are negative exponential whereas repair time distribution is arbitrary.
(vii) After repair, unit acts like a new one.
4.4 NOTATION AND STATES OF THE SYSTEMS

R = Unit in F mode and under replacement
U₀ = a new unit under order
Fw₀ = unit in F mode and waiting for ordering a new unit

Considering these symbols the system may be in any one of the following states:

\[ S₀ = (N₀, N₀), \quad S₁ = (N₀, Fr), \quad S₂ = (Fwᵣ, Fr) \]
\[ S₃ = (N₀, U₀), \quad S₄ = (Fr, U₀), \quad S₅ = (N₀, R) \]
\[ S₆ = (Fwᵣ, R), \quad S₇ = (Fw₀, U₀), \quad S₈ = (Fw₀, R) \]

4.5 OTHER SYMBOLS

\( α, a \) : constant failure rate of a normal unit and maximum repair time of a failed unit
\( β, δ \) : constant delivery rate of an ordered unit and replacement rate of a failed unit.

\( g₁(.), G₁(.) \): p.d.f and c.d.f of repair time of a failed unit other notations are same as given in paper [9].

Possible transitions among different states, along with the transition rates, are shown in fig. 4.1

4.6 TRANSITION PROBABILITIES AND SOJOURN TIME

Simple probabilistic considerations yield the following expressions for distribution functions of transition times.

\[ Q_{01}(t) = [1 - e^{-αt}]; \quad Q_{10}(t) = \int_0^t e^{-(α+βt)} dG₁(t); \]
\[ Q_{11}^{(2)}(t) = \int_0^t e^{αt} [1-e^{-αt}] dG₁(t); \quad Q_{12}(t) = \int_0^t e^{-(α+βt)} G₁(t) dt \]
\[ Q_{13}(t) = aQ_{12}(t)/α; \quad Q_{14}^{(2)}(t) = a \int_0^t e^{-αt} (1-e^{-αt}) G₁(t) dt \]
\[ Q_{14}(t) = α \int_0^t e^{-(α+βt)} dt; \quad Q_{35}(t) = β \int_0^t e^{-(α+βt)} G₁(t) dt; \]
\[ Q_{13}(t) = \int_0^t e^{-(α+βt)} dG₁(t); \quad Q_{46}(t) = β \int_0^t e^{-(α+βt)} G₁(t) dt; \]
\[ Q_{47}(t) = \alpha \int_0^t e^{-(a+\beta)t} \tilde{G}_1(t) \, dt; \quad Q_{50}(t) = \delta \int_0^t e^{-(a+\beta)t} \, dt; \]
\[ Q_{56}(t) = \alpha \int_0^t e^{-(a+\delta)t} \, dt; \quad Q_{78}(t) = [1 - e^{-a t}] \]
\[ Q_{61}(t) = [1 - e^{-\delta t}] = Q_{83}(t) \]

[1-15]

The non-zero elements \( P_{ij} \) obtained by letting \( t \to \infty \) in [1-15] are:

\[ P_{01} = P_{61} = P_{78} = P_{83} = 1 \]
\[ P_{10} = g_1^* (a+\alpha); \quad P_{12}^{(2)} = g_1^* (a) - g_1^* (a+\alpha) \]
\[ P_{13} = a [1-g_1^*(a+p\alpha)]/ (a+p\alpha) \]
\[ P_{14}^{(2)} = [1-g_1^*(a)] - a [1-g_1^*(a+p\alpha)]/ (a+p\alpha) \]
\[ P_{34} = \alpha / (a+\beta); \quad P_{35} = \beta / (a+\beta) \]
\[ P_{43} = g_1^*(a+\beta); \quad P_{46} = \beta [1-g_1^*(a+\beta)]/ (a+\beta) \]
\[ P_{47} = a [1-g_1^*(a+\beta)]/ (a+\beta); \quad P_{50} = \delta / (a+\delta) \]
\[ P_{56} = \alpha / (a+\delta); \]

[16-28]

The mean sojourn times \( \mu_i \) in state \( S_i \) are:

\[ \mu_0 = 1 / 2\alpha; \quad \mu_1 = [1-g_1^*(a+p\alpha)]/ (a+p\alpha) \]
\[ \mu_3 = 1 / (a+\beta); \quad \mu_4 = [1-g_1^* (a+\beta)]/ (a+\beta) \]
\[ \mu_5 = 1 / (a+\beta); \quad \mu_6 = \mu_8 = 1 / \delta \]
\[ \mu_7 = 1 / \beta; \]

[29-36]
4.7 TIME TO SYSTEM FAILURE

To obtain the distribution function $\pi_i(t)$ of the time to system failure with starting state $S_i \times E$ ($i = 0, 1, 3, 5$), we regard the down states $S_2, S_4$ and $S_6$ as absorbing. Using

$$\pi_0(t) = Q_{01}(t) \pi_1(t)$$

$$\pi_1(t) = Q_{10}(t) \pi_0(t) + Q_{13}(t) \pi_3(t) + Q_{12}(t)$$

$$\pi_3(t) = Q_{35}(t) \pi_3(t) + Q_{34}(t)$$

$$\pi_5(t) = Q_{50}(t) \pi_0(t) + Q_{56}(t)$$

Taking Laplace–Stieltjes Transform of equations [37-40] and solving for $\tilde{\pi}_0(s)$, we have,

$$\tilde{\pi}_0(s) = \frac{N_1(s)}{D_1(s)}$$

where

$$N_1(s) = \tilde{Q}_{01}[\tilde{Q}_{12} + \tilde{Q}_{13}(\tilde{Q}_{34} + \tilde{Q}_{35}\tilde{Q}_{56})]$$

$$D_1(s) = 1 - \tilde{Q}_{01}(\tilde{Q}_{10} + \tilde{Q}_{13}\tilde{Q}_{35}\tilde{Q}_{50})$$

(where we have omitted the argument $s$ for brevity)

Hence, Starting with state $S_0$, MTSF is

$$E(T) = -\left(\frac{d}{ds}\right)\tilde{\pi}_0(s)\bigg|_{s=0} = \frac{D_1'(O) - N_1'(O)}{D_1(O)} = \frac{\mu_0 + \mu_1 + P_{13}(\mu_3 + \mu_5P_{35})}{1 - P_{10} - P_{13}P_{35}P_{50}}$$

4.8 AVAILABILITY ANALYSIS

Let $A_i(t)$ be the probability that the system initially in state $S_i$ is up at epoch $t$. The recursive relations for $A_i(t)$ are:

$$A_0(t) = e^{-2\alpha t} + q_{01}(t)A_1(t)$$
\[ A_1(t) = e^{-(a+\alpha)t} + q_{10}(t)A_0(t) + q_{11}(t)A_1(t) + q_{13}(t)A_3(t) + q_{14}(t)A_4(t) \]

\[ A_3(t) = e^{-(a+\beta)t}q_{34}(t)A_4(t) + q_{35}(t)A_5(t) \]

\[ A_4(t) = q_{43}(t)A_3(t) + q_{46}(t)A_6(t) + q_{47}(t)A_7(t) \]

\[ A_5(t) = e^{-(\alpha+\delta)t} + q_{50}(t)A_0(t) + q_{56}(t)A_6(t)A_4(t) \]

\[ A_6(t) = q_{61}(t)A_1(t) \]

\[ A_7(t) = q_{78}(t)A_8(t) \]

\[ A_8(t) = q_{83}(t)A_3(t) \]

Where

\[ M_0(t) = e^{-2\alpha t}, \quad M_1(t) = e^{(\alpha+\delta)t}, \quad M_2(t) = e^{-(\alpha+\beta)t}, \quad M_3(t) = e^{-(\alpha+\delta)t} \]

Taking Laplace transforms of [43-50] and solving for \( A_0^*(s) \), we have

\[ A_0^*(s) = N_2(s) / D_2(s), \]

Where

\[ N_2(s) = \text{...} \]
\[ D_2(s) = \text{...} \]
$N_1(s) = q_0^*(s)[q_{13} + q_{14} (q_{43} + q_{47}q_{78}q_{83})](M_3 + M_4q_{43}) -$

$(q_{43} + q_{47}q_{78}q_{83})^* (M_3[ q_{34}(1 - q_{11}^{*2}) + q_{14}[q_{35}q_{56}q_{61}]) +$

$M_4[q_{14}] + M_0[1 - q_{11}^{*2} - q_{14}^{*2}q_{46} + q_{13}q_{34}q_{46} + q_{34}$

$D_2(s) = 1 - q_{11}^{*2} - q_{61}^{*2} (q_{14}^{*2}q_{34}q_{46} + q_{14}[q_{35}q_{56}] - q_{14}^{*2}[q_{43} + q_{47}q_{78}q_{83})$

$[q_{34}(1 - q_{11}^{*2}) + q_{14}^{*2}q_{35}q_{56}q_{61} - q_{14}^{*2}(q_{10}q_{34} - q_{14}^{*2}q_{35}q_{50}) -$

$q_{01}^{*2}(q_{10}^{*2} + q_{13}^{*2}q_{35}q_{50})$

Hence, starting from state $S_0,$ the steady state availability of the system is

$A_0(\infty) = \lim_{s \to 0} s A_0(s) = N_2(0) / D_2(0)$

$N_2(0) = \mu_0[p_{35}(1 - p_{11}^{*2}) + p_{14}^{*2}p_{46}(1 - p_{35}p_{56}) + p_{34}p_{46}(p_{10} + p_{14}^{*2})$

$- p_{35}p_{56}(p_{13} + p_{14}^{*2})] + [p_{13} + p_{14}^{*2}(1 - p_{46})](\mu_3 + \mu_5p_{35} +$

$\mu_1[1 - p_{34}(1 - p_{46})]$)

$D_2(0) = [p_{13}p_{34} + p_{14}^{*2}][\mu_4 + \mu_6(p_{46} + p_{47}) + \mu_7p_{47}] +$

$+ [1 - p_{34}(1 - p_{46})](\mu_1 + \mu_0p_{10}) +$

$[p_{13} + p_{14}^{*2}(1 - p_{46})][\mu_3 + p_{35}(\mu_5 + \mu_0p_{50} + \mu_6p_{56})]$}

The expected up time of the system in $(0, t]$ is

$\mu_{up}(t) = \int_0^t A_0(u) \, du$
so that $\mu_{up}^*(s) = A_0^*(s) / s$

the expected down time of the system in $(0,t]$ is

$\mu_d(t) = t - \mu_{up}(t)$

so that $\mu_d^*(s) = \mu_{up}^*(s) / s^2$

4.9 BUSY PERIOD ANALYSIS

(a) Expected busy period of the repairman in repair in $(0, t]$

Let $W_i(t)$ denote the probability that the system initially under repair in state $S_i$ remains in the same state at least time $t$ or passes to non-regenerative state and then continues to remain there under repair without visiting to any regenerative state including itself. By probabilistic considerations, we have

$$W_1(t) = e^{-at} \bar{G}_1(t) ; \quad W_4(t) = e^{(a+b)t} \bar{G}_1(t) \quad [53-54]$$

Recursive relations $B_i(t)$, the probability that the system starting from state $S_i$ is busy at time $t$, are:

$$B_0(t) = q_{01}(t) \odot A_1(t)$$

$$B_1(t) = W_1(t) + q_{10}(t) \odot B_0(t) + q_{11}(t) \odot B_1(t) + q_{13}(t) \odot B_3(t) + q_{14}(t) \odot B_4(t)$$

$$B_3(t) = q_{34}(t) \odot B_4(t) + q_{35}(t) \odot B_5(t)$$

$$B_4(t) = W_4(t) + q_{43}(t) \odot B_3(t) + q_{46}(t) \odot B_6(t) + q_{47}(t) B_7(t)$$

$$B_5(t) = q_{50}(t) \odot B_0(t) + q_{56}(t) \odot B_6(t)$$
\[ B_6(t) = q_{61}(t) \otimes B_1(t) \]

\[ B_7(t) = q_{78}(t) \otimes B_8(t) \]

\[ B_8(t) = q_{83}(t) \otimes B_3(t) \]

Taking Laplace Transforms of [55-62] and computing the relevant elements of the inverse matrix, we have

\[ B'_6(s) = \frac{N_3(s)}{D_2(s)} \quad [63] \]

Where

\[ N_3(s) = q_{11}^*(1 - q_{34}^* q_{43}^* + q_{47}^* q_{78}^* q_{83}^*) + \]

\[ W_4^*(q_{14}^* + q_{13}^* q_{34}^*) \]

In the long run, the fraction of time for which the system is under repair is given by

\[ B_6(\infty) = \lim_{t \to \infty} B_6(t) = \lim_{s \to 0} B'_6(s) = \frac{N_3(0)}{D'_2(0)} \quad [64]\]

The expected busy period of the repairman in repair in \((0, t]\) is

\[ \mu_b(t) = \int_0^t B_0(u) \, du \]

so that \( \mu_b^*(s) = \frac{B_6^*(s)}{s} \)

(b) Expected busy period of the repairman in replacement in \((0, t]\):

Let \( N_6(t) \) denote the probability that the repairman busy with replacement of the unit initially in regenerative state \( S_i \) and remains busy at epoch \( t \) with out transiting to any other regenerative state. By probabilities arguments, we have

\[ N_5(t) = e^{(a+\delta)}; \quad N_6(t) = e^{\delta t} = N_6(t) \quad [65-66] \]

We define \( R_i(t) \), the probability that at time \( t \) the server is busy with
replacement of the operative unit by the newly delivered unit given that the system starting from regenerative state \( S_i \) at \( t = 0 \). By probabilistic arguments, we have the following recursive relations for \( R_\delta(t) \)

\[
R_0(t) = q_{01}(t) \otimes R_1(t)
\]

\[
R_1(t) = q_{10}(t) \otimes R_0(t) + q_{11}^{(2)}(t) \otimes R_1(t) + q_{13}(t) \otimes R_3(t) + R_1^{(2)}(t) \otimes R_4(t)
\]

\[
R_3(t) = q_{34}(t) \otimes R_4(t) + q_{35}(t) \otimes R_5(t)
\]

\[
R_4(t) = q_{43}(t) \otimes R_3(t) + q_{46}(t) \otimes R_6(t) + q_{47}(t) \otimes R_7(t)
\]

\[
R_5(t) = N_5(t) + q_{50}(t) \otimes R_0(t) + q_{56}(t) \otimes R_6(t)
\]

\[
R_6(t) = N_6(t) + q_{61}(t) \otimes R_1(t)
\]

\[
R_7(t) = q_{78}(t) \otimes R_8(t)
\]

\[
R_8(t) = N_8(t) + q_{83}(t) \otimes R_3(t)
\]

[67-74]

Taking Laplace transforms of [68-74] and computing the relevant elements of the inverse matrix, the Laplace transform of \( R_\delta(t) \) is seen to be

\[
R_\delta^*(s) = \frac{N_4(s)}{D_2(s)}
\]

[75]

where

\[
N_4(s) = q_{01} q_{35} [q_{13}^{*1} + q_{14}^{*2}(q_{43}^{*2} + q_{47}^{*2} q_{78}^{*2} q_{83}^{*2})] (N_5^{*2} + N_6 q_{50}^{*2}) + q_{01}(q_{14}^{*2} + q_{34}^{*2})(N_6 q_{46}^{*2} + N_8 q_{47}^{*2} q_{78}^{*2})
\]

the expected busy period of the repairman in replacement in \((0, t]\) is

\[
\mu_R(t) = \int_0^t R_\delta(u) \, du
\]

so that \( \mu_b^*(s) = R_\delta^*(s) / s \)
4.10 EXPECTED NUMBER OF ORDERS FOR THE NEW UNIT IN (0, t]

Let \( V_0(t) \) be the expected number of orders for the new unit in \((0, t]\) given that the system entered regenerative state \( S_i(t) \) at \( t = 0 \). By probabilistic arguments, we have

\[
V_0(t) = Q_{01}(t) V_1(t)
\]

\[
V_1(t) = Q_{01}(t) V_0(t) + Q_{11}(t) V_1(t) + Q_{13}(t) [1 + V_3(t)] + Q_{14}(t) [1 + V_4(t)]
\]

\[
V_3(t) = Q_{34}(t) V_4(t) + Q_{35}(t) V_5(t)
\]

\[
V_4(t) = Q_{43}(t) V_3(t) + Q_{46}(t) V_6(t) + Q_{47}(t) V_7(t)
\]

\[
V_5(t) = Q_{50}(t) V_0(t) + Q_{56}(t) V_6(t)
\]

\[
V_6(t) = Q_{61}(t) V_1(t)
\]

\[
V_7(t) = Q_{78}(t) V_8(t)
\]

\[
V_8(t) = Q_{83}(t) [1 + V_3(t)]
\]

We have

\[
V_0(s) = N_5(s) / D_3(s)
\]

\[
N_5(s) = (\tilde{\alpha}_{13} + \tilde{\alpha}_{14}^{(2)}) [1 - (\tilde{\alpha}_{34} \tilde{\alpha}_{43} + \tilde{\alpha}_{47} \tilde{\alpha}_{78} \tilde{\alpha}_{83})] + \tilde{\alpha}_{01} \tilde{\alpha}_{47} \tilde{\alpha}_{78} \tilde{\alpha}_{83} \tilde{\alpha}_{13} \tilde{\alpha}_{34} + \tilde{\alpha}_{14}^{(2)}
\]
In steady state, number of orders for the new unit per unit of time is given by

\[ V_0(\infty) = \lim_{t \to \infty} \frac{V_0(t)}{t} \lim_{s \to 0} V_0(s) = N_3(0) / b_3(0) = 4.11 \]

**PARTICULAR CASES**

**CASE 1. WHEN REPAIR TIME DISTRIBUTION IS TAKEN TO BE NEGATIVE EXPONENTIAL** i.e. \( g_1(t) = r_1 e^{-r_1 t} \). Then the expression for

\[ E(T), A_0(\infty), B_0(\infty), R_0(\infty), \text{and } V_0(\infty), \]

become:

\[ E(T) = L_1 / D_2; \quad A_0(\infty) = L_2 / D_3 \]

\[ B_0(\infty) = L_3 / D_3; \quad R_0(\infty) = L_4 / D_3 \]

\[ V_0(\infty) = L_5 / D_3; \]

Where

\[ L_1 = (\alpha + \beta) (\alpha + \delta) (a + \alpha + r_1) + 2 \alpha [ (\alpha + \delta) (\alpha + \beta) + a (\alpha + \delta + \beta)] \]

\[ L_2 = \beta \delta (a + r_1) (a + \alpha + \beta + r_1) (\beta (\alpha + \delta) (a + \alpha + r_1) + 2 \alpha (2a + 2 \delta + \beta)) \]

\[ L_3 = 2 \alpha \beta \delta (a + \beta) (a + \alpha + \beta + r_1) [a + \alpha + r_1] + a \alpha \]

\[ L_4 = 2a \alpha (\alpha + \beta) (a + \alpha + \delta + r_1) + (a + \alpha + \beta + r_1) \]

\[ L_5 = 2a \alpha \beta \delta (a + \beta) (a + \alpha + \delta + r_1) + [\beta (a + \alpha + r_1) + a \alpha] \]
\[ D_2 = 2 \alpha \left[ (r_1 + \alpha + \alpha) (\alpha + \beta) - r_1 (\alpha + \beta) (\alpha + \beta) + ab \beta \right] \]

\[ D_3 = (a + \delta) (a + \alpha + \beta + r_1) \left[ \beta^2 \delta [2 \alpha (a + \alpha + r_1) + r_1 (a + r_1)] \right. \]
\[ + 2 \alpha \alpha^2 \right] \left[ \beta (a + \delta) + \alpha \delta \right] \alpha + \alpha + \delta + r_1 \left[ 2 \alpha (a + \delta) (\beta + \delta)^2 \right] \]

**CASE 2:** If we assume that whenever an operating unit fails, it is rejected and an order is placed immediately for a new unit to replace the failed unit i.e. \( G_1(t) = 0 \) and \( \alpha = \infty \), then we have

\[ E(T) = \left[ (\alpha + \beta) (3 \alpha + \delta) + 2 \alpha \delta \right] / 2 \alpha^2 (\alpha + \beta + \delta) \]

\[ A_0(\infty) = \beta \delta [(\alpha + \delta) (2 \alpha + \delta) + \alpha \beta] / D_4 \]

\[ R_0(\infty) = 2 \alpha \beta (\alpha + \beta) (\alpha + \delta) / D_4 \]

\[ V_0(\infty) = 2 \alpha \beta \delta (\alpha + \beta) (\alpha + \delta) / D_4 \]

\[ D_4 = 2 \alpha (\alpha + \beta) (\alpha + \delta) (\alpha + \delta) \beta^2 \delta^2 \]

**4.12 PROFIT ANALYSIS**

We are now in position to obtain the profit function of the system considering mean up time of the system and expected busy period of the server in repair and replacement. The expected total profit function incurred in \((0, t]\) is

\[ G(t) = \text{expected total revenue in } (0, t] \]
\[ - \text{expected total service cost in } (0, t] \]
\[ = C_1 \mu_{up}(t) - C_2 \mu_{u}(t) - C_3 \mu_{r}(t) - C_4 V_0(t) \]

The expected total profit per unit time in steady state is

\[ G = \lim [G(t)/t] = \lim s^2 G^*(s) \]

Where \( C_1 \) is the revenue per unit up time \( C_2, C_3, C_4 \), are the costs per unit time in repair, replacement of the failed unit by the new ordered unit and \( C_4 \) is the cost per order for a new unit to replace the old one.
STATE TRANSITION DIAGRAM

Figure no. 41

Down State

Up State
\( \delta = 0.004, \alpha = 3, r_1 = 0.6 \)