CHAPTER 5
5.1 INTRODUCTION

Non-response, is a growing problem in surveys of human populations is that not all persons sampled actually respond to the survey due to non-availability of sampling units. The problem of non-response in sample survey occurs mainly in agriculture, socio-economic and biomedical researches.

Let \((y_i, x_i)\) be the values of \(i^{th}\) unit of the study character \(y\) and the auxiliary character \(x\) for a population of size \(N\) with population mean \(\bar{Y}\) and \(\bar{X}\). Assume that the population is divided into two groups, those will respond at the first attempt belong to the response class and those who will not respond called non-response class. Let \(N_1\) and \(N_2\) be the number of units in the population that belong to the response class and the non-response class respectively \((N_1+N_2 = N)\). Let \(n_1\) be the number of units responding in a simple random sample of size \(n\) drawn from the population through simple random sample without replacement and \(n_2\) be the number of units not responding in the sample. We may regard
the sample of \( n_1 \) respond as a simple random sample from respondent class \( U^{(1)} = (1, 2, \ldots, i_1, \ldots, N_1) \) of \( N_1 \) units and the sample of \( n_2 \) as a simple random sample from non-response class \( U^{(2)} = (1, 2, \ldots, i_2, \ldots, N_2) \) of \( N_2 \) units. Again suppose \( r = \frac{n_2}{k}, k > 1 \) units are selected by making extra efforts. Therefore, we have \( n_1 + r \) observations on the \( y \) character in place of \( n \).

The literature contains many ideas for reducing, estimating and adjusting for non-response bias. Most of these approaches have used the traditional concept of design based probabilities to justify inference. One of simplest approach was proposed by Hansen and Hurwitz (1946) themselves. They defined the estimator for population mean using \( n_1 + r \) observation on \( y \) character by \( \bar{y}^* \) given as

\[
\bar{y}^* = \frac{n_1}{n} \bar{y}_1 + \frac{n_2}{n} \bar{y}_2
\]

where \( \bar{y}_1 \) and \( \bar{y}_2 \) are the sample means based on \( n_1 \) and \( r \) units respectively. The estimator \( \bar{y}^* \) is unbiased and has variance

\[
V(\bar{y}^*) = \frac{1-f}{n} S_y^2 + \frac{W_2(k-1)}{n} S_{y(2)}^2
\]

where \( f = \frac{n}{N} \), \( W_i = \frac{N_i}{N} \), \( i = 1, 2 \), \( S_y^2 \) and \( S_{y(2)}^2 \) are the variances for the whole population and for the non-response group of the population.
If \( \overline{X} \) is known and we have incomplete information on \( y \) and \( x \), then the conventional ratio estimators proposed by Hansen and Hurwitz (1946) is

\[
\bar{y}_n^* = \frac{\bar{y}^*}{\overline{X}} \overline{X}
\]

(5.1.3)

[See also Cochran (1977)]. The alternate ratio estimator, using \( \overline{X} \) and the complete information on \( X \) and incomplete information on \( y \) for \( n \) sample units, proposed by Rao (1986) is

\[
\bar{y}_{r2}^* = \frac{\bar{y}^*}{\overline{X}} \overline{X}
\]

(5.1.4)

where \( \overline{x} = \frac{1}{n} \sum_{j=1}^{n} x_{j} \) is sample mean of size \( n \),

\( \bar{x}^* = \frac{n_{1}}{n} \overline{x}_{1} + \frac{n_{2}}{n} \overline{x}_{2} \), \( \overline{x}_{1} \) and \( \overline{x}_{2} \) denote the sample mean of \( X \) character based on \( n_{1} \) and \( r \) units respectively.

The conventional and alternate regression estimators \( \bar{y}_{ln}^* \) and \( \bar{y}_{lr2}^* \) [Rao (1990)] are defined by

\[
\bar{y}_{ln}^* = \bar{y}^* + b^*(\overline{X} - \overline{x}^*)
\]

(5.1.5)

\[
\bar{y}_{lr2}^* = \bar{y}^* + b^{**}(\overline{X} - \overline{x})
\]

(5.1.6)

where \( b^* = \frac{\hat{S}_{yx}}{\hat{S}_{x}^2} \); \( b^{**} = \frac{\hat{S}_{yx}}{\hat{S}_{x}^2} \), \( \hat{S}_{x}^2 \) and \( \hat{S}_{yx} \) are estimates of \( S_{x}^2 \) and \( S_{yx} \).

In case when \( \overline{X} \) is known and we have incomplete information on \( y \) and \( x \) character for the selected sample units,
the regression estimator (derived from the difference estimator by replacing the optimum value of the constant by its estimate) is given by

\[ \bar{y}_i^* = \bar{y}^* + b^{**}(X - \bar{x}^*) \quad (5.1.7) \]

where

\[ b^{**} = \frac{(1 - \hat{f}) \hat{s}_{yx} + \frac{W_2 (k - 1)}{n} \hat{s}_{yx(2)}}{1 - \frac{1}{n} \hat{s}_x^2 + \frac{W_2 (k - 1)}{n} \hat{s}^2 x(2)} \]

For expressing MSE, we use following notations

Let

\[ S_y^2 = \sum_{i(1)=1}^{N_1} \{ y_{i(1)} - \bar{y}_{i(1)} \}^2, \quad S_x^2 = \sum_{i(1)=1}^{N_1} \{ x_{i(1)} - \bar{x}_{i(1)} \}^2, \]

\[ S_u^2 = \sum_{i(1)=1}^{N_1} \{ x_{i(1)}^2 - \mu^2_{2i(1)} (x) \}^2, \]

\[ S_{yx} = \sum_{i(1)=1}^{N_1} \{ y_{i(1)} - \bar{y}_{i(1)} \} \{ x_{i(1)} - \bar{x}_{i(1)} \} \]

\[ S_{yu} = \sum_{i(1)=1}^{N_1} \{ y_{i(1)} - \bar{y}_{i(1)} \} \{ x_{i(1)}^2 - \mu^2_{2i(1)} \}, \]

\[ S_{xu} = \sum_{i(1)=1}^{N_1} \{ x_{i(1)} - \bar{x}_{i(1)} \} \{ x_{i(1)}^2 - \mu^2_{2i(1)} \}, \]

\[ S_y^2(2) = \sum_{i(2)=1}^{N_2} \{ y_{i(2)} - \bar{y}_{i(2)} \}^2, \]

\[ S_u^2(2) = \sum_{i(2)=1}^{N_2} \{ x_{i(2)}^2 - \mu^2_{2i(2)} (x) \}^2, \]

\[ S_{yx}(2) = \sum_{i(2)=1}^{N_2} \{ y_{i(2)} - \bar{y}_{i(2)} \} \{ x_{i(2)} - \bar{x}_{i(2)} \} \]

100
\[
S_{yu(2)} = \sum_{i(2)}^{N_2} \{y_{i(2)} - \bar{y}_{i(2)}\} \{x_{i(2)}^2 - \mu_{2i(2)}\},
\]

\[
S_{xu(2)} = \sum_{i(2)}^{N_2} \{x_{i(2)} - \bar{x}_{i(2)}\} \{x_{i(2)}^2 - \mu_{2i(2)}\}
\]

\[
\text{Cov}(\bar{y}^*, \bar{x}^*) = \frac{1-f}{n} S_{yx} + \frac{W_2(k-1)}{n} S_{yx(2)}
\]

\[
V(\bar{x}^*) = \frac{1-f}{n} S_x^2 + \frac{W_2(k-1)}{n} S_x^2
\]

The mean squared errors of \(\bar{y}_{1r1}, \bar{y}_{1r2}\) and \(\bar{y}_{1r}\) to terms of order \(n^{-1}\) are

\[
M(\bar{y}_{1r1}) = \frac{(1-f)}{n} S_{\bar{y}}^2 (1 - \rho^2) + \frac{W_2(k-1)}{n} (S_{\bar{y}(2)}^2 + \beta^2 S_{\bar{y}(2)}^2 - 2\beta S_{y(2)})
\]

(5.1.8)

\[
M(\bar{y}_{1r2}) = \frac{(1-f)}{n} S_{\bar{y}}^2 (1 - \rho^2) + \frac{W_2(k-1)}{n} S_{\bar{y}(2)}^2
\]

(5.1.9)

and

\[
M(\bar{y}_{1r}) = \left(\frac{1-f}{n} S_{\bar{y}}^2 + \frac{W_2(k-1)}{n} S_{\bar{y}(2)}^2\right) - \frac{(\text{Cov}(\bar{y}^*, \bar{x}^*))^2}{V(\bar{x}^*)}
\]

(5.1.10)

and \(\beta\) is the regression coefficient of \(y\) on \(x\) for the whole population. In section 5.2 an estimator has been proposed which is more efficient than all the above estimators.

**5.2 PROPOSED ESTIMATORS AND ITS PROPERTIES**

Let population mean \(\bar{X}\) and second raw moment \(\mu'_2(x)\) of auxiliary variable \(x\) be known. Let \(\bar{x}\) and \(m'_2(x)\) be sample mean and second raw moments based on a sample of size \(n\).
Let \( n_2 \) units out of a sample of size \( n \) are non-respondents. Then we define the population mean \( \overline{Y} \) as
\[
\overline{Y}_g^* = y^* + \delta_1 (\overline{X} - \overline{x}^*) + \delta_2 (\mu_2'(x) - m_2'(x))
\] (5.2.1)
where \( \delta_1 \) and \( \delta_2 \) are suitable constants.

Following Sukhatme et al. (1992), it can easily be seen that

\[
\text{Cov}(\overline{y}^*, \overline{x}) = \text{Cov}(E(\overline{y}^*, \overline{x}) | n_1, n_2) + \text{E}\{\text{Cov}(\overline{y}^*, \overline{x}) | n_1, n_2\}
\]

because \( \text{Cov}(\overline{y}^*, \overline{x}) | n_1, n_2 = 0 \)

Similarly \( \text{Cov}(\overline{y}^*, m'_2(x)) = \text{Cov}(\overline{y}, m'_2(x)) \)
\[
= \frac{1-f}{n} S_{yx}
\]

The MSE of \( \overline{y}_g^* \) are respectively given by

\[
M(\overline{y}_g^*) = \frac{1-f}{n} \left[ S^2_y + \delta_1^2 S^2_x + \delta_2^2 S^2_u - 2 \delta_1 S_{yx} - 2 \delta_2 S_{yu} + 2 \delta_1 \delta_2 S_{xu} \right]
+ \frac{(K-1)W_2}{n} \left[ S^2_{y(2)} + \delta_1^2 S^2_{x(2)} + \delta_2^2 S^2_{u(2)} - 2 \delta_1 S_{yx(2)} - 2 \delta_2 S_{yu(2)} + 2 \delta_1 \delta_2 S_{xu(2)} \right]
\] (5.2.2)

The optimum values of \( \delta_1 \) and \( \delta_2 \) for which minimize

\[
M_0(\overline{y}_g^*) \text{ are given by}
\]
\[
\delta_{010} = \frac{S^*_{yx} S^*_{u} - S^*_{yu} S^*_{xu}}{S^*_{x} S^*_{u} - S^*_{xu}}
\] (5.2.3)
\[ \delta_{020} = \frac{S_{yx}^* S_{xu}^* - S_{yu}^* S_x^*}{S_x^* S_u^* - S_{xu}^*} \]  

(5.2.4)

where \( S_x^* = (1-f)S_x^2 + W_2 (k-1)S_x(2) \)

\[ S_u^2 = (1-f)S_u^2 + W_2 (k-1)S_u(2) \]

\[ S_{yx}^* = (1-f)S_{yx} + W_2 (k-1)S_{yx}(2) \]

\[ S_{yu}^* = (1-f)S_{yu} + W_2 (k-1)S_{yu}(2) \]

\[ S_{xu}^* = (1-f)S_{xu} + W_2 (k-1)S_{xu}(2) \]

and the corresponding minimum MSE is obtained as

\[ M(\bar{y}_g^*) = \left( \frac{1}{n} \right) S_y^* [1 - \rho^* 2 - G^* 2] \]  

(5.2.5)

\[ \rho^* = \frac{S_{yx}^*}{S_y^* S_y^*}, \quad G^* = \frac{k_{12}^*(y, x) + \rho^* \beta_1^*(x) - 2 k_{12}^*(y, x) \rho^* \sqrt{\theta_1^*(x)}}{\beta_2^*(x) - \beta_1^*(x)} \]

\( \beta_1(x) = \frac{S_{xu}^*}{S_x^*}, \quad \beta_2(x) = \frac{S_u^*}{S_u^*}, \quad k_{12}(y, x) = \frac{S_{yu}^*}{S_y^* S_u^*} \)

Comparing (5.1.8) with (5.2.5), we find that

\[ M_0(\bar{y}_g^*) \text{opt.} < M_0(\bar{y}_{lr1}) \]  

(5.2.6)
Table 6.1 shows that for $K = 2, 4, 8$ (i.e. $r = 8, 4$) the proposed estimator $\bar{y}_g$ is highly efficient than all the existing estimators.

5.4 COST ASPECT

Consider a cost function

$$C(n; n_1, n_2') = C_0 n + C_1 n_1 + C_2 n_2'$$  \hspace{1cm} (5.4.1)

Where $C_0$ is the cost per initial sample of size $n$, $C_1$ is the cost per unit of processing the results from the first attempt and $C_2$ is the cost per unit of getting and processing the data in the second stratum, clearly, $C$ is a random variable and the expected cost

$$C = E(C(n; n_1, n_2')) = C_0 n + C_1 n W_1 + \frac{C_2 n W_2}{k}$$  \hspace{1cm} (5.4.2)

Let us consider the estimators $\bar{y}_g^*$ under above cost function. If cost of survey is fixed, the optimum values of $k$ are given by

$$k_g^* = \left[ \frac{C_2 (D_1 - W_2 S_{y(2)}^2)}{S_{y(2)}^2 (C_0 + C_1 W_1)} \right]^{1/2}$$

The optimum values of $k$ for the estimators $\bar{y}_g^*, \bar{y}_1^*, \bar{y}_2^*, \bar{y}_{1r1}$ and $\bar{y}_{1r2}$ are given by

$$k_1^* = \left[ \frac{C_2 (S_y^2 - W_2 S_{y(2)}^2)}{S_{y(2)}^2 (C_0 + C_1 W_1)} \right]^{1/2}$$

$$k_{r1}^* = \left[ \frac{C_2 (S_{yR}^2 - W_2 S_{yR(2)}^2)}{S_{yR(2)}^2 (C_0 + C_1 W_1)} \right]^{1/2}$$
where

\[ S_{yxR}^2 = S_{y}^2 + R^2 S_{x}^2 - 2 R S_{yx} \]
\[ S_{yxR(2)}^2 = S_{y(2)}^2 + R^2 S_{x(2)}^2 - 2 R S_{yx(2)} \]
\[ S_{yx1}^2 = S_{y}^2 (1 - \rho^2) \]
\[ S_{yx1(2)}^2 = S_{y(2)}^2 + \beta^2 S_{x(2)}^2 - 2 \beta S_{yx(2)} \]
\[ D_1 = S_{y}^2 (1 - \rho^2 - G^2) \]

Let \( V_0 \) be fixed variance of \( \bar{y}^* \). Then the optimum values of \( n \) for \( \bar{y}^* \), \( \bar{y}_{r1}^* \), \( \bar{y}_{r2}^* \), \( \bar{y}_{l1}^* \) and \( \bar{y}_{l2}^* \) are respectively found to be

\[ n_0 = \frac{S_{y}^2 + W_2 (k_0 - 1) S_{y(2)}^2}{V_0 + \left( \frac{S_{y}^2}{N} \right)} \]
\[ n_{0\eta} = \frac{\{S_{y}^2 + R^2 S_{x}^2 - 2 R S_{yx}\} + W_2 (k_0 - 1) S_{yxR(2)}^2}{V_0 + \left( \frac{S_{y}^2 + R^2 S_{x}^2 - 2 R S_{yx}}{N} \right)} \]
\[ n_{0\text{tr}_2} = \frac{\{S_Y^2 + R^2 S_X^2 - 2RS_{yx}\} + w_2 (k_0 - 1)S_{y(2)}^2}{V_0 + \frac{S_Y^2}{N}} \]

\[ n_{0\text{ln}} = \frac{S_Y^2 (1 - \rho^2) + w_2 (k_0 - 1)S_{y(2)}^2}{V_0 + \frac{S_Y^2 (1 - \rho^2)}{N}} \]

\[ n_{0\text{lr}_2} = \frac{S_Y^2 (1 - \rho^2) + w_2 (k_0 - 1)S_{y(2)}^2}{V_0 + \frac{S_Y^2 (1 - \rho^2)}{N}} \]

\[ n_{0g} = \frac{S_Y^2 (1 - \rho^2 - G^2) + w_2 (k_0 - 1)S_{y(2)}^2}{V_0 + \frac{S_Y^2 (1 - \rho^2 - G^2)}{N}} \]

for large \( n \), values of denominators in the expression of the sample size \( n \) would be almost same. Therefore, we have

\[ n_{0g} \leq n_{0\text{tr}_2} \leq n_{0\text{ln}} \leq n_{0\text{lr}_2} \leq n_{0\text{ln}} \leq n_0 \]

When variance \( V_0 \) is fixed for above population, sample size for all the above estimators is given below
Let $C_0 = 0.1$, $C_1 = 0.4$, $C_2 = 4.5$

Table-5.2

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$V_0 = \frac{S^2}{100}$</th>
<th>$V_0 = \frac{S^2}{200}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0$</td>
<td>67</td>
<td>88</td>
</tr>
<tr>
<td>$n_1$</td>
<td>20</td>
<td>34</td>
</tr>
<tr>
<td>$n_2$</td>
<td>16</td>
<td>27</td>
</tr>
<tr>
<td>$n_{R_1}$</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>$n_{R_2}$</td>
<td>14</td>
<td>25</td>
</tr>
<tr>
<td>$n_g$</td>
<td>9</td>
<td>15</td>
</tr>
</tbody>
</table>

From the above table we conclude that whenever variance decreases, sample size of the above estimators increases and sample size for the proposed estimator is smaller than all sample sizes for other estimators.
CHAPTER 6
6.1. INTRODUCTION

Auxiliary information is utilized at the planning, designing as well as estimation stage of sample survey. A variety of techniques have been developed for increasing precision of estimators of population mean/total. Estimators so defined, are by ratio and regression methods of estimation, which take the advantage of high correlation between the study and auxiliary variables. Hansen, Hurwitz and Madow (1953) proposed difference method of estimation for estimating population mean. Bedi and Hajela (1984) proposed Searls (1964) type estimator of population mean after replacing sample mean by difference type estimator and found it to be more efficient than difference type estimator for small sample sizes. Recently, Dubey and Singh (2001) presented improved difference type estimator of \( \bar{Y} \) by using the known value of \( \bar{X} \) and investigated the efficiency of their estimator over usual estimators of mean. Their estimator is better than other usual estimators if \( \bar{X} \) is closer to \( \bar{Y} \). Similarly, it is possible that under suitable conditions, estimator for population variance \( (S^2_Y) \) of study character \( y \) could be improved by exploiting information on the auxiliary character \( x \).
Assume that the finite population consists of N identifiable units \((U_1, U_2, U_3, \ldots, U_N)\), taking the values \((y_1, y_2, y_3, \ldots, y_N)\) on study variable \(y\) and \((x_1, x_2, x_3, \ldots, x_N)\) on auxiliary variable. Let a sample of size \(n\) is drawn with SRSWOR to observe \(\{y_i, x_i\}\), \(i = 1, 2, \ldots, n\).

Define \(s^2_y = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_i - \bar{y})^2\), \(\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i\) which are an unbiased estimator of

\[S^2_y = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})^2, \quad \bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i\]

respectively. Further, \(s^2_x\) and \(S^2_x\) are defined accordingly.

Again \(K_i\), \(i = 1, 2, \ldots, 9\). Singh et al (1973), Searls and Intarapanich (1990) suggested an estimator

\[s^2_{y1} = K_1 s^2_y\] \hspace{1cm} (6.1.1)

Where \(K_1\) is a suitably constant which depends upon the coefficient of kurtosis of \(y\). The estimator \(s^2_{y1}\) is more efficient than \(S^2_y\), if sample size \(n\) is small.

The estimator \(s^2_{y2}\) has been discussed by Isaki (1983)

\[s^2_{y2} = s^2_y + K_2 (S^2_x - s^2_x)\] \hspace{1cm} (6.1.2)

Singh et al (1988) suggested an estimator of \(S^2_y\) as

\[s^2_{y3} = K_3 s^2_y + K_4 (S^2_x - s^2_x), \quad K_3 + K_4 \neq 0\] \hspace{1cm} (6.1.3)
Srivastava and Jhajj (1980) extended the results of Das and Tripathi (1978) by proposing a generalized estimator. This utilize the known value of population mean and variance of auxiliary character. This estimator is equally efficient as difference type estimator

\[
s^2_{y4} = s^2_y + K_4 \overline{(\bar{X} - \bar{X})} + K_5 (S^2_x - s^2_x) \tag{6.1.4}\]

which is more efficient than \(s^2_{y3}\) for small sample size.

Dubey and Kant (2004) proposed estimator of \(S^2_y\) as

\[
s^2_{y5} = K_6 s^2_y + K_7 s^2_x + (1 - K_6 - K_7) S^2_x \tag{6.1.5}\]

The estimator \(s^2_{y5}\) is more efficient than all the above estimators, if the value of \(S^2_y\) is near to \(S^2_x\). Again, Sharma (2004) suggested an estimator of \(S^2_y\) as

\[
s^2_{y6} = K_8 s^2_y + K_9 (S^2_x - s^2_x) + (1 - K_{10}) K_{11} S^2_x \tag{6.1.6}\]

### 6.2 PROPOSED ESTIMATOR

Let \(\hat{\sigma}^2_y\) and \(\hat{\sigma}^2_x\) be the unbiased estimator of \(\sigma^2_y\) and \(\sigma^2_x\) under any sampling design. Then we propose the estimator of \(\sigma^2_y\) as

\[
\hat{\sigma}^2_{yg} = \varphi_1 \hat{\sigma}^2_y + \varphi_2 (\bar{X} - \hat{X}) + \varphi_3 (\sigma^2_x - \hat{\sigma}^2_x) + (1 - \varphi_1) \varphi_4 \sigma^2_x \tag{6.2.1}\]

where \(\varphi_1, \varphi_2, \varphi_3\) and \(\varphi_4\) are suitable constants. It can be seen that \(\hat{\sigma}^2_{yg}\) includes following estimators as its particular cases.
\[ \hat{\sigma}_{yg}^2 = \varphi_1 \hat{\sigma}_y^2 \text{, for } \varphi_2 = \varphi_3 = \varphi_4 = 0 \]  
(6.2.1)

\[ \hat{\sigma}_{yg_2}^2 = \hat{\sigma}_y^2 + \varphi_3 (\hat{\sigma}_x^2 - \hat{\sigma}_X^2) \text{, for } \varphi_1 = 1 \& \varphi_2 = 0 \]  
(6.2.2)

\[ \hat{\sigma}_{yg_3}^2 = \varphi_1 \hat{\sigma}_y^2 + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) \text{, for } \varphi_4 = \varphi_2 = 0 \]  
(6.2.3)

\[ \hat{\sigma}_{yg_4}^2 = \hat{\sigma}_y^2 + \varphi_2 (\bar{X} - \hat{X}) + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) \text{ for } \varphi_1 = 1 \]  
(6.2.4)

\[ \hat{\sigma}_{yg_5}^2 = \varphi_1 \hat{\sigma}_y^2 + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) + (1 - \varphi_1) \sigma_x^2 \text{, for } \varphi_4 = 1 \& \varphi_2 = 0 \]  
(6.2.5)

\[ \hat{\sigma}_{yg_6}^2 = \varphi_1 \hat{\sigma}_y^2 + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) + (1 - \varphi_1) \varphi_4 \sigma_x^2, \text{ for } \varphi_2 = 0 \]  
(6.2.7)

\[ \hat{\sigma}_{yg_7}^2 = \varphi_1 \hat{\sigma}_y^2 + \varphi_2 (\bar{X} - \hat{X}) + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) \text{ for } \varphi_4 = 0 \]  
(6.2.8)

\[ \hat{\sigma}_{yg_8}^2 = \varphi_1 \hat{\sigma}_y^2 + \varphi_2 (\bar{X} - \hat{X}) + \varphi_3 (\sigma_x^2 - \hat{\sigma}_X^2) + (1 - \varphi_1) \sigma_x^2 \]  
\text{ for } \varphi_4 = 1 \]  
(6.2.9)

For expressing mean square error of \( \hat{\sigma}_{yg}^2 \), we use the following notations:

\[ \eta_{lg}(x) = \frac{\text{Cov}(\hat{\sigma}_x^2, X)}{V^3(\hat{X})} \], \( \eta_{lg}(x) = \frac{V(\hat{\sigma}_x^2)}{V^2(\hat{X})} \), \( \eta_{lg}(y) = \frac{V^2(\hat{\sigma}_y^2)}{V^2(\hat{Y})} \),

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\[ \eta_{4g}(y, x) = \frac{\text{Cov}(\hat{\sigma}_x^2, \hat{X})}{\sqrt{V(X) V(Y)}} \], \quad \eta_{5g}(y, x) = \frac{\text{Cov}(\hat{\sigma}_x^2, \hat{\sigma}_y^2)}{V(X) V(Y)} \]

\[ \psi = \frac{\sigma_x^2}{\sigma_y^2}, L_g^2(y, x) = \frac{\eta_{5g}(y, x)^2}{\eta_{2g}(x)} \]

\[ H_g^2(y, x) = \left[ \frac{\eta_{5g}(y, x) - \eta_{4g}(y, x) \sqrt{\eta_{1g}(x)}}{\eta_{2g}(x) (\eta_{2g}(x) - \eta_{1g}(x))} \right]^2 \]

\[ F_g^2(y, x) = L_g^2(y, x) + H_g^2(y, x) \]

\[ C(\hat{\sigma}_y^2) = \frac{\text{S.E.}(\hat{\sigma}_y^2)}{\sigma_y^2}, \text{S.E.}(\hat{\sigma}_y^2) = \sqrt{V(\hat{\sigma}_y^2)} \]

The proposed estimator has bias

\[ B(\hat{\sigma}_{yg}^2) = (1 - \phi_1)(\varphi_4 \psi - 1) \sigma_y^2 \] (6.2.10)

and MSE

\[ M(\hat{\sigma}_{yg}^2) = (1 - \phi_1)^2(1 - \varphi_4 \psi)^2 \sigma_y^4 + \varphi_1^2 V(\hat{\sigma}_y^2) + \varphi_2^2 V(\hat{X}) + \varphi_3^2 V(\hat{\sigma}_x^2) - 2 \varphi_1 \varphi_2 \text{Cov}(\hat{\sigma}_y^2, \hat{X}) - 2 \varphi_1 \varphi_3 \text{Cov}(\hat{\sigma}_y^2, \hat{\sigma}_x^2) + 2 \varphi_2 \varphi_3 \text{Cov}(\hat{\sigma}_x^2, \hat{X}) \] (6.2.11)

The values of \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) for which \( M(\hat{\sigma}_{yg}^2) \) is minimized, are given by

\[ \varphi_{01} = \frac{(1 - \varphi_4 \psi)^2}{[(1 - \varphi_4 \psi)^2 + C^2(\hat{\sigma}_y^2)\{1 - F_g^2(y, x)\}]} \] (6.2.12)

\[ \varphi_{02} = \varphi_{01} \varphi_{05} \] (6.2.13)

\[ \varphi_{03} = \varphi_{01} \varphi_{06} \] (6.2.14)

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where

\[
\varphi_{05} = \frac{[V(\hat{\sigma}_y^2)\text{Cov}(\hat{\sigma}_y^2, \hat{\mu}) - \text{Cov}(\hat{\sigma}_y^2, \hat{\mu})\text{Cov}(\hat{\sigma}_y^2, \hat{\mu}^2)]}{V(\hat{\mu})V(\hat{\sigma}_y^2) - \text{Cov}^2(\hat{\sigma}_y^2, \hat{\mu})}
\]

(6.2.15)

\[
\varphi_{06} = \frac{[V(\hat{\mu})\text{Cov}(\hat{\sigma}_x^2, \hat{\sigma}_y^2) - \text{Cov}(\hat{\sigma}_x^2, \hat{\mu})\text{Cov}(\hat{\sigma}_y^2, \hat{\mu})]}{V(\hat{\mu})V(\hat{\sigma}_x^2) - \text{Cov}^2(\hat{\sigma}_x^2, \hat{\mu})}
\]

(6.2.16)

Thus, the minimum MSE of \( \hat{\sigma}_{yg}^2 \) is

\[
M_0(\hat{\sigma}_{yg}^2) = \frac{V(\hat{\sigma}_{yg}^2)(1 - \varphi_{4\gamma})^2(1 - \Phi_{2g}(y,x))}{[(1 - \varphi_{4\gamma})^2 + C^2(\hat{\sigma}_{yg}^2)(1 - \Phi_{2g}(y,x))]}
\]

(6.2.17)

### 6.3 Efficiency Comparisons

The minimum variance of estimators \( \hat{\sigma}_{y1}^2, \hat{\sigma}_{y2}^2, \hat{\sigma}_{y3}^2, \hat{\sigma}_{y4}^2, \hat{\sigma}_{y5}^2, \hat{\sigma}_{y6}^2, \hat{\sigma}_{y7}^2, \) and \( \hat{\sigma}_{y8}^2 \) are respectively as follows

\[
V_0(\hat{\sigma}_{y1}^2) = \frac{V(\hat{\sigma}_{y}^2)}{1 + C^2(\hat{\sigma}_{y}^2)}
\]

(6.3.1)

\[
M_0(\hat{\sigma}_{y2}^2) = V(\hat{\sigma}_{y}^2) [1 - L_{2g}(y,x)]
\]

(6.3.2)

\[
M_0(\hat{\sigma}_{y3}^2) = \frac{V(\hat{\sigma}_{y}^2) [1 - L_{2g}(y,x)]}{1 + Q_{lg}}
\]

(6.3.3)
\[ M_0(\hat{\sigma}^2_{y_4}) = V(\hat{\sigma}^2_y) [1 - F_g^2(y, x)] \quad (6.3.4) \]

\[ M_0(\hat{\sigma}^2_{y_5}) = \frac{V(\hat{\sigma}^2_y) [1 - L_g^2(y, x)]}{1 + Q_{2g}} \quad (6.3.5) \]

\[ M_0(\hat{\sigma}^2_{y_6}) = \frac{V(\hat{\sigma}^2_y) [1 - L_g^2(y, x)]}{1 + Q_{3g}} \quad (6.3.6) \]

\[ M_0(\hat{\sigma}^2_{y_7}) = \frac{V(\hat{\sigma}^2_y) [1 - F_g^2(y, x)]}{1 + Q_{4g}} \quad (6.3.7) \]

\[ M_0(\hat{\sigma}^2_{y_8}) = \frac{V(\hat{\sigma}^2_y) [1 - F_g^2(y, x)]}{1 + Q_{5g}} \quad (6.3.8) \]

Here

\[ Q_{1g} = C^4(\hat{\sigma}^2_y) [1 - L_g^2(y, x)] \]
\[ Q_{2g} = \frac{C^2(\hat{\sigma}^2_y) [1 - L_g^2(y, x)]}{(1 - \psi)^2} \]
\[ Q_{3g} = \frac{C^2(\hat{\sigma}^2_y) [1 - L_g^2(y, x)]}{(1 - \Phi_4 \psi)^2} \]
\[ Q_{4g} = C^2(\hat{\sigma}^2_y) [1 - F_g^2(y, x)] \]
\[ Q_{5g} = \frac{C^2(\hat{\sigma}^2_y) [1 - F_g^2(y, x)]}{(1 - \psi)^2} \]

Now it can be seen that

\[ V(\hat{\sigma}^2_{y_4}) - M_0(\hat{\sigma}^2_{y_4}) = \frac{V(\hat{\sigma}^2_y)(1 - \Phi_4 \psi)^2 \sigma_y^4 F_g^2(y, x) + \Phi_4 \psi(2 - \Phi_4 \psi)V(\hat{\sigma}^2_y)M_0(\hat{\sigma}^2_{y_4})}{\{\sigma_y^4 + V(\hat{\sigma}^2_y)\} + [(1 - \Phi_4 \psi)^2 \sigma_y^4 + M_0(\hat{\sigma}^2_{y_4})]} \]

\[ > 0 \text{ if } 0 < \Phi_4 \psi < 2 \quad (6.3.9) \]

\[ V(\hat{\sigma}^2_{y_2}) - M_0(\hat{\sigma}^2_{y_2}) = \frac{V(\hat{\sigma}^2_y)(1 - \Phi_4 \psi)^2 \sigma_y^4 H_g^2(y, x) + V(\hat{\sigma}^2_y)M_0(\hat{\sigma}^2_{y_2})}{(1 - \Phi_4 \psi)^2 \sigma_y^4 + M_0(\hat{\sigma}^2_{y_2})} \]

\[ > 0 \quad (6.3.10) \]

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\[
M_0(\hat{\sigma}_{yg3}^2) - M_0(\hat{\sigma}_{yg}^2) = \frac{V(\hat{\sigma}_y^2) \sigma_y^4 H^2(y,x)(1-\varphi_4)^2 + \varphi_4 \psi (2-\varphi_4 \psi)M_0(\hat{\sigma}_{yg7}^2) V(\hat{\sigma}_{yg2}^2)}{[\sigma_y^4 + V(\hat{\sigma}_{yg2}^2)][(1-\varphi_4 \psi)^2 \sigma_y^4 + M_0(\hat{\sigma}_{yg7}^2)]}
\]

\[
> 0, \text{ if } 0 < \varphi_4 \psi < 2 \quad (6.3.11)
\]

\[
M_0(\hat{\sigma}_{yg4}^2) - M_0(\hat{\sigma}_{yg}^2) = \frac{M_0(\hat{\sigma}_{yg7}^2)}{[(1-\varphi_4 \psi)^2 \sigma_y^4 + M_0(\hat{\sigma}_{yg7}^2)]}
\]

\[
> 0 \quad (6.3.12)
\]

\[
M_0(\hat{\sigma}_{yg5}^2) - M_0(\hat{\sigma}_{yg}^2) = \frac{V(\hat{\sigma}_y^2) \sigma_y^4 H^2(y,x)(1-\psi)^2 (1-\varphi_4 \psi)^2 + \{(-\psi)^2 - (1-\varphi_4 \psi)^2\} M_0(\hat{\sigma}_{yg3}^2) V(\hat{\sigma}_{yg2}^2)}{[(1-\psi)^2 \sigma_y^2 + V(\hat{\sigma}_{yg2}^2)][(1-\varphi_4 \psi)^2 \sigma_y^4 + M_0(\hat{\sigma}_{yg7}^2)]}
\]

\[
> 0 \quad (6.3.13)
\]

A sufficient condition for \(M_0(\hat{\sigma}_{yg3}^2) - M_0(\hat{\sigma}_{yg}^2)\) is always positive is that

\[
1 < \varphi_4 < (2\psi^{-1} - 1) \quad (6.3.14)
\]

Pandey and Singh (1977) used guessed value of population variance. Let \(\sigma_{y_{\text{min}}}^2\) be the value of \(\sigma_y^2\), which may be easily guessed from past data or repeated surveys. Therefore condition (6.3.12) will always be satisfied if \(\varphi_4\) is taken between one and \(2(\psi_{\text{max}}^{-1} - 1)\), where \(\psi_{\text{max}} = \frac{\sigma_x^2}{\sigma_{y_{\text{min}}}^2}\).

Again,
It can easily be seen that the proposed estimator is more efficient than all the above estimator if conditions (6.3.9) and (6.3.17) satisfied.

\[ M_0(\hat{o}^2_{yg6}) - M_0(\hat{o}^2_{yg7}) = \frac{V(\hat{o}^2_y) H_g^2(y, x) (1 - \psi_4)^2}{[(1 - \psi_4)^2 + C^2(\hat{o}^2_y)(1 - F_g^2(y, x))][(1 - \psi_4)^2 + C^2(\hat{o}^2_y)(1 - F_g^2(y, x))]} > 0 \]  

(6.3.15)

\[ M_0(\hat{o}^2_{yg7}) - M_0(\hat{o}^2_{yg7}) = \frac{M_0^2(\hat{o}^2_{yg7})}{[1 - \psi_4^4 \sigma_y^4 + M_0(\hat{o}^2_{yg7})]} > 0 \]  

(6.3.16)

\[ M_0(\hat{o}^2_{yg8}) - M_0(\hat{o}^2_{yg8}) = \frac{M_0^2(\hat{o}^2_{yg8}) \sigma_y^4 \psi^2 (1 - \psi^2 - (1 - \psi_4)^2)}{[\sigma_y^4 (1 - \psi)^2 + M_0(\hat{o}^2_{yg7})][(1 - \psi_4^4 \sigma_y^4 + M_0(\hat{o}^2_{yg7})]} > 0, \text{ if } 1 < \psi_4 < (2\psi^{-1} - 1) \]  

(6.3.17)

It can easily be seen that the proposed estimator is more efficient than all the above estimator if conditions (6.3.9) and (6.3.17) satisfied.

### 6.4 Optimum Class of Estimators of \( \sigma_y^2 \)

For getting estimates of \( \psi_i \), \( i = 1, 2, 3, 5, 6 \), we replace population values by sample values in (6.2.7) to (6.2.11). Substituting that values in (6.2.12), we find optimum class of estimators of \( \sigma_y^2 \) as

\[ \hat{o}^2_{ya} = \psi_{01} \hat{o}^2_{ygk} + (1 - \psi_{01}) \sigma_x^2 \]  

(6.4.1)
where \( \sigma^2_{\hat{y}_{ak}} = \sigma^2 + \hat{\phi}_{05} (\bar{X} - \hat{X}) + \hat{\phi}_{06} (\sigma_x^2 - \hat{\sigma}_x^2) \) \hspace{1cm} (6.4.2)

\( \hat{\phi}_{01}, \hat{\phi}_{05} \) and \( \hat{\phi}_{06} \) are respectively sample estimates of \( \varphi_{01}, \varphi_{05} \) and \( \varphi_{06} \).

6.5 SPECIAL CASE : Simple Random Sampling Without Replacement

In case of simple random sampling without replacement procedure, \( \sigma^2_y = \frac{N-1}{N} S_y^2 \). Therefore, we propose estimator of \( S_y^2 \) as

\[
s^2_{yR} = \varphi_{1R} S_y^2 + \varphi_{2R} (\bar{X} - x) + \varphi_{3R} (S_x^2 - s_x^2) + (1 - \varphi_{1R}) \varphi_{4} S_x^2
\]

\hspace{1cm} (6.5.1)

For getting MSE of \( s^2_{yR} \), we use following notations

\[
\mu_{rs} = E (\bar{Y} - \bar{y})^r (\bar{X} - x)^s,
\]

\[
\beta_1 (x) = \frac{\mu_{03} (y, x)}{\mu_{02}^3 (y, x)}, \qquad \beta_2 (x) = \frac{\mu_{04} (y, x)}{\mu_{02}^2 (y, x)}, \qquad \beta_2 (y) = \frac{\mu_{40} (y, x)}{\mu_{20}^2 (y, x)}
\]

\[
\eta_1 (y, x) = \frac{\mu_{21} (y, x)}{\sqrt{\mu_{02} (y, x) \mu_{20} (y, x)}}, \qquad \eta_2 (y, x) = \frac{\mu_{22} (y, x)}{\mu_{02} (y, x) \mu_{20} (y, x)}
\]

\[
A = \frac{(N-1) \binom{N}{n} n \binom{n-N-1}{n-1}}{(n-1) \binom{N-3}{n}}, \quad B = \frac{(N-1) \binom{N-n-1}{n-1}}{(N-3) \binom{N-3}{n}} N,
\]

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\[ M = \frac{N^2 n^3 + 6 N^2 - 3 n^3}{(n-1) N (N-3)}, \]

\[ D = \frac{N^2 n^3 \cdot N^2 + 2 N - n - 1}{(n-1) N (N-3)}, \]

\[ \beta_2^*(x) = A \beta_2 (x) - M, \quad \beta_2^*(y) = A \beta_2 (y) - M, \]

\[ \eta_2^*(y, x) = (A \eta_2(y, x) + 2 B \rho^2 - D), \quad L^2(y, x) = \frac{\eta_2(y, x)^2}{\beta_2^*(x)} \]

\[ F^2(y, x) = L^2(y, x) + H^2(y, x), \quad \psi_s = \frac{S_x^2}{S_y^2} \]

\[ H^2(y, x) = \left[ \frac{\eta_2^*(y, x) \beta_1^*(x) - \eta_1(y, x) \beta_2^*(x)}{\beta_2^*(x) \{ \beta_2^*(x) - \beta_1(x) \}} \right]^2, \quad \theta_1 = \frac{n-2}{N-2} \]

The optimum values of \( \varphi_{iR}, i = 1, 2, 3 \) are as

\[ \varphi_{01R} = \frac{(1 - \varphi_4 \psi)^2}{\left[ (1 - \varphi_4 \psi)^2 + (1 - \theta_1) \{ \beta_2^*(y) - F^2(y, x) \} \right]} \] (6.5.2)

\[ \varphi_{02} = \varphi_{01} \left[ \frac{\beta_2^*(x) \mu_21(y, x) \mu_{02}^2(y, x) - \mu_22(y, x) \mu_{03}(y, x)}{\mu_02(y, x) \{ \beta_2^*(x) - \beta_1(x) \}} \right] \] (6.5.3)

\[ \varphi_{03} = \varphi_{01} \left[ \frac{\mu_22(y, x) \mu_{02}(y, x) - \mu_21(y, x) \mu_{03}(y, x)}{\mu_02^3(y, x) \{ \beta_2^*(x) - \beta_1(x) \}} \right] \] (6.5.4)

and corresponding minimum MSE of \( s_{yR}^2 \) is

\[ M_0(s_{yR}^2) = \frac{(1 - \theta_1) S_x^4 \{ \beta_2^*(y) - F^2(y, x) \} (1 - \varphi_4 \psi)^2}{n + (1 - \theta_1) Q_1} \] (6.5.5)
The estimator (6.5.1) includes estimators $s_{y1}^2$ to $s_{y8}^2$, defined in section 6.1 and MSE of these estimators reduces to

\[ V_0 (s_{y1}^2) = \frac{(1 - \theta_1)S_y^4 \beta_2^*(y)}{n + (1 - \theta_1)\beta_2^*(y)} \]  
(6.5.9)

\[ M_0 (s_{y2}^2) = \frac{1}{n} (1 - \theta_1)S_y^4 \{ \beta_2^*(y) - L^2(y,x) \} \]  
(6.5.10)

\[ M_0 (s_{y3}^2) = \frac{(1 - \theta_1)S_y^4 \{ \beta_2^*(y) - L^2(y,x) \}}{n + (1 - \theta_1)Q_2} \]  
(6.5.11)

\[ M_0 (s_{y4}^2) = \frac{1}{n} (1 - \theta_1)S_y^4 \{ \beta_2^*(y) - F^2(y,x) \} \]  
(6.5.12)

\[ M_0 (s_{y5}^2) = \frac{(1 - \theta_1)S_y^4 \{ \beta_2^*(y) - L^2(y,x) \}}{n + (1 - \theta_1)Q_3} \]  
(6.5.13)

\[ M_0 (s_{y6}^2) = \frac{(1 - \theta_1)S_y^4 \{ \beta_2^*(y) - L^2(y,x) \}}{n + (1 - \theta_1)Q_4} \]  
(6.5.14)

\[ M_0 (s_{y7}^2) = \frac{(1 - \theta_1)S_y^4 \{ \beta_2^*(y) - F^2(y,x) \}}{n + (1 - \theta_1)Q_5} \]  
(6.5.15)

\[ M_0 (s_{y8}^2) = \frac{(1 - \theta_1)S_y^4 \beta_2^*(y)\{1 - F^2(y,x)\}}{(n - 1) + (1 - \theta_1)Q_6} \]  
(6.5.16)

where $Q_1 = \frac{\{ \beta_2^*(y) - F^2(y,x) \}}{(1 - \varphi_4 \psi)^2}$, $Q_2 = \{ \beta_2^*(y) - L^2(y,x) \}$. 

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Comparing (6.5.8) to (6.5.16) with (6.5.5), it can easily seen that the proposed estimator is more efficient than existing estimators.

For symmetrically distributed variables where odd order moments reduce to zero, \( s^2_{yR} \) is equally efficient as \( s^2_y \).

### 6.6 EMPIRICAL STUDY

Let us consider the population from Tripathi et al. (2001) which consists summarized data of 142 cities of India with population (number of persons) 1,00,000 and above. Let \( x \) be census population in the year 1961 (in 00's) and \( y \) be census population in the year 1971 (in 00's). Values of the required population parameters are given below:

- \( \bar{Y} = 4015.2183 \)
- \( \bar{X} = 2900.3872 \)
- \( S_y = 8479.338 \)
- \( S_x = 6372.441 \)
- \( \rho = 0.9948 \)
- \( \beta_2(x) = 48.1567 \)
- \( \beta_1(x) = 35.7945 \)
- \( \eta_1(y,x) = 6.1772 \)
- \( \beta_2(y) = 40.8536 \)
- \( \eta_2(y,x) = 43.7615 \)

The actual mean square error of the estimator are calculated considering sample of size \( n = 20,30,50 \) drawn with SRSWOR from the population. Relative efficiencies (R.E.) of
different estimators of $S_y^2$ with respect to conventional estimator $S_y^2$, defined by $[V(s_y^2)/M(.)] \times 100$ are displayed in the following table:

**TABLE-6.1**

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$n = 20$</th>
<th>$n = 30$</th>
<th>$N = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{y1}^2$</td>
<td>275.166</td>
<td>273.732</td>
<td>152.717</td>
</tr>
<tr>
<td>$s_{y2}^2$</td>
<td>3697.862</td>
<td>3697.096</td>
<td>3696.320</td>
</tr>
<tr>
<td>$s_{y3}^2$</td>
<td>3873.029</td>
<td>3804.205</td>
<td>3749.081</td>
</tr>
<tr>
<td>$s_{y4}^2$</td>
<td>3956.910</td>
<td>3954.831</td>
<td>3941.025</td>
</tr>
<tr>
<td>$s_{y5}^2$</td>
<td>4132.077</td>
<td>4088.995</td>
<td>3749.782</td>
</tr>
<tr>
<td>$s_{y7}^2$</td>
<td>4622.722</td>
<td>4262.620</td>
<td>3974.782</td>
</tr>
<tr>
<td>$s_{y8}^2$</td>
<td>4881.775</td>
<td>4661.742</td>
<td>4364.884</td>
</tr>
</tbody>
</table>
TABLE-6.2

<table>
<thead>
<tr>
<th></th>
<th>3873.029</th>
<th>4132.079</th>
<th>3804.206</th>
<th>4088.719</th>
<th>3749.082</th>
<th>3997.782</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>3904.837</td>
<td>4179.533</td>
<td>3833.223</td>
<td>4124.991</td>
<td>3763.374</td>
<td>4080.074</td>
</tr>
<tr>
<td>.2</td>
<td>3990.198</td>
<td>4249.248</td>
<td>3875.851</td>
<td>4178.276</td>
<td>3784.371</td>
<td>4029.072</td>
</tr>
<tr>
<td>.4</td>
<td>4098.630</td>
<td>4357.682</td>
<td>3942.154</td>
<td>4261.156</td>
<td>3817.029</td>
<td>4061.728</td>
</tr>
<tr>
<td>.6</td>
<td>4280.822</td>
<td>4539.874</td>
<td>4053.559</td>
<td>4400.413</td>
<td>3817.029</td>
<td>4116.601</td>
</tr>
<tr>
<td>.8</td>
<td>4622.722</td>
<td>4881.776</td>
<td>4262.621</td>
<td>4661.742</td>
<td>4064.155</td>
<td>4308.851</td>
</tr>
<tr>
<td>1</td>
<td>5384.784</td>
<td>5643.841</td>
<td>4728.599</td>
<td>5244.218</td>
<td>4204.394</td>
<td>4449.091</td>
</tr>
<tr>
<td>1.2</td>
<td>7697.296</td>
<td>7956.336</td>
<td>6142.631</td>
<td>7011.772</td>
<td>4908.088</td>
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<tr>
<td>1.4</td>
<td>22578.69</td>
<td>22837.822</td>
<td>15242.156</td>
<td>18386.268</td>
<td>9382.932</td>
<td>9627.544</td>
</tr>
<tr>
<td>1.6</td>
<td>636324.4</td>
<td>636586.21</td>
<td>390529.413</td>
<td>487498.416</td>
<td>194233.8</td>
<td>194475.23</td>
</tr>
<tr>
<td>2</td>
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<td>14385.995</td>
<td>1007.414</td>
<td>1192.619</td>
<td>6837.377</td>
<td>7082.036</td>
</tr>
<tr>
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<td>6934.176</td>
<td>5517.597</td>
<td>6230.473</td>
<td>4593.024</td>
<td>4837.273</td>
</tr>
<tr>
<td>2.4</td>
<td>5083.743</td>
<td>5342.797</td>
<td>4544.521</td>
<td>5014.119</td>
<td>4007.131</td>
<td>4251.828</td>
</tr>
</tbody>
</table>

Above table demonstrates that the proposed estimator is more efficient than all the above estimator for different values of $\varphi_4$. When $\varphi_4 = 1.8$, proposed estimator $s^2_{yg}$ and $s^2_{y6}$ gives highest efficiency. For $\varphi_4 = 0$, $s^2_{yg}$ is equally efficient as and $s^2_{y6}$ respectively. At $\varphi_4 = 1.0$ efficient than other existing estimators. and $s^2_{y6}$ is equally efficient as and respectively.