Chapter 6

Summary, Conclusion and Future Prospects

In this thesis work, we have been able to do a comprehensive study of mesons particularly in the light-heavy flavour sector. Various issues and challenges and attempts in understanding the physics and dynamics of the quarks at the hadronic dimensions are discussed in the initial chapters of this thesis. There exist theoretical and experimental evidences to uphold the various predictions of quantum chromodynamics such as the hadron spectroscopy, the exotic states, etc.

There is strong motivation to understand the effective interaction heavy quarks (and anti-quarks) since it is expected to be "cener" than between light quarks. For heavy particles the non-relativistic constituents quark model is more acceptable, the perturbative QCD contributions (such as one gluon exchange) is more adequate and chiral fields are less important.

The spectroscopic properties of mesons containing at least one heavy flavour quark/antiquark are studied in a potential scheme describe by coulomb plus power potential with variational approach $CPP - VS$. With the gaussian trial wave function (GTW), we have computed ground state masses, the fermi momentum parameters and the value of the wave function at the origin for charm and beauty
mesons as well as for charmonium and bottomonium. The computation has been performed for different inter-quark potential by varying the power index $\nu$ from 0.5 to 2.0. Our results on fermi momentum as seen from Table 3.1 lie in the range of 0.277 GeV for $D$ mesons to 0.354 GeV for $B_s$ mesons. These values, particularly in the case of $B$ mesons (0.303 GeV) are in close agreement with the values commonly used in experimental analysis [Hwang et al, 1996; Ajay K Rai et al, 2002]. We found that the potential form with power $\nu = 0.5$ provides better agreement with the experimental values of the properties of $q\bar{Q}$ mesons consistently. This study not only suggests the importance of the fermi momentum parameter, but also the potential forms for the understanding of the dynamics of the light-heavy systems. Our results for the spin-averaged ground state masses, ratios of the chromomagnetic splitting between the vector-pseudoscalar pairs are in good agreements with the respective experimental values.

Though the $q\bar{Q}$ systems are well described by the $CPP_{\nu} - VS(GTW)$ scheme with $\nu = 0.5$, the heavy-heavy (systems $QQ$) seem to be better described for $0.7 < \nu < 1.4$ according the Table 3.2.

The fermi momentum parameter $p_F$ is most essential parameter for the of weak decays of light-heavy flavour hadrons in spectator model. It represents the momentum distribution of the spectator quark inside a meson. In this way the spectator model incorporates the bound state effects and reduced the strong dependance of the quark mass in the decay width of the light-heavy flavour mesons. It is also the maximum kinematically allowed value of momentum in the computation of the lepton energy spectrum of the meson decay. So, for the study of weak interaction decay properties, the Fermi momentum parameter is very important and it accounts for the strong interaction effects as it is determined from the hadron spectroscopy.

Pseudoscalar or leptonic decay constant is another important input parameter for the annihilation width of the heavy flavour mesons. Phenomenologically it is
also important to have reliable estimates of these decay constants, because they appear in many processes from which we can learn about quantities of fundamental importance to the standard model, such as the quark mixing matrix. In the quark model, the pseudoscalar decay constant, $f_p$, is related to the amplitude for the annihilation of the $q\bar{Q}$ pair and can express in terms of the Wave function of the $q\bar{Q}$ system. The relation between $f_p$ and the ground state wave function at the origin $\psi(0)$ is given by the Van-Royen-Weisskof formula (V-W) including the colour factor.

Some discrepancy in the values of ratios of pseudo-scalar decay constants suggests the limitations of the V-W formula and the need to incorporate contributions due to relativistic effects, radiative correction finite heavy quark mass, etc. However, the overall order in which they vary is in accordance with the lattice predictions [Ma and McKellar, 1998]. In conclusion, the variational approach employed here is found to be successful in the study of heavy-light flavour mesons with a coulomb plus power potential. The treatment of light flavour relativistically and heavy flavour non-relativistically also seems to be appropriate in the light of successful predictions of the various properties of light-heavy flavoured systems. The parameters and the results obtained here can be useful in the study of the leptonic and semi-leptonic decays of these mesons.

The decay rates of $c\bar{c}$ and $b\bar{b}$ mesons have been studied with contributions from different correction terms. The correction based on hard processes involved in the decays are quantitatively studied in the frame work of different phenomenological potential models. The potential models which are successful in predicting the ground state properties of $c\bar{c}$ and $b\bar{b}$ systems are listed in Table 3.4 along with their predictions of the mesonic masses and square of the radial wave functions at the origin.

It is to be noted from Table 3.8 and 3.9 that Cornell predictions overestimate while predictions from $CPP_v - VS(GTW)$, $\nu \sim 0.9$ to 2.0, under estimate the
decay widths of $0^{--} \rightarrow \gamma \gamma$ and $1^{--} \rightarrow l^+l^-$ of $c\bar{c}$ and $b\bar{b}$ systems. The failure of $CPP_\nu - VS(GTW)$ in this case may be due to the relatively low value of the radial wave function (GTW) at the origin. This discrepancy of $CPP_\nu - \nu E(GTW)$ scheme has motivated to study the heavy-heavy flavour system with coulomb plus power potential using another trial wave function. A natural choice is to consider the hydrogenic trial wave function (HTW). This choice of HTW has enabled us to compute consistently few lower excited state of $S$-wave and $P$-wave masses using virial theorem. The values of the fermi momentum ($\bar{p}$) radial wave function at the origin ($R(0)$) and the spin average (center of weight) masses of $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems computed in this scheme ($CPP_\nu - VT(HTW)$) are presented in Table 3.10 to Table 3.15. Considerable enhancement in the value of $R(0)$ have been obtained in this scheme compared to those of $CPP_\nu - VS(GTW)$. The spin-hyperfine and spin-orbit part of one gluon exchange potential (OGEP) are employed here to obtain the different $S$-wave and $P$-wave masses. The results with different choices of the potential power index $\nu$ from 0.5 to 2.0 are presented in Table 3.29 and 3.17 for $c\bar{c}$ systems in Tables 3.18 and 3.19 for $c\bar{b}$ systems and in Tables 3.20 and 3.21 for $b\bar{b}$ systems systems. In these calculations the potential strength $A$ (in GeV$^{\nu+1}$) has been taken constant numerically to minimize free parameters required for the study. Once again better agreement with known experimental values and other theoretical predictions are found around the potential index 0.9 to 1.0 for $c\bar{c}$ and $c\bar{b}$ systems. While similar agreement for $b\bar{b}$ system is found around the $\nu$ values 0.9 to 1.3. The shift on the potential index that yield the $S$-wave and $P$-wave masses may be due to the inadequacy of keeping the potential parameter $A$, numerically same for different choices of the potential index $\nu$, as well as due to the variational approach employed for the study. To sort out this discrepancy, we further study these systems by fixing the values of $A$ for each choices of $\nu$ so as to get the experimental ground state spin-average (center of weight) masses of the $Q\bar{Q}$ systems as discussed above. Once the value of $A$ is known for each choice of $\nu$ from the center of mass of the ground state of the chosen $Q\bar{Q}$ ($Q \in c, b$) meson, their excited states are computed with the respective $A$-values.

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The results obtained here using (CPPva - VT(HTW)) are represented in Table 3.22 & 3.23 for cc systems in Tables 3.24 & 3.25 for cb systems and in Tables 3.26 and 3.27 for bb systems. The values of $R(0)$ computed here shows marginal increase for $\nu = 0.5$ and 0.7 and marginal decrease for $\nu \geq 0.9$ for cc and cb systems. While, in case of bb systems marginal decrease in the value of $R(0)$ for $\nu = 0.5$ to 2.0 are observed. The behavior of $R(0)$ for the three specific cases studied here, (1)CPP$_\nu$ - VS(GTW) (2) CPP$_\nu$ - VT(HTW) and (3) CPP$_\nu$A - VT(HTW) are shown in Fig 6.1.

From this graph, it is quit obvious that CPP$_\nu$A - VT(HTW) must be the right choice for the study of decay properties of the heavy flavour mesons. The GTW
Approach provide values of $R(0)$ much lower than the other two cases. The $R(0)$ values of $CPP_\nu - VT(HTW)$ show sharp increase with the change of $\nu$ and that can lead to overestimated predictions of the decay rates.

It is interesting to see the behaviour of $A$ obtained for $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems against the power index $\nu$ plotted in Fig 3.1, where the three curves corresponding to $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ intersect at common numerical value of $A = 0.15$ at the potential index, $\nu = 1.1$. For the $c\bar{c}$ and $c\bar{b}$ systems in the range of $0.9 \leq \nu \leq 1.5$ can have the $A$-values very close to each other. In the case of $b\bar{b}$ system, distinctly separate behaviour of $A$ with $\nu$ are found. From QCD point of view, the quantities that decides the value of the potential parameters may be the energy scale and the strong running coupling constant $\alpha_s$. The $\alpha_s$ for $c\bar{c}$ and $b\bar{b}$ systems employed in the present study vary from 0.300 to 0.233. So, the numerically constant value of $A$ for $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ suggests that the potential parameter $A$ is independent of $\alpha_s$ value and the energy scale at the potential index $\nu = 1.1$. It is one of an important findings of the present study. Its implication with respect to QCD at the hadronic dimensions need to be looked into.

Using the spin hyperfine and spin orbit part of the OGEP, we have been able to predict the low-lying $S$-wave and $P$-wave masses of $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems successfully. The computed masses are listed in Tables 3.28 to 3.30 and are compared with existing experimental values as well as other theoretical predictions.

The pseudoscalar decay constant of these mesons are also computed and are listed in Table 5.3 along with their ground state masses.

Though the ground state masses of these mesons $\eta_c$, $B_c$ and $\eta_b$ are not found to vary much with respect to the change in $\nu$, under $CPP_\nu A - VT(HTW)$ scheme, larger variations in the predictions of its pseudoscalar decay constant $f_p$ with the potential index $\nu$ are found. These variations of $f_p$ with $\nu$ and potential parameter $A$ are seen explicitly in Fig 6.2. The finite mass of the heavy quark depencance on
the decay constants of $q\bar{Q}$ mesons can be evaluated within the potential scheme studied here.

Employing the masses of the pseudoscalar and vector mesons and using the predicted radial wave function at the origin for $c\bar{c}$ and $b\bar{b}$ systems from different potential models including $CPP_A - VT(HTW)$ listed in Table 3.32 and 3.33, the di-gamma and leptonic widths are computed. The different correction terms discussed earlier and the predictions using NRQCD formalism are also computed. The results are presented in Table 3.34 for $\eta_c \to \gamma\gamma$ in Table 3.35 for $\eta_b \to \gamma\gamma$. It can be concluded that from the Tables that the $CPP_A - VT(HTW)$ results are now in accordance with other potential model predictions as well as with the
experimental values.

The theoretical predictions of the decay widths for $J/\psi \rightarrow l^+l^-$ and $\Upsilon \rightarrow \gamma l^+l^-$ are presented in Tables 3.36 and 3.37 respectively. The results of $CPP_{\nu A} - VT(HTW)$ are found to be in accordance with other potential model predictions with the correction terms as well as with the widths computed using NRQCD formalism.

It is quite inspiring to note that the ERHM predictions of the di-gamma decay widths of $\eta_c$ and leptonic decay widths of $J/\psi$ and $\Upsilon$ are in good agreement with the respective experimental results with out any corrections.

The $\Gamma_{NRQCD}$ widths for $\eta_c \rightarrow \gamma\gamma$ and $J/\psi \rightarrow l^+l^-$ computed using the parameters of the phenomenological models, BT, Log and $CPP_{\nu=0.7A} - VT(HTW)$ are in excellent agreement with the respective experimental values (see Tables 3.34 and 3.36). Their predictions using the conventional formula $\Gamma_0$ and $\Gamma_{VW}$ without and with corrections ($\Gamma_7$) are far from the experimental values. The predicted decay widths of $\eta_b \rightarrow \gamma\gamma$ by most of the models except the cornell and $CPP_{\nu > 1.5A} - VT(HTW)$ are in accordance with other theoretical values computed here, as well as with the theoretical predictions of Bodwin et al (0.364 keV) [Bodwin et al, 1995] and Fabiano (0.49 keV) [Fabiano, 2003].

These predictions thus guide the experimental determination of the the properties of $\eta_b$. In the case of $\Upsilon \rightarrow l^+l^-$ decay widths, given in Table 3.37, illustrate the success of NRQCD formalism as most of the phenomenological model predictions with the exception of Cornell, $CPP_{\nu A} - VT(HTW)$ for $\nu < 0.7$ and $\nu \geq 1.5$ and BT, fall in good accord with the experimental results of $1.32 \pm 0.5keV$. The results of $CPP_{\nu A} - VT(HTW)$ for the $\nu$ values 0.5 to 0.9 without the corrections ($\Gamma_{VW}$) and for $\nu$ values 1.1 to 2.0 with corrections ($\Gamma_7$) are close to the experimental values of 1.32 keV. Looking in to the success of NRQCD formalism for the decay rates of $c\bar{c}$ and $b\bar{b}$ mesons studied here, we have predicted their light-hadron c decay width also. These results given in Tables 3.38 for $c\bar{c}$ systems and in Table 3.39
for $b\bar{b}$ systems will be useful in the experimental analysis of the hadronic decay of these mesons.

The present study of the decay rates quarkonia with coulomb plus power potential for different power index clearly indicates that the importance of QCD related corrections very much depends on the phenomenological potential employed. The success of $CPP_{VT} + VT(HTW)$ in the determination of the $S$ and $P$ wave masses and decay rates of $c\bar{c}$ and $b\bar{b}$ systems provide future scopes to study various transition rate and excited states of these mesonic systems. With the masses and wave functions of the heavy flavour mesons at hand, it would be rather simple to compute various transition rates such as $E1$ and $M1$ in these mesons. Such computations largely form the applications of the present study. The decay rates and branching ratios of heavy flavour mesonic bound states are important ingredients in our present understanding of QCD.

Another aspect of the present study is that the multi-quark states as di-hadronic molecules. These exotic tetra, penta and hexa-quarks states have been predicted in the framework of QCD and related phenomenological models. Though such states certainly should exist are numerous, their decay widths are broad and hence buried within the background of lighter mesons like pions. The situations are worse at the light flavour sector. So, it was expected that in heavy-flavour sector such as $c\bar{s}uud$ penta quark or $ssuudd$ ($\lambda-N$) hexa quark states are more stable which decay only weakly. Several other authors [Manohar A, 1993; Gell-Mann and Nussinov S, 2003] suggested that tetra quarks, particularly the $cqq\bar{q}$ or $c\bar{c}qq$ are even more stable and more likely to be discovered. The recently observed narrow resonance near 3.87 GeV in the BELLE Collaboration in electron-positron collision through the $B$-meson decay $B^\pm \rightarrow K^\pm X$ [Abe K et al, 2003] followed by the decay $X \rightarrow J/\psi \pi^+ \pi^-$ [Choi S K et al, 2003] suggests that this state has the quantum number of a $D - D^*$ in $S$-wave molecular resonance. This may be the first among a host of other $Q\bar{q}qQ$ exotic dimesonic states in the light-heavy flavour sector. The discovery was confirmed by the CDF Collaboration using proton-antiproton
Another exotic hadronic state recently reported is a penta-quark $\bar{s}uudd$ resonance, $\Theta^+(1540$ MeV) by LEPS group at SPRING-8 [Nakano et al, 2003] with a width smaller than 20 MeV produced by $\gamma + n \rightarrow K^- + \Theta^+$ reaction. Several other groups have also confirmed the results [CDF Collaboration]. These experimental discoveries of exotic hadrons open up considerable interest in the study of hadron molecules.

We have computed the binding energies and masses of exotic hadron resonances by treating them as di-hadronic molecules based on the confinement scheme for mesons and baryons. The present work is motivated by the recent experimental evidence for a narrow baryon resonance with strangeness $s = +1$, $J^P = (1/2)^-$, at the energy range $1.54 \pm 0.01$ GeV/$c^2$ that decays into $K^+$ and neutron [Nakano T et al, 2003]. This decay identifies the baryon resonance as a meson-baryon molecule [Nakano T et al, 2003]. The penta quark state as a single multiquark hadron can decay to many other possible channels which have not seen experimentally. This strongly supports the hadron molecular idea. Also such a multiquark state is expected to be at higher energy than observed 1.5 GeV range. However, the exact differences between them require more intense study. The present attempt on hadron molecules presented here is just a first step in this direction.

The di-hadronic interaction has been taken as that due to the asymptotic expression of the confined one gluon exchange interaction among the quarks [Vijayakumar K B et al, 1993]. While the hadronic masses have been computed using the ERHM model. The Van-der-Waal type of di-hadronic binding energies have been computed using the ERHM basis with a residual strong interaction strength $k_{mol}$. This parameter was fixed through the experimentally known penta quark state as a $K - N$ hadron molecule [Nakano T et al, 2003]. As we have considered the residual confined gluon exchange potential for the molecular interaction, it must be same for different compositions of other di-hadrons. Thus, the study provides
us a parameter free predictions of various low-laying di-hadronic hadron molecular states. The computed results are shown in Tables 4.2, 4.3 and 4.4 respectively. Our predictions shown in Tables 4.2 and 4.3 are compared and identified with some of the experimentally known exotic hadronic states. These exotic states are those whose spin parity do not match with the expected quark antiquark structure for mesons and 3-quark structure of baryons. Accordingly, the pseudoscalar di-mesonic and vector-vector di-mesonic combinations will have the parity and charge conjugation $PC$ as $++$ while for the combinations of pseudoscalar-vector di-meson state will have $PC$ value $+-$, as shown in Table 4.2. The experimental low lying candidates with the predicted $J^{PC}$ values are chosen for comparison.

We found many $0^{++}$ di-mesonic states in the energy range $0.363$ GeV to $1.913$ GeV in the light flavour ($u, d, s$) sector. Many of these states are identified with experimentally known exotic mesonic states. Accordingly we identify $h_1(1.170$ GeV) $1^{+-}$ and $f_0(0.980$ GeV) $0^{+-}$ states as the $\pi - K^*$ and $K - K$ di-mesonic states. Other di-mesonic states with charmed D-mesons are also being studied. The $D_1(2.42$ GeV), $D_{s1}(2.540$ GeV) and $X(3.870$ GeV) are identified here as $\pi - D^*$, $K - D^*$ and $D - D^*$ di-mesonic states respectively. Many of these predicted states could be experimentally seen.

The low-lying penta quark states as the meson-baryon di-hadronic molecular naturally provide us negative parity baryons. These states in the $u, d, s$ sector have been studied and are tabulated in Table 4.3. The non-strange meson-baryon molecular resonances in the mass range $1.180-2.247$ GeV are predicted. The well-known $\Lambda(1.405$ GeV) [Lutz M et al, 2002; Nacher J et al, 1999] is very close to predicted $\pi - \Sigma$ di-hadronic molecule. Its decay (100 %) in to $\Sigma \pi$ mode [Particle Data group, (2002)] also suggest it to be the meson-baryon molecule. Other states that we could identify are the $\Sigma(1.750$ GeV) $(1/2)^-$ baryon or $K - \Sigma(1.737$ GeV) state. The resonance $(1.850$ GeV) observed by N G Kelker et al [Kelker N G et al, 2003] in a time delay analysis of $K^+N$ reaction is close to our $K - \Delta$ di-hadronic molecular state. As $\Delta$ is just spin excitation of $N$, the time delay analysis of $K - \Delta$
can throw more information regarding some of these states.

The binding energies of the di-baryonic systems (NN to \(\Sigma\Sigma\)) in the \(u, d, s\), sector tabulated in Table 4.4 are in the range of 112-120 MeV compared to 92-99 MeV of the meson-baryon penta quark systems and 64-123 MeV of the di-mesonic tetraquark systems. The predicted masses of the di-baryonic systems are in the range of 1.990-2.907 GeV as against other theoretical predictions of 2.17 - 2.46 GeV [Garcilo et al, 1999].

As the interaction among the hadrons are understood at least at the low energy limit, by the Yukawa type, we have studied the d-hadronic binding energy with the Yukawa type interaction. The study provides us a parameter free predictions of various di-hadronic hadron molecular states. Our predictions are given in Table 4.10.

Here, we found many \(0^{++}\) di-mesonic states in the energy range 3.81 GeV to 10.59 GeV in the heavy/light flavour \((s,c,b)\) sector. Many of these di-hadronic molecular states with the predicted spin-parity are expected to be seen in future experiments.

The \(X(3872\pm0.6\pm0.5)\) is extremely close to \(D^0D^*\) threshold 3871.2 \(\pm\) 0.7 MeV. Its binding energy is likely to be predicted as 0.4 MeV [Braeten E et al, 2004]. This is the smallest binding energy of any \(S\) wave di-hadron bound state. The next smallest is the deuteron (proton-neutron di-baryonic state) with binding energy 2.2 MeV. For two hadrons whose low energy interactions are mediated by pion exchange (Yukawa type), the natural scale for the binding energy of a di-hadronic molecule is \((m^2_\mu/2\mu)\), where \(\mu\) is the reduced mass of the two hadrons. For \(D - D^*\) molecule, this scale is about 10 MeV, so the binding \(E_b\) is at least an order of magnitude smaller than the natural low energy scale. If the binding energy of the di-hadron, say \(X\) is so small, the low energy universality implies that the \(X\) or \(D - D^*\) system has properties that are determined by the \(D - D^*\) scattering length \(a\).
and are insensitive to shorter distance scale of QCD. The universal binding energy of the molecule is related to the scattering length [Braaten E et al, 2004]

\[ E_b = m_{h_1} + m_{h_2} - m_x \approx (2\mu a^2)^{-1} \]

where \( \mu = \frac{m_{h_1}m_{h_2}}{m_{h_1} + m_{h_2}} \), the reduced mass of the di-hadronic molecule.

From the present computed value of \( E_b \), (Yukawa model) of the di-hadronic systems, we can get the scattering length \( a \) according to the relation given above. This parameter is important in identifying the exotic tetra-quark, penta-quark states as di-hadronic molecules. Molecular states can be confirmed by observing an enhancement in the \( h_1 h_2 \) (say \( D^0 D^{0*} \)) invariant mass distribution [Braaten E et al, 2004]. Detailed account of this estimations from the binding energy computed here form the future scope of the present study.

For the \( B \) and \( B_c \) mesons studied in this thesis, its decays are particularly interesting for the following reasons

1. Due to the QCD asymptotic freedom and the large masses and momenta released by heavy flavors, electroweak and strong interactions are closely correlated. Their interplay in perturbative calculations can be further improved by the renormalization group methods.

2. A new symmetry - called heavy flavor symmetry (HFS) - appears in an effective Lagrangian derived from QCD in the limit \( M \to \infty \) (\( M \) being the heavy quark mass). This symmetry allows the determination of the form factors involved in the exclusive decay modes. Some of these predictions play a crucial role in determining the \( CKM \) matrix elements. The \( 1/M \) expansion provides a solid theoretical framework for the spectator model in which only the heavy quark undergoes decay while the light constituents are spectators. Semileptonic decays are the best way to understand many properties of the \( b \)-flavored hadrons and to measure \( V_{cb} \) and \( V_{ub} \).
3. The physics of heavy flavors plays an essential role in CP violation too, and may open windows on the mechanism of the gauge symmetry breaking, i.e. at the Higgs sector.

The semileptonic decays offer an extremely favorable testing ground for both perturbative QCD, radiative corrections and nonperturbative QCD effects such as decay constants, form factors, and the best possible estimations of the CKM matrix elements.

With the mass parameters of the beauty and the charm quark fixed from the study of its spectra, we have successfully computed the semi-leptonic and non-leptonic decay widths of $B$-meson. Details of the QCD renormalizations of weak decays of these heavy mesons are discussed in chapter 5. The results of $B \to l^- X$ and $B \to X$ as given in Table 5.1 show the expected relationship $\Gamma_{NL} \approx 3 \Gamma_{SL}$.

We have also studied the decays of $B_c$ meson within the spectator model. The parameters such as the charm and beauty quark masses used in our study of heavy flavour mesons and the pseudoscalar decay constants $f_{B_c}$ obtained here for $\nu \simeq 1.1$ are found to be very appropriate in the calculation of the decay widths. The partial widths obtained here are compared with those obtained though the Bethe-Salpeter approach (ALV) [Vhady A L et al, 1996] as well as that from a relativistic quark model SG [Godfrey S et al, 2004] in Table 5.2. We obtained a higher branching ratio in the $b$-decay channel compared to other approaches as seen from in Table 5.2. We get about 63% as the branching fractions of b-quark decay, about 30% as that of c-quark decay and about 7% in the annihilation channel. However, the CKM mixing matrix elements $V_{cb}$ and $V_{cs}$ used as free parameters in all the three models are different but lie within the range given in particle data group [Hagiwara et al, 2002]. We have re-computed the results of (ALV) [Vhady A L et al, 1996] and SG [Godfrey S et al, 2004] by considering the values of $V_{cb}$ and $V_{cs}$ used by us. The results are shown within brackets below their values in Table 5.2. About 11% increase in the cases of ALV and 8% increase in the case of SG to
the total width are found. The lifetime of $B_c^+$ predicted by the present calculation is found to be in good agreement with the experimental values as well as that by the Bethe Salpeter method (ALV) (See Table 5.3). The predicted values from relativistic model (SG) is found to be far from the experimental values as well as other theoretical models.

The $B_c$ meson whose properties such as its mass, pseudoscalar decay constant and partial weak decay widths are studied here is one of a specially interesting heavy flavour meson. It not only add to $B$ mesons for studying the decay of the flavour, $b$ and to $D$ mesons for studying the decays of flavour $c$, but also offers a unique place where one can study the two heavy flavour $b$ and $c$ simultaneously in there mesons. The leptonic radiative decays, exclusive and inclusive of semi-leptonic decays for various models such as $\Gamma(B_c \rightarrow B_s + \ell^+\nu\ell)$, $\Gamma(B_c \rightarrow \eta_c + \ell^+\nu\ell)$, $\Gamma(B_c \rightarrow \tau\nu\gamma)$ etc. are some of the future prospects of the present study.

In conclusion, a simple nonrelativistic variational method with potential $-\frac{\alpha_s}{r} + Ar^\nu$ employed in the present study is found to be quite successful in predicting various properties of heavy flavour mesons. The method can be useful to study various hadronic and radiative transitions of the charm-beauty system.

The present study has also open up many interesting aspects of heavy flavour dynamics from the points of view of non-perturbative QCD effects. The NRQCD formalism employed in the study of the di-gamma and leptonic decays can be extended for other decays of these mesons also. Particular hadronic channel must be included in the study of the light hadronic decay widths of the heavy flavour hadrons.

The behaviour of potential parameters for the right predictions of the spectroscopy must be understood from the non-perturbative QCD models such as the HQET and the instantons etc. The relationship between the confined gluon exchange potential studied in the frame work of ERHM and the instanton induced interac-
tions at the hadronic dimensions are to be established. Though the present study related to heavy flavour mesons, the heavy flavour baryonic counterpart is largely understudied. Thus, there exists vast scope in the study of these baryons within the potential scheme presented in this thesis.

The study of di-hadronic molecules with more reasonable hadron-hadron interactions are to be studied by incorporating the spin-hyperfine interactions as well as other lower mesonic degree of freedom. This definitely will help to understand the basic strong interactions at the hadronic level. From the binding energy of the di-hadronic systems studied here, the respective scattering lengths and the di-hadronic elastic scattering amplitudes can be determined before establishing the molecular structure of many of the exotic hadrons.

The branching ratios of the various decay modes of heavy-heavy flavour mesons are of fundamental interest in heavy flavour physics. Theoretically predicted branching ratios in many cases help the experimental analysis of the decay of heavy flavour mesons. In short, heavy flavour hadrons can offer many interesting physics in the future experiments and theoretical efforts.