CHAPTER 4

VECTOR FUNCTION SPACES

Let $X$ be a compact Hausdorff space and $B$ be a Banach algebra. Let $C(X;B)$ denote the algebra of all $B$-valued, continuous functions on $X$. In this chapter, we define and discuss different types of Bishop and Šilov decompositions for a closed subspace of $C(X;B)$. If $A$ is a closed subspace of $C(X)$, then the algebraic tensor product $A \otimes B$ of $A$ and $B$ can be looked upon as a subspace of $C(X;B)$. We concentrate our study mainly, on the subspaces of the form $A \otimes B$.

1. General Case

Let $X$ be a compact Hausdorff space and $B$ be a commutative Banach algebra with identity $e$. Let $m(B)$ denote the maximal ideal space of $B$. Then $C(X;B)$, the set of all $B$-valued, continuous functions on $X$, is a commutative Banach algebra under pointwise operations and the norm given by

$$||f|| = \sup_{x \in X} ||f(x)||_B, \quad f \in C(X;B).$$

The function $1 \otimes e$ defined by $(1 \otimes e)(x) = e (x \in X)$ is the identity of $C(X;B)$. 

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For $f \in \mathcal{C}(X)$ and $b \in B$, the function $f \circ b$ defined by $(f \circ b)(x) = f(x)b$ is in $\mathcal{C}(X; B)$. Thus both $\mathcal{C}(X)$ and $B$ are embedded in $\mathcal{C}(X; B)$ and hence $\mathcal{C}(X; B)$ separates the points of $X$ and contains vector constants (i.e., functions of the form $1 \circ b$, $b \in B$). Let $\mathcal{C}(X) \circ B$ be the space of all finite linear combinations of functions of the form $f \circ b$, where $f \in \mathcal{C}(X)$ and $b \in B$. Then it is shown in [19] that $\mathcal{C}(X) \odot B = \mathcal{C}(X; B)$, where $\mathcal{C}(X) \odot B$ denotes the closure of $\mathcal{C}(X) \circ B$ in $\mathcal{C}(X; B)$. Taking $B = \mathbb{C}$, we get the standard algebra $\mathcal{C}(X)$.

Definitions 4.1.1. (i) If $V$ is a closed subspace of $\mathcal{C}(X; B)$ containing vector constants, then we say that $V$ is a vector function space on $X$.

(ii) If $V$ is a closed subalgebra of $\mathcal{C}(X; B)$ containing vector constants and separating the points of $X$, then $V$ is said to be a vector function algebra on $X$.

For example, let $A$ be a (complex) function space on $X$. For $f \in A$ and $b \in B$, define $f \circ b$ as above, then $f \circ b$ is in $\mathcal{C}(X; B)$. The algebraic tensor product $A \otimes B$ of $A$ and $B$ is the space of all finite linear combinations of functions of the form $f \circ b$, where $f \in A$ and $b \in B$. Then $A \hat{\otimes} B$, the uniform closure of $A \circ B$ in $\mathcal{C}(X; B)$, is a vector function space on $X$. If $A$ is a function algebra on $X$, then $A \hat{\otimes} B$ is a vector function algebra on $X$. 
In the earlier chapters, we have discussed the Bishop and Silov decompositions for subalgebras and subspaces of scalar-valued, continuous functions on $X$. We start with defining these types of decompositions for a vector function space on $X$.

Let $V$ denote a vector function space on $X$. Then Feyel and Pradelle [15, Definition 7] have defined a set of antisymmetry for $V$ as follows.

**Definition 4.1.2** [15]. A subset $K$ of $X$ is said to be a set of antisymmetry in $(FP)$-sense or an $(FP)$-antisymmetric set for $V$ if whenever $f \in \mathcal{N}(V|_K)$ and $f$ is real-valued, then $f$ is constant; where $\mathcal{N}(V|_K) = \{ f \in C(K) : (f \circ e)g \in V|_K \text{ for all } g \in V|_K \}$.

The collection of all maximal $(FP)$-antisymmetric sets for $V$ forms a decomposition of $X$. We call this decomposition the Bishop decomposition in $(FP)$-sense for $V$ and denote it by $\mathcal{B}_{FP}(V)$.

We have seen in second chapter that if $A$ is a function space on $X$, then Edwards [9] has also defined a set of antisymmetry for $A$ with the help of the multiplier of $A$.

If $V$ is a vector function space on $X$, then we define $\mathcal{M}(V) = \{ f \in C(X;\mathbb{E}) : fg \in V \text{ for every } g \in V \}$ and $\mathcal{N}(V) = \{ f \in C(X) : (f \circ e)g \in V \text{ for every } g \in V \}$.
Then $M(V)$ and $N(V)$ are closed subalgebras of $C(X;B)$ and $C(X)$ respectively. Also, $M(V) \subseteq V$ and if $V$ is an algebra, then $M(V) = V$.

**Definitions 4.1.3.** (i) A subset $K$ of $X$ is called a set of antisymmetry in $(E)$-sense or an $(E)$-antisymmetric set for $V$ if $K$ is a set of antisymmetry for $N(V)$, i.e., if $f \in N(V)$ and $f|_K$ is real-valued, then $f|_K$ is constant.

The collection of all maximal $(E)$-antisymmetric sets for $V$ is a decomposition of $X$. We call this decomposition the Bishop decomposition in $(E)$-sense for $V$ and denote it by $\mathcal{K}_E(V)$.

(ii) A subset $K$ of $X$ is called a set of antisymmetry in $(B)$-sense or a $(B)$-antisymmetric set for $V$ if whenever $f \in M(V)$ and $(\phi \circ f)|_K$ is real-valued for each $\phi \in m(B)$, then $f|_K$ is constant, where $m(B)$ denotes the maximal ideal space of $B$.

The collection of all maximal $(B)$-antisymmetric sets for $V$ forms a decomposition of $X$. We call this decomposition the Bishop decomposition in $(B)$-sense for $V$ and denote it by $\mathcal{K}_B(V)$.

Next, we define the Šilov decompositions for $V$.

**Definitions 4.1.4.** (i) Let $V$ be a vector function space on $X$ and $(N(V))_R$ be the set of all real-valued functions in $N(V)$. A set of constancy of $(N(V))_R$ is called an $(E)$-Šilov set for $V$. 
The collection of all maximal \((E)\)-Silov sets for \(V\) is a decomposition of \(X\). This decomposition is called the Silov decomposition in \((E)\)-sense for \(V\) and we denote it by \(\mathcal{F}_E(V)\).

If we define an \((FF)\)-Silov set for \(V\), then it turns out to be the same as an \((E)\)-Silov set for \(V\), as in the scalar case.

(ii) For a vector function space \(V\) on \(X\), let \(\mathcal{M}(V)_\mathbb{R} = \{ f \in \mathcal{M}(V) : \phi \circ f \text{ is real-valued for all } \phi \in \mathcal{M}(\mathbb{E}) \}\). A set of constancy of \(\mathcal{M}(V)_\mathbb{R}\) is called an \((S)\)-Silov set for \(V\).

The decomposition consisting of all maximal \((S)\)-Silov sets for \(V\) is called the \(\text{Silov decomposition in } (S)\)-sense for \(V\) and it is denoted by \(\mathcal{F}(V)\).

As usual, we shall use \(\mathcal{K}_{\mathbb{R}}, \mathcal{K}, \mathcal{K}_E\) and \(\mathcal{F}, \mathcal{F}_E\) in place of \(\mathcal{K}_{\mathbb{R}}(V), \mathcal{K}(V), \mathcal{K}(V)\) and \(\mathcal{F}_E(V), \mathcal{F}(V)\) respectively, when there is no confusion about the vector space.

Remarks 4.1.5. (i) If \(B = \mathbb{C}\), then \(\mathcal{K}_{\mathbb{R}}, \mathcal{K}_E\) and \(\mathcal{F}_E\) coincide with the Bishop decomposition in \((FP)\)-sense, Bishop decomposition in \((E)\)-sense and the Silov decomposition respectively, as defined in chapter 2 (Definitions 2.1.1, 2.1.2 and 2.1.3). Since in this case \(\mathcal{M}(V) = \mathcal{M}(V)\) and \(\mathcal{M}(\mathbb{E})\) contains only identity map, \(\mathcal{K}_E = \mathcal{K}\) and \(\mathcal{F}_E = \mathcal{F}\).

(ii) It is clear from the definitions that \(\mathcal{K} \subset \mathcal{F}\) and \(\mathcal{K}_E \subset \mathcal{F}_E\).
Proposition 4.1.6. Let $V$ be a vector function space on $X$.

(i) $\mathcal{X}_{\mathbb{F}} < \mathcal{X}_E$, $\mathcal{Y} < \mathcal{X}_E$ and $\mathcal{F} < \mathcal{F}_E$.

(ii) If $V$ is an algebra, then $\mathcal{X} < \mathcal{X}_{\mathbb{F}}$ and hence $\mathcal{X} < \mathcal{X}_{\mathbb{F}} < \mathcal{X}_E$.

Proof. (i) First we show that $\mathcal{X}_{\mathbb{F}} < \mathcal{X}_E$. Let $K \in \mathcal{X}_{\mathbb{F}}$. It is enough to show that $K$ is an $(E)$-antisymmetric set for $V$. Let $f \in \mathcal{N}(V)$ and $f|_K$ be real-valued. Then $f|_K \in \mathcal{N}(V)|_K \subset \mathcal{N}(V|_K)$. Since $K$ is an $(FP)$-antisymmetric set for $V$, $f|_K$ is constant. Hence $K$ is an $(E)$-antisymmetric set for $V$.

To show that $\mathcal{X} < \mathcal{X}_E$, suppose that $H \in \mathcal{X}$. Let $g \in \mathcal{N}(V)$ and $g|_H$ be real-valued. Then it is clear that $g \circ e \in \mathcal{N}(V)$. For $\phi \in m(\mathbb{E})$, $(\phi \circ (g \circ e))|_H = (\phi(e)g)|_H = g|_H$ and so, $\phi \circ (g \circ e)$ is real-valued on $H$ for each $\phi \in m(\mathbb{E})$. Hence $(g \circ e)|_H$ is constant, i.e., $g|_H$ is constant which proves that $H$ is an $(E)$-antisymmetric set for $V$. Consequently, $\mathcal{X} < \mathcal{X}_E$.

Similarly, we can show that $\mathcal{Y} < \mathcal{Y}_E$.

(ii) Suppose $V$ is an algebra. Let $K \in \mathcal{X}$. Also, let $f \in \mathcal{N}(V|_K)$ and $f$ be real-valued. Then $f \circ e \in \mathcal{N}(V|_K)$. Since $V$ is an algebra, $\mathcal{N}(V|_K) = V|_K$. So, there exists $g \in V$ such that $g|_K = f \circ e$. For $\phi \in m(\mathbb{E})$, $(\phi \circ g)|_K = \phi \circ (g|_K) = \phi \circ (f \circ e) = f$. Thus $g \in V$ and $(\phi \circ g)|_K$ is real-valued for each $\phi \in m(\mathbb{E})$. Therefore, $g|_K$ is constant, i.e., $f$ is constant and hence $K$ is an $(FP)$-antisymmetric set for $V$. Thus $\mathcal{X} < \mathcal{X}_{\mathbb{F}}$. 

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In general, \( \kappa, \kappa_{FP} \) and \( \kappa_E \) are not equal. For, in case of \( B = C \), we have seen an example (Example 2.3.6(b)) in which \( \kappa_{FP} \neq \kappa_E \). Hence \( \kappa_{FP} \neq \kappa_E = \kappa \). The following example shows that \( \kappa \) and \( \kappa_E \) need not be equal.

**Example 4.1.7.** Let \( D = \{ z \in C : |z| \leq 1 \} \) and \( B = C[0,1] \). Let \( V = \{ f \in C(D;B) : f_o \in A(D) \} \), where \( f_o(z) = (f(z))o \), \( z \in D \). Then \( V \) is a vector function algebra on \( D \) \([37, \text{Example 1.33}]. \) First we show that \( \kappa(V) = A(D) \), the disk algebra on \( D \). Let \( f \in \kappa(V) \). Then \( f \circ e \in V \), since the identity \( 1 \circ e \in V \). Therefore, \( (f \circ e)_o = f \in A(D) \). Hence \( \kappa(V) \subseteq A(D) \). Conversely, let \( g \in A(D) \) and \( h \in V \). Then \( ((g \circ e)h)_o = (g \circ e)_o h_o = gh_o \) is in \( A(D) \). Thus \( (g \circ e)h \in V \) and \( g \in \kappa(V) \). Hence \( A(D) \subseteq \kappa(V) \).

We know that \( \kappa(A(D)) = \kappa(A(D)) = \{ D \} \). Since \( \kappa_E(V) = \kappa(N(V)) \) and \( \kappa_E(V) = \kappa(N(V)) \), we get \( \kappa_E(V) = \kappa_E(V) = \{ D \} \). So, it is clear from the definition of \( \kappa_{FP} \) that \( \kappa_{FP}(V) = \{ D \} \).

Note that for \( B = C[0,1] \), \( m(B) = \{ \phi_t : 0 \leq t \leq 1 \} \), where \( \phi_t(b) = b(t) \) for \( b \in C[0,1] \).

Claim. \( \kappa(V)_R = \{ h \in V : \phi_t \circ h \) is real-valued for \( 0 \leq t \leq 1 \} \) separates the points of \( D \).

Let \( z_1, z_2 \in D \) such that \( z_1 \neq z_2 \). Then \( \text{Re}z_1 \neq \text{Re}z_2 \) or \( \text{Im}z_1 \neq \text{Im}z_2 \). Define \( f \) and \( g \) on \( D \) by \( (f(z))_t = t\text{Re}z \) and \( (g(z))_t = t\text{Im}z \) for \( 0 \leq t \leq 1, z \in D \). Then \( f, g \in C(D;B) \) and \( f_o = g_o = 0 \). Thus \( f, g \in V \). Also, \( \phi_t \circ f \) and \( \phi_t \circ g \) are...
real-valued functions on $D$ for $0 \leq t \leq 1$. So, $f, g \in (MC(V))^R$.

Since $\phi_t \circ f$ or $\phi_t \circ g$ separates $z_1$ and $z_2$ for $0 < t \leq 1$, $f$ or $g$ separates $z_1$ and $z_2$.

Hence $\mathcal{F}(V) = \{ \{z\} : z \in D \} = \mathcal{K}(V)$. Thus we get $\mathcal{K}(V) \not\subseteq \mathcal{K}_E(V) = \mathcal{K}_{FP}(V)$ and $\mathcal{F}(V) \not\subseteq \mathcal{F}_E(V)$.

The significance of the Bishop decomposition is that it has the $(D)$-property for a complex function algebra. We start with the definition of the $(D)$-property for vector function spaces.

Definition 4.1.8. Let $V$ be a vector function space on $X$ and $\mathcal{S}$ be a decomposition of $X$. We say that $\mathcal{S}$ has the $(D)$-property for $V$ if $f \in \mathcal{C}(X;B)$ and $f|_E \in (V|_E)^\mathcal{S}$ for each $E \in \mathcal{S}$, then $f \in V$, where $(V|_E)^\mathcal{S}$ denotes the uniform closure of $V|_E$ in $\mathcal{C}(E;B)$.

Remarks 4.1.9. (i) It is easy to see that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two decompositions of $X$ with $\mathcal{S}_1 \subset \mathcal{S}_2$ and if $\mathcal{S}_1$ has the $(D)$-property for $V$, then $\mathcal{S}_2$ also has the $(D)$-property for $V$.

(ii) The decomposition $\mathcal{S} = \{ \{x\} : x \in X \}$ has the $(D)$-property for $V$ if and only if $V = \mathcal{C}(X;B)$.

In [33, Theorem 1], Machado has proved the following.
Theorem 4.1.10. Let $V$ be a vector subspace of $\mathcal{C}(X;\mathcal{E})$ which is a module over a unital subalgebra $A$ of $\mathcal{C}(X)$. Fix $f$ in $\mathcal{C}(X;\mathcal{E})$. Then there is a maximal set of antisymmetry $K$ for $A$ such that

$$\inf \left\{ ||f-g|| : g \in V \right\} = \inf \left\{ ||(f-g)|_K|| : g \in V \right\}.$$ 

Since $V$ is a module over the algebra $\mathcal{N}(V)$ of $\mathcal{C}(X)$ and $\mathcal{K}(\mathcal{N}(V)) = \mathcal{K}_F(V)$, we get the next corollary.

Corollary 4.1.11. Let $V$ be a vector function space on $X$ and $f \in \mathcal{C}(X;\mathcal{E})$. Then there is $K \in \mathcal{K}_F$ such that

$$\inf \left\{ ||f-g|| : g \in V \right\} = \inf \left\{ ||(f-g)|_K|| : g \in V \right\}.$$ 

Remarks 4.1.12. (i) It follows from the Corollary 4.1.11 that $\mathcal{K}_F$ has the (D)-property for $V$. In fact, Feyel and Pradelle [15, Theorem 8] have proved that even $\mathcal{K}_{FP}$ also has the (D)-property for $V$.

(ii) In general, $\mathcal{K}$ does not have the (D)-property for $V$, as Example 4.1.7 together with Remark 4.1.9(ii) shows.

As a corollary of the following proposition, we prove that $\mathcal{F}$ is the finest u.s.c. decomposition with the (D)-property.

Proposition 4.1.13. Let $V$ be a vector function space on $X$. If a decomposition $\mathcal{S}$ of $X$ has the (D)-property for $V$, then $\mathcal{S}$ has the (D)-property for $\mathcal{N}(V)$. 

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Proof. Let $\mathcal{S}$ be a decomposition of $X$ with the (D)-property for $V$. Let $f \in C(X)$ and $f|_E \in (N(V)|_E)^\uparrow$ for each $E \in \mathcal{S}$. Let $g \in V$. To show that $(f \circ e)g \in V$, it is enough to show that $((f \circ e)g)|_E \in (V|_E)^\uparrow$ for each $E \in \mathcal{S}$, as $\mathcal{S}$ has the (D)-property for $V$. Let $E \in \mathcal{S}$. Since $f|_E \in (N(V)|_E)^\uparrow$, there exists a sequence $\{f_n\}$ in $N(V)$ such that $\{f_n\}$ converges uniformly to $f$ on $E$. Then $(f_n \circ e)g \in V$ for each $n$ and $\{(f_n \circ e)g\}$ converges to $(f \circ e)g$ on $E$. Thus $((f \circ e)g)|_E \in (V|_E)^\uparrow$. Hence $(f \circ e)g \in V$ and $f \in N(V)$.

Corollary 4.1.14. Let $V$ be a vector function space on $X$ and $\mathcal{S}$ be an u.s.c. decomposition of $X$ with the (D)-property for $V$. Then $\mathcal{S}_E \prec \mathcal{S}$ and hence $\mathcal{S} \prec \mathcal{S}$.

Proof. Since $\mathcal{S}$ has the (D)-property for $V$, by Proposition 4.1.13, $\mathcal{S}$ has the (D)-property for $N(V)$ which is a closed subalgebra of $C(X)$. Also, $\mathcal{S}$ is an u.s.c. decomposition of $X$. So, by Theorem 2.1.7, $\mathcal{S}(N(V)) \prec \mathcal{S}$, i.e., $\mathcal{S}_E \prec \mathcal{S}$.

To define other properties of the decomposition of $X$, we need some preliminaries. For details, we refer to ([8], [43]).

For a compact Hausdorff space $X$, let $B(X)$ denote the class of all Borel subsets of $X$ and $M(X)$ be the space of all regular, complex-valued Borel measures on $X$. Let $m$ be a
mapping of $\mathcal{B}(X)$ into $B^*$, the dual of $B$. For each $b \in B$, define the scalar-valued set function $\langle b, m \rangle$ by $\langle b, m \rangle(E) = \langle b, m(E) \rangle = m(E)(b)$ ($E \in \mathcal{B}(X)$). If $\langle b, m \rangle \in \mathcal{M}(X)$ for each $b \in B$, then $m$ is said to be a weak*-regular, $B^*$-valued measure on $X$. For such an $m$, define the total variation measure $|m|$ of $m$ by

$$|m|(E) = \sup \sum_{i=1}^{n} |m(E_i)|,$$

where the supremum is taken over all finite Borel partitions $\{E_1, E_2, \ldots, E_n\}$ of $E$. It is easy to see that $|m|$ is a positive Borel, regular measure on $X$. If $|m|(X)$ is finite, then $m$ is said to be of bounded variation. Equipped with the natural linear operations and with the norm of $m$ given by $||m|| = |m|(X)$, the set of all weak*-regular, $B^*$-valued measures of bounded variation forms a Banach space which we shall denote by $\mathcal{M}(X; B^*)$. It is shown in [53] that $\mathcal{M}(X; B^*)$ is the dual of $\mathcal{C}(X; E)$.

Let $V$ be a vector function space on $X$. Then

$$V^\perp = \{ m \in \mathcal{M}(X; B^*) : \int_X fdm = 0 \text{ for all } f \in V \}.$$ 

**Definition 4.1.15.** Let $V$ be a vector function space on $X$ and $\mathcal{G}$ be a decomposition of $X$. We say that $\mathcal{G}$ has the (GA)-property for $V$ if for each $m \in b(V)^\perp$, $\text{supp } m \subseteq E$ for some $E \in \mathcal{G}$, where $b(V)^\perp$ is the set of all extreme points of the unit ball of $V^\perp$. 164
Now, we define peak sets and p-sets for $V$.

Definitions 4.1.16. Let $V$ be a vector function space on $X$ and $F$ be a closed subset of $X$.

(i) $F$ is said to be a peak set for $V$ if there exists $f$ in $V$ such that $\|f|_F\| = e$ and $||f(x)|| < 1$ for all $x \in X - F$. The intersection of peak sets is called a generalized peak set for $V$.

(ii) $F$ is called a p-set for $V$ if $\mu \in V^\perp$ implies that $\mu|_F \in V^\perp$, where $\mu(F \cap G) = \mu(F \cap G)$ for every Borel subset $G$ of $X$.

We observe certain basic properties of peak sets and p-sets.

Proposition 4.1.17. Let $V$ be a vector function space on $X$ and $F$ be a closed subset of $X$. Then $V|_F$ is closed in $C(F; E)$ if and only if there is a number $\lambda \geq 1$ such that $||\mu + V^\perp|_F|| \leq \lambda ||\mu + V^\perp||$ for all $\mu \in M(F; B^\times)$, where

\[ V^\perp_F = \{ m \in M(F; B^\times) : \int_X fdm = 0 \text{ for all } f \in V|_F \} \]

Proof. The proof of the scalar-valued case [29, Theorem 38, p.188] works just as well in vector-valued situation.

Corollary 4.1.18. Let $V$ be a vector function space on $X$ and $F$ be a p-set for $V$. Then $V|_F$ is closed in $C(F; E)$.
This follows from the definition of p-set and Proposition 4.1.17.

It is well known that if $A$ is a complex function algebra on $X$, then a generalized peak set is always a p-set [29, Theorem 40, p.190]. We prove a somewhat stronger result.

**Proposition 4.1.19.** Let $V$ be a vector function space on $X$. If $F$ is a generalized peak set for $M(X)$, then $F$ is a p-set for $V$.

**Proof.** Let $F$ be a generalized peak set for $M(X)$. Let $\mu \in V^\perp$ and $\varepsilon > 0$. Then there exists an open set $U$ in $X$ such that $|\mu|(U-F) < \varepsilon$. Since $F$ is a generalized peak set for $M(X)$, there is a peak set $K$ for $M(X)$ with $F \subset K \subset U$. Hence $|\mu|(K-F) < \varepsilon$. Let $f \in M(X)$ be a peaking function for $K$. Let $\chi^* = \chi_k \ast e$, where $\chi_k$ denotes the characteristic function of $K$. Then $f^n$ converges pointwise and boundedly to $\chi^*$. Let $g \in V$. Since $f^n \in M(X)$, $f^n g \in V$. Thus

$$\int_X \mu g = \int_X \chi_k \ast e \, d\mu = \lim_{n \to \infty} \int_X f^n \, d\mu = 0,$$

as $\mu \in V^\perp$. Also, let $\varepsilon > 0$. Then

$$\left| \int_X \mu g \right| = \left| \int_X \chi_k \ast e \, d\mu \right| = \left| \int_X f^n \, d\mu \right| < \varepsilon \left| g \right|.$$

Therefore,

$$\int_F \mu g < \varepsilon \left| g \right|.$$ 

Since $\varepsilon$ is arbitrary, $\int_F \mu g = 0$. Hence $\mu \in V^\perp$. Therefore, $\int_F \mu g = 0$. 

Hence $\mu \in V^\perp$. 

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For a vector function algebra $V$, we have $V = \mathcal{M}(V)$ and hence we get the following.

**Corollary 4.1.20.** If $V$ is a vector function algebra on $X$, then a generalized peak set for $V$ is a p-set for $V$.

**Remark 4.1.21.** In general, we do not know whether a p-set for a vector function algebra $V$ is also a generalized peak set for $V$. But, for the special case in which $V = A \hat{\otimes} B$, where $A$ is a (complex) function space on $X$, we shall prove this result in the next section.

**Definition 4.1.22.** Let $V$ be a vector function space on $X$ and $\mathcal{S}$ be a decomposition of $X$. We say that $\mathcal{S}$ has the (S)-property for $V$ if $F \subseteq X$ is a p-set for $V$ and $F$ is saturated with $\mathcal{S}$, then $\mathcal{S} \cap F$ has the (D)-property for $V|_F$.

**Remark 4.1.23.** By the same argument as given for the scalar-valued case in [20, Theorem 1.2], one can prove that for a decomposition $\mathcal{S}$ of $X$, (GA)-property $\Rightarrow$ (S)-property $\Rightarrow$ (D)-property for a vector function space $V$ on $X$.

We have noted earlier that $X_{FP}$ has the (D)-property for $V$. We prove that $X_{FP}$ has, in fact, the (GA)-property for $V$. 

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Proposition 4.1.24. Let $V$ be a vector function space on $X$. Then $\mathcal{K}_{FP}$ has the (GA)-property for $V$.

Proof. Let $\mu \in \mathcal{B}(V^1)^e$ and $S = \text{supp } \mu$. It is enough to show that $S$ is an (FP)-antisymmetric set for $V$. Let $f \in N(V|_S)$ and $f$ be real-valued. We may assume that $0 \leq f \leq \frac{1}{2}$. Let $\eta = (f \circ e)\mu$. Then $\eta \in \mathcal{M}(X;B^*)$. Let $g \in V$. Then $((f \circ e)\delta)|_S \in V|_S$. So, there exists $h \in V$ such that $((f \circ e)\delta)|_S = h|_S$. Thus

$$\int_S gd\eta = \int_S g(f \circ e)\delta d\mu = \int_X h d\mu = \int_X h d\mu = 0,$$

as $\mu \in V^1$. Hence $\eta \in V^1$. It can be checked that

$$|| (f \circ e)\mu || + || (e-f \circ e)\mu || = 1.$$

Let $\eta_1 = \frac{(f \circ e)\mu}{|| (f \circ e)\mu ||}$ and $\eta_2 = \frac{(e-f \circ e)\mu}{|| (e-f \circ e)\mu ||}$. Then

$$\eta_1, \eta_2 \in \mathcal{B}(V^1) \text{ and } \mu = || (f \circ e)\mu || \eta_1 + || (e-f \circ e)\mu || \eta_2.$$ 

Since $\mu \in \mathcal{B}(V^1)^e$, $\mu = \eta_1 = \eta_2$. Hence $(f \circ e)|_S$ is constant or $f$ is constant. Thus $S$ is an (FP)-antisymmetric set for $V$.

Remark 4.1.25. Since $\mathcal{K}_{FP} < \mathcal{K}_E < \mathcal{F}_E$, $\mathcal{K}_E$ and $\mathcal{F}_E$ also have the (GA)-property for $A$. By Remark 4.1.23, $\mathcal{K}_{FP}$, $\mathcal{K}_E$ and $\mathcal{F}_E$ have the (S)-property and the (D)-property for a vector function space $V$ (Remark 4.1.12(i)).

Next, we wish to show that $\mathcal{K}_{FP}$ is the finest Hausdorff decomposable decomposition with the (S)-property. We start with a lemma.
Lemma 4.1.26. Let $V$ be a vector function space on $X$ and $F$ be a closed subset of $X$. If $F$ is a $p$-set for $N(V)$, then $F$ is a $p$-set for $V$.

Proof. Let $F$ be a $p$-set for $N(V)$. Since $N(V)$ is a closed subalgebra of $C(X)$, $F$ is a generalized peak set for $N(V)$. Hence, by the same argument as given in Proposition 4.1.19, we can show that $F$ is a $p$-set for $V$.

Recall (Definition 2.2.4) that a decomposition $\mathcal{S}$ of $X$ is Hausdorff decomposable if $Y = X/\mathcal{S}$ is Hausdorff decomposable. Also, if $\mathcal{K}_\alpha$ is a decomposition of $Y$, then $\tilde{\mathcal{K}}_\alpha = \{ \tilde{E}_\alpha = q^{-1}(E_\alpha) : E_\alpha \in \mathcal{K}_\alpha \}$ is a decomposition of $X$ associated with $\mathcal{K}_\alpha$, where $q : X \rightarrow Y$ denotes the quotient map.

With the help of Proposition 4.1.13 and Lemma 4.1.26, we can prove the following theorem exactly as we have proved it for a complex function space in chapter 2 (Theorem 2.2.5).

Theorem 4.1.27. Let $V$ be a vector function space on $X$ and $\mathcal{S}$ be a decomposition of $X$ such that $\mathcal{S} < \mathcal{K}_{fp}$ and $\mathcal{S}$ has the $(S)$-property for $V$. Then $\mathcal{K}_{fp} = \tilde{\mathcal{K}}_{\sigma(Y)}$, where $\tilde{\mathcal{K}}_{\sigma(Y)}$ is the decomposition of $X$ associated with the decompositon $\mathcal{K}_{\sigma(Y)}$ of $Y$ for $Y = X/\mathcal{S}$. 169
We know that a decomposition ℳ of X is Hausdorff decomposable if ℳ(Y) = ℳ for Y = X/ℳ. Since ℳ has the (S)-property for V, by Theorem 4.1.27, ℳ is Hausdorff decomposable and as in chapter 2 (Corollary 2.2.6), we get the following corollary.

**Corollary 4.1.28.** Let V be a vector function space on X and ℳ < ℳ be a decomposition of X with the (S)-property for V. Then ℳ = ℳ if and only if ℳ is Hausdorff decomposable.

**Remark 4.1.29.** Since ℳ(V) = ℳ(N(V)), ℳ(V) = ℳ(N(V)) and members of ℳ(N(V)) and ℳ(N(V)) are p-sets for N(V), by Lemma 4.1.26, members of ℳ(V) and ℳ(V) are p-sets for V also. Further, as in chapter 2 (Remark 2.2.8(ii)), members of ℳ(V) are p-sets for V.

Note that if ℳ(V) is finite, then ℳ(V) = ℳ(V), as ℳ(V) and ℳ(V) are the Šilov and Bishop decompositions for N(V). We shall show that if ℳ(V) is finite, then ℳ(V) = ℳ(V).

**Proposition 4.1.30.** Let V be a vector function space on X. If ℳ(V) has finitely many members, then ℳ(V) = ℳ(V). In particular, if ℳ(V) has finitely many members, then ℳ(V) = ℳ(V).

**Proof.** We use the argument similar to that given for function algebras (Theorem 1.1.15). Let ℳ(V) = {F₁, F₂, ..., Fₙ}. It is enough to show that each Fᵢ, 1 ≤ i ≤ n, is
a (B)-antisymmetric set for \( V \). For \( F_i \neq F_j \), there exists \( f_{ij} \in (\mathcal{M}(V))_R \) such that \( f_{ij} = e \) on \( F_i \) and \( f_{ij} = 0 \) on \( F_j \). Take \( f_i = f_{i2} f_{i3} \ldots f_{in} \). Then \( f_i \in (\mathcal{M}(V))_R \), \( f_i = e \) on \( F_i \) and \( f_i = 0 \) on \( F_j \), for all \( j \neq i \). Similarly, for each \( i \), \( 1 \leq i \leq n \), there exists \( f_i \in (\mathcal{M}(V))_R \) such that \( f_i = e \) on \( F_i \) and \( f_i = 0 \) on \( F_j \), for all \( j \neq i \).

Fix \( F_i \in \mathcal{F}(V) \). Let \( g \in \mathcal{M}(V) \) and \( (\phi \circ g)|_{F_i} \) be real-valued for all \( \phi \in \mathfrak{m}(B) \). Then \( f_i g \in \mathcal{M}(V) \), \( (\phi \circ (f_i g))|_{F_j} = (\phi \circ f_i)|_{F_j} (\phi \circ g)|_{F_j} = 0 \), for all \( j \neq i \) and \( (\phi \circ (f_i g))|_{F_i} = (\phi \circ g)|_{F_i} \) is real-valued, i.e., \( \phi \circ (f_i g) \) is real-valued for all \( \phi \in \mathfrak{m}(B) \). Thus \( f_i g \in (\mathcal{M}(V))_R \) and therefore, \( (f_i g)|_{F_i} \) is constant, i.e., \( g|_{F_i} \) is constant. Hence \( F_i \) is a (B)-antisymmetric set for \( V \).

Finally, we define the essential set of a vector function space.

**Definition 4.1.31.** Let \( V \) be a vector function space on \( X \) and \( I \) be the largest closed ideal of \( C(X;B) \) contained in \( V \). Then, for each \( \phi \in \mathfrak{m}(B) \), \( I_\phi = \{ \phi \circ f : f \in I \} \) is an ideal of \( C(X) \). Let \( E_\phi = \{ x \in X : g(x) = 0 \text{ for all } g \in I_\phi \} \), for each \( \phi \in \mathfrak{m}(B) \). Then \( E = \cap \{ E_\phi : \phi \in \mathfrak{m}(B) \} \) is called the essential set of \( V \).
The existence of the largest closed ideal of $C(X;E)$ which is contained in $V$ can be shown as it is shown for complex function algebras.

Proposition 4.1.32. Let $V$ be a vector function space on $X$ and $E(V)$, $E(M(V))$ and $E(N(V))$ denote respectively, the essential sets of $V$, $M(V)$ and $N(V)$. Then

(i) $E(V) = E(M(V))$ and

(ii) $E(V) \subseteq E(N(V))$.

Proof. (i) It is enough to show that the following are equivalent.

(a) $I$ is the largest closed ideal of $C(X;E)$ contained in $V$.

(b) $I$ is the largest closed ideal of $C(X;E)$ contained in $M(V)$.

Suppose (a) holds. To show that $I \subseteq M(V)$, let $f \in I$ and $g \in V$. Then $fg \in I \subseteq V$. Thus $f \in M(V)$ and $I \subseteq M(V)$.

If there is a closed ideal $J$ such that $I \subseteq J \subseteq M(V)$, then $J \subseteq V$, as $M(V) \subseteq V$. Therefore, $I = J$ and (b) holds.

Conversely, assume that (b) holds. Then, clearly $I \subseteq V$. If $I \subseteq J \subseteq V$ for some closed ideal $J$ of $C(X;E)$, then as above, $J \subseteq M(V)$. Hence $I = J$ and (a) holds.

(ii) Let $J$ be the largest closed ideal of $C(X)$ contained in $N(V)$. Then $J \otimes E$ is a closed ideal of $C(X;E)$ and $J \otimes E \subseteq V$. So, $J \otimes E \subseteq I$, where $I$ denotes the largest closed ideal of $C(X;E)$ contained in $V$. Let $x \in E(V)$ and $\phi \in m(E)$. Then $x \in E_{\phi}$, i.e., $(\phi \circ f)(x) = 0$ for all $f \in I$. Therefore, $g(x) = (\phi \circ (g \otimes e))(x) = 0$ for all $g \in J$. Hence $x \in E(N(V))$. 

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We do not know whether $E(V) = E(N(V))$ holds good. But for $V = A \hat{\otimes} B$, we shall prove this result in the next section.

2. Special Case

Let $A$ denote a complex function space on $X$, $B$ denote a commutative Banach algebra with identity $e$ and $m(B)$ denote the maximal ideal space of $B$. Let $A \hat{\otimes} B$ denote the uniform closure of the algebraic tensor product $A \otimes B$ of $A$ and $B$ in $C(X;B)$. Then $A \hat{\otimes} B$ is a vector function space on $X$. For a function algebra $A$ on $X$, the idea of a vector function algebra $A \# B$ is defined in [37, Definition 1.32]. Similarly, for a complex function space $A$ on $X$, we define

$$A \# B = \{ f \in C(X;B) : \phi \circ f \in A \text{ for all } \phi \in m(B) \}.$$ 

Then $A \# B$ is also a vector function space on $X$ and it can be checked that $A \hat{\otimes} B \subseteq A \# B$. If $A$ is a function algebra on $X$, then $A \hat{\otimes} B$ and $A \# B$ are vector function algebras on $X$. Further, if $B$ is a function algebra on $m(B)$, then $A \hat{\otimes} B$ and $A \# B$, defined in chapter 0, can be regarded as vector function algebras $A \hat{\otimes} B$ and $A \# B$ with the identification defined as $(f \otimes g)(x) = f(x)g$, for $f \in A$, $g \in B$.

In this section, we discuss the Bishop and Šilov decompositions for vector function spaces of the types $A \hat{\otimes} B$ and $A \# B$. Also, we study the essential set of $A \hat{\otimes} B$ in detail.
Unless otherwise mentioned, A denotes a complex function space on X and B denotes a commutative semisimple Banach algebra with identity e.

Remark 4.2.1. In general, $A \hat{\otimes} B \not\subseteq A \# B$. For, let A be a function algebra on X. Milne [42] has proved that if the function algebra $A \hat{\otimes} B = A \# B$ for all function algebras B, then A must have the approximation property and if every function algebra has the approximation property, then every Banach space has the same property. But there exists a Banach space without approximation property [14]. Therefore, there must exist function algebras A and B such that $A \hat{\otimes} B \not\subseteq A \# B$.

Since the Bishop (Šilov) decompositions for A, $A \hat{\otimes} B$ and $A \# B$ are the decompositions of the same space X, it is natural to investigate the relations between them. For a complex function space A on X, let $\mathcal{K}_{FP}(A)$, $\mathcal{K}_{E}(A)$ and $\mathcal{F}(A)$ denote respectively, the Bishop decomposition in (FP)-sense, the Bishop decomposition in (E)-sense and the Šilov decomposition for A.

Proposition 4.2.2. Let A be a complex function space on X. Then $\mathcal{N}(A) = \mathcal{N}(A \hat{\otimes} B) = \mathcal{N}(A \# B)$, where $\mathcal{N}(A \hat{\otimes} B)$ and $\mathcal{N}(A \# B)$ are defined as in section 1. Consequently, $\mathcal{K}_{E}(A \hat{\otimes} B) = \mathcal{K}_{E}(A \# B) = \mathcal{K}_{E}(A)$ and $\mathcal{F}_{E}(A \hat{\otimes} B) = \mathcal{F}_{E}(A \# B) = \mathcal{F}(A)$. 

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Proof. We shall show that $N(A \otimes B) \subseteq N(A \# B) \subseteq N(A) \subseteq N(A \otimes B)$.

Let $f \in N(A \otimes B)$ and $g \in A \# B$. Fix $\phi \in m(B)$. Then $\phi \circ g \in A$.

Therefore, $(\phi \circ g) \circ e \in A \otimes B$. Since $f \in N(A \otimes B)$, $(f \circ e)(\phi \circ g) \circ e \in A \otimes B$ which is contained in $A \# B$. So, $\psi \circ (\psi \circ g) \circ e \in A$ for each $\psi \in m(B)$, i.e., $f(\phi \circ g) \subseteq A$.

Thus $\phi \circ (f \circ e)g \subseteq A$. Since this is true for each $\phi \in m(B)$, $(f \circ e)g \in A \# B$ and so, $f \in N(A \# B)$. Hence $N(A \otimes B) \subseteq N(A \# B)$.

To show that $N(A \# B) \subseteq N(A)$, let $f \in N(A \# B)$ and $g \in A$. Then $g \circ e \in A \# B$. So, $(f \circ e)(g \circ e) \in A \# B$, i.e., $fg \circ e \in A \# B$. Therefore, $\phi \circ (fg \circ e) \subseteq A$ for each $\phi \in m(B)$ or $fg \subseteq A$. Thus $f \in N(A)$.

Finally, let $f \in N(A)$ and $g \in A \otimes B$. Then $g = \sum_{i=1}^{n} g_i \otimes b_i$, where $g_i \in A$ and $b_i \in B$ for $i = 1, 2, \ldots, n$.

Since $f \in N(A)$, $fg_i \subseteq A$ for each $i = 1, 2, \ldots, n$. Therefore, $(f \circ e)g \subseteq A \otimes B$. Now, let $h \in A \otimes B$. Then $h = \lim_{n \to \infty} h_n$, where $h_n \in A \otimes B$. As we have shown, $(f \circ e)h_n \subseteq A \otimes B$ and so, $(f \circ e)h = (f \circ e)(\lim_{n \to \infty} h_n) = \lim_{n \to \infty} (f \circ e)h_n \subseteq A \otimes B$. Hence $f \in N(A \otimes B)$ and we get $N(A \subseteq N(A \otimes B)$.

The next proposition gives the relations between $N(A \otimes B) \subseteq N(A \# B)$ and $N(A)$ and $N(A \otimes B)$, $N(A \# B)$ and $N(A)$. 175
Proposition 4.2.3. Let $A$ be a complex function space on $X$. Then $N(A) \otimes B \subset M(A \otimes B) \subset M(A \# B) \subset N(A) \# B$. Further, we get

(i) $\mathcal{N}(A \otimes B) = \mathcal{N}(A \# B) = \mathcal{N}(A)$

(ii) $\mathcal{M}(A \otimes B) = \mathcal{M}(A \# B) = \mathcal{M}(A)$.

Proof. Let $f \in N(A) \otimes B$. Then $f = \sum_{i=1}^{n} f_i \otimes b_i$, where $f_i \in N(A)$ and $b_i \in B$ for $i = 1, 2, \ldots, n$. Let $g \in A \otimes B$. Then $g = \sum_{j=1}^{m} g_j \otimes b_j$ for $g_j \in A$ and $b_j \in B$ for $j = 1, 2, \ldots, m$.

Since $f_i \in N(A)$, $f_i g_j \in A$ for all $i$ and $j$. Thus $fg = \sum_{i,j} f_i g_j \otimes b_i b_j \in A \otimes B$. If $h \in A \otimes B$, then $h = \lim_{n \to \infty} h_n$, for $h_n \in A \otimes B$. Hence $fh = \lim_{n \to \infty} (fh_n) \in A \otimes B$ and so, $f \in M(A \otimes B)$. Thus $N(A) \otimes B \subset M(A \otimes B)$ which is closed. Hence $N(A) \otimes B \subset M(A \otimes B)$.

To show that $M(A \otimes B) \subset M(A \# B)$, let $f \in M(A \otimes B)$ and $g \in A \# B$. Fix $\phi \in m(B)$. Since $M(A \otimes B) \subset A \otimes B \subset A \# B$, $\phi \circ f \in A$. We prove that $\phi \circ f \in N(A)$. Let $h \in A$. Then $f(h \otimes e) \in A \otimes B$ and so, $\phi \circ (f(h \otimes e)) \in A$, i.e., $(\phi \circ f)h \in A$ which shows that $\phi \circ f \in N(A)$. Thus $\phi \circ (fg) = (\phi \circ f)(\phi \circ g) \in A$, as $\phi \circ g \in A$. Since $\phi \in m(B)$ is arbitrary, $fg \in A \# B$. Hence $f \in M(A \# B)$ and $M(A \otimes B) \subset M(A \# B)$.

Let $f \in M(A \# B)$ and $\phi \in m(B)$. To prove that $f \in N(A) \# B$, we must show that $\phi \circ f \in N(A)$. Clearly
\( \phi \circ f \in A \). Let \( h \in A \). Then \( f(h \circ e) \in A \# B \) and therefore, 
\( (\phi \circ f)h = \phi \circ (f(h \circ e)) \in A \). Hence \( \phi \circ f \in \mathcal{N}(A) \). Thus 
\( \mathcal{M}(A \# B) \subseteq \mathcal{N}(A) \# B \).

Finally, we shall prove (i) and (ii).

(i) Since \( \mathcal{F}(A) \# B) = \mathcal{F}(A \# B) = \mathcal{F}(A) \# B \) (Proposition 4.2.2), we show that if \( K \in \mathcal{K}_e(A) \), then \( K \) is a \((B)\)-antisymmetric set for \( A \# B \). Let \( K \in \mathcal{K}_e(A) \), \( f \in \mathcal{M}(A \# B) \) and \( (\phi \circ f)|_K \) be 
real-valued for all \( \phi \in m(B) \). Since \( \mathcal{M}(A \# B) \subseteq \mathcal{N}(A) \# B \), 
\( \phi \circ f \in \mathcal{N}(A) \) for all \( \phi \in m(B) \). Therefore, \( (\phi \circ f)|_K \) is 
constant for each \( \phi \in m(B) \). Hence \( f|_K \) is constant, as \( B \) is 
semisimple. Thus \( K \) is a \((B)\)-antisymmetric set for \( A \# B \) and 
we get \( \mathcal{F}(A \# B) = \mathcal{K}_e(A) \) and also (ii) holds.

Remarks 4.2.4. (i) By Proposition 4.2.2 and 4.2.3, we have 
\( \mathcal{K}_e(A) = \mathcal{K}_e(A \# B) = \mathcal{K}_e(A \# B) = \mathcal{K}_e(A \# B) = \mathcal{K}_e(A \# B) \) and 
\( \mathcal{F}(A) = \mathcal{F}(A \# B) = \mathcal{F}(A \# B) = \mathcal{F}_e(A \# B) = \mathcal{F}_e(A \# B) \) for a 
complex function space \( A \) on \( X \). Hence, by Remark 4.1.25, all 
the above decompositions have the \((G)\)-property and hence 
the \((S)\)-property and the \((D)\)-property for \( A \# B \) and \( A \# B \).

(ii) If \( A \) is a function algebra on \( X \), then \( \mathcal{K}(A) = \mathcal{K}(A \# B) = 
\mathcal{F}_p(A \# B) = \mathcal{F}(A \# B) \), since \( \mathcal{K}(A) \# B < \mathcal{F}_p(A \# B) < \mathcal{K}_e(A \# B) \). 
Also, \( \mathcal{F}(A) = \mathcal{F}(A \# B) = \mathcal{F}_e(A \# B) \), by the above remark. The 
same is true for \( A \# B \) also.
Next, we compare the Bishop decompositions in (FP)-sense.

**Proposition 4.2.5.** Let $A$ be a complex function space on $X$ and $K$ be a closed subset of $X$. Then $N(A \hat{\otimes} B|_K) \subset N(A|_K)$ and $N(A \# B|_K) \subset N(A|_K)$. Consequently, $\mathcal{K}_{FP}(A) \subset \mathcal{K}_{FP}(A \hat{\otimes} B)$ and $\mathcal{K}_{FP}(A) \subset \mathcal{K}_{FP}(A \# B)$.

**Proof.** Let $f \in N(A \hat{\otimes} B|_K)$ and $g|_K \in A|_K$, where $g \in A$. Then $(f \otimes e)(g \otimes e)|_K \in A \otimes B|_K$. Therefore, there exists $h \in A \otimes B$ such that $(f \otimes e)(g \otimes e) = h$ on $K$, i.e., $f g \otimes e = h$ on $K$. For $\phi \in m(B)$, $\phi \circ h \in A$ and so, $(f g \otimes e)|_K = (\phi \circ (f g \otimes e))|_K = (\phi \circ h)|_K \in A|_K$. Hence $f \in N(A|_K)$. Thus $N(A \hat{\otimes} B|_K) \subset N(A|_K)$.

Similarly, we can prove that $N(A \# B|_K) \subset N(A|_K)$.

We shall use Corollary 4.1.28 to show the equality of $\mathcal{K}_{FP}(A)$ and $\mathcal{K}_{FP}(A \hat{\otimes} B)$. For that, we must prove that $\mathcal{K}_{FP}(A)$ has the (S)-property for $A \otimes B$. Consequently, we get $\mathcal{K}_{FP}(A \hat{\otimes} B) \subset \mathcal{K}_{FP}(A \# B)$.

**Proposition 4.2.6.** If $A$ is a complex function space on $X$, then $\mathcal{K}_{FP}(A)$ has the (GA)-property for $A \otimes B$.

**Proof.** Let $\mu \in \mathcal{B}(A \otimes B)^e$ and $S = \text{supp } \mu$. It is enough to show that $S$ is an (FP)-antisymmetric set for $A$. Let $f \in N(A|_S)$ and $f$ be real-valued. Let $\eta = f \mu$. Then $\eta \in M(X; B^*)$. Let
\( g \in A \ast B \). Then \( g = \sum_{i=1}^{n} g_i \ast b_i \), where \( g_i \in A \) and \( b_i \in B \) for all \( i \). Since \( f \in \mathcal{N}(A|_S) \), \( f g|_S \in A|_S \) for each \( i \). Therefore, 
\[
fg|_S = \left( \sum_{i=1}^{n} f g_i \ast b_i \right)|_S \in A \ast B|_S.
\]
Thus \( \int g d\nu = \int g f d\mu = 0 \), as \( \mu \in (A \ast B)^\perp \). Hence \( \eta \in (A \ast B)^\perp \). By the same argument as given in Proposition 4.1.24, we get \( f|_S \) is constant. Hence \( S \) is an \((FP)\)-antisymmetric set for \( A \).

**Corollary 4.2.7.** For a complex function space \( A \) on \( X \),
\[
\mathcal{K}_{FP}(A) = \mathcal{K}_{FP}(A \ast B).
\]

**Proof.** Since \((GA)\)-property \(\Rightarrow (S)\)-property, by Proposition 4.2.6, \( \mathcal{K}_{FP}(A) \) has the \((S)\)-property for \( A \ast B \). Also, \( \mathcal{K}_{FP}(A) \) is Hausdorff decomposable and by Proposition 4.2.5,
\[
\mathcal{K}_{FP}(A) \subset \mathcal{K}_{FP}(A \ast B).
\]
Hence, by Corollary 4.1.28,
\[
\mathcal{K}_{FP}(A) = \mathcal{K}_{FP}(A \ast B).
\]

We do not know whether \( \mathcal{K}_{FP}(A) = \mathcal{K}_{FP}(A \# B) \) is true, in general.

Let \( \mathcal{C} \) be a decomposition of \( X \). Then it is natural to ask whether the \((D)\)-property of \( \mathcal{C} \) for \( A \), \( A \ast B \) and \( A \# B \) are related.

**Proposition 4.2.8.** Let \( A \) be a complex function space on \( X \) and \( \mathcal{C} \) be a decomposition of \( X \). Consider the following statements.
(i) \( \mathcal{S} \) has the (D)-property for \( A \otimes B \).

(ii) \( \mathcal{S} \) has the (D)-property for \( A \# B \).

(iii) \( \mathcal{S} \) has the (D)-property for \( A \).

Then (i) \( \Rightarrow \) (ii) \( \Leftrightarrow \) (iii).

Proof. Suppose (i) holds. Let \( f \in \mathcal{C}(X;B) \) and \( f|_E \in (A \# B)|_E \) for each \( E \in \mathcal{S} \). Fix \( \phi \in m(B) \) and \( E \in \mathcal{S} \). It can be checked on the same line as in Lemma 0.1.4 that \( (A \# B)|_E \) \( \subset \) \( (A|_E \otimes B) \) and so, \( f|_E \in (A|_E \otimes B) \). Thus \( \phi \circ f|_E \in (A|_E \otimes B) \) and so, \( (\phi \circ f)|_E \in (A \otimes B|_E \otimes B) \). This is true for each \( E \in \mathcal{S} \). Since (i) holds, \( (\phi \circ f)|_E \in (A \otimes B|_E \otimes B) \). Therefore, for each \( \psi \in m(B) \), \( \phi \circ g \circ f \in A \otimes B \). Hence \( f \in A \# B \). Thus \( \mathcal{S} \) has the (D)-property for \( A \# B \).

Now, suppose (ii) holds. Let \( g \in \mathcal{C}(X) \) and \( g|_E \in (A|_E \otimes B) \) for each \( E \in \mathcal{S} \). Then \( g \circ e \in \mathcal{C}(X;B) \) and \( (g \circ e)|_E \in (A|_E \otimes B) \) for each \( E \in \mathcal{S} \). But \( (A|_E \otimes B) \) \( \subset \) \( (A \otimes B|_E \otimes B) \) \( \subset \) \( (A \# B)|_E \). Therefore, \( (g \circ e)|_E \in (A \# B)|_E \) for each \( E \in \mathcal{S} \). Since \( \mathcal{S} \) has the (D)-property for \( A \# B \), \( g \circ e \in A \# B \). Hence \( g \in A \) and (iii) holds.

Finally, suppose that \( \mathcal{S} \) has the (D)-property for \( A \).

Let \( h \in \mathcal{C}(X;B) \) and \( h|_E \in (A \# B)|_E \) for each \( E \in \mathcal{S} \). Then \( h|_E \in (A|_E \otimes B) \) and so, \( (\phi \circ h)|_E \in (A|_E \otimes B) \) for each \( E \in \mathcal{S} \) and \( \phi \in m(B) \). Since (iii) holds, \( \phi \circ h \in A \) for each \( \phi \in m(B) \). Hence \( h \in A \# B \) which proves that (iii) \( \Rightarrow \) (ii).
In general, we do not know whether (iii) ⇒ (i). However, in presence of some additional condition on $\mathcal{S}$, (iii) ⇒ (i), as we see in the next proposition.

**Proposition 4.2.9.** Let $\mathcal{S}$ be an u.s.c. decomposition of $X$. If $\mathcal{S}$ has the (D)-property for a complex function space $A$, then $\mathcal{S}$ has the (D)-property for $A \hat{\otimes} B$.

**Proof.** Suppose that $\mathcal{S}$ has the (D)-property for $A$. Then, by Theorem 2.1.7, $\mathcal{F}(A) < \mathcal{S}$. By Remark 4.2.4(i), $\mathcal{F}(A \hat{\otimes} B) < \mathcal{S}$ and also $\mathcal{F}(A \hat{\otimes} B)$ has the (D)-property for $A \hat{\otimes} B$. Hence $\mathcal{S}$ has the (D)-property for $A \hat{\otimes} B$.

We have already defined the $p$-sets and (S)-property for a vector function space on $X$. To compare the (S)-property for $A$, $A \hat{\otimes} B$ and $A \# B$, first we find out the relation between their $p$-sets and peak sets.

**Proposition 4.2.10.** Let $A$ be a complex function space on $X$ and $F$ be a closed subset of $X$. Then the following are equivalent.

(i) $F$ is a peak set (generalized peak set) for $A$.

(ii) $F$ is a peak set (generalized peak set) for $A \hat{\otimes} B$.

(iii) $F$ is a peak set (generalized peak set) for $A \# B$.

**Proof.** Suppose that (i) holds, i.e., $F$ is a peak set for $A$. Let $f \in A$ be a peaking function for $F$, i.e., $f|_F = 1$ and
\[ |f(x)| < 1 \text{ for } x \in X - F. \text{ Then } f \circ e \in A \otimes B, \quad (f \circ e)_{|F} = e \text{ and for } x \in X - F, \quad |(f \circ e)(x)| = |f(x)e| = |f(x)| < 1. \] Thus \( f \circ e \in A \otimes B \) is a peaking function for \( F \) and (ii) holds.

Since \( A \otimes B \subset A \# B \), (ii) \( \Rightarrow \) (iii). Finally, assume that (iii) holds. Let \( g \in A \# B \) be a peaking function for \( F \), i.e., \( g|_F = e \) and \( |g(x)| < 1 \) for \( x \in X - F \). Let \( \phi \in m(B) \).

Then \( \phi \circ g \in A, \quad (\phi \circ g)|_F = \phi(e) = 1 \) and for \( x \in X - F, \)
\[
|\phi \circ g(x)| = |\phi(g(x))| \leq |\phi||g(x)| < 1. \] Thus \( \phi \circ g \) is a peaking function for \( F \) and (i) holds.

Since a generalized peak set is the intersection of peak sets, the equivalence for a generalized peak set follows from the equivalence proved for a peak set.

**Lemma 4.2.11.** Let \( A \) be a complex function space on \( X \). Then

(i) \( \mu \in A^\perp \Rightarrow \mu \otimes \phi \in (A \# B)^\perp \) for all \( \phi \in m(B) \), where \( (\mu \otimes \phi)(G) = \mu(G)\phi \) for every Borel subset \( G \) of \( X \).

(ii) \( \eta \in (A \otimes B)^\perp \Rightarrow \eta \circ b \in A^\perp \) for all \( b \in B \), where \( (\eta \circ b)(G) = \eta(G)(b) \) for every Borel subset \( G \) of \( X \).

**Proof.** (i) Let \( \mu \in A^\perp \) and \( f \in A \# B \). For \( \phi \in m(B) \), \( \phi \circ f \in A \).

Hence \( \int_X f d(\mu \otimes \phi) = \int_X (\phi \circ f) d\mu = 0 \). Thus \( \mu \otimes \phi \in (A \# B)^\perp \), for each \( \phi \in m(B) \).

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(ii) Let $\eta \in (A \otimes B)^\perp$ and $g \in A$. Then, for $b \in B$, $g \circ b \in A \otimes B$ and hence $\int_X gd(\eta \circ b) = \int_X (g \circ b) d\eta = 0$. Thus $\eta \circ b \in A^\perp$, for each $b \in B$.

Remark 4.2.12. Since $(A \# B)^\perp \subset (A \otimes B)^\perp$, it also follows from Lemma 4.2.11 that, $\mu \in A^\perp \Rightarrow \mu \circ \phi \in (A \otimes B)^\perp$ for all $\phi \in m(B)$ and $\eta \in (A \# B)^\perp \Rightarrow \eta \circ b \in A^\perp$ for all $b \in B$.

Proposition 4.2.13. Let $A$ be a complex function space on $X$ and $F$ be a closed subset of $X$. Consider the following statements.

(i) $F$ is a $p$-set for $A \# B$.

(ii) $F$ is a $p$-set for $A$.

(iii) $F$ is a $p$-set for $A \otimes B$.

Then (i) $\Rightarrow$ (ii) $\iff$ (iii).

Proof. (i) $\Rightarrow$ (ii). Suppose that $F$ is a $p$-set for $A \# B$. Let $\mu \in A^\perp$ and $\phi \in m(B)$. Then, by Lemma 4.2.11(i), $\mu \circ \phi \in (A \# B)^\perp$. Hence $(\mu \circ \phi)_F \in (A \# B)^\perp$. Thus for $g \in A$, $\int_F gd\mu = \int_F (g \circ e)d(\mu \circ \phi) = 0$ which shows that $\mu_F \in A^\perp$. Hence $F$ is a $p$-set for $A$.

(ii) $\Rightarrow$ (iii). Assume that $F$ is a $p$-set for $A$. Let $\eta \in (A \otimes B)^\perp$. Then, by Lemma 4.2.11(ii), $\eta \circ b \in A^\perp$ for each
\[ f = \sum_{i=1}^{n} f_i \otimes b_i, \text{ where } f_i \in A \text{ and } b_i \in B \text{ for each } i. \]

Therefore,

\[ \int f d\eta = \sum_{i=1}^{n} \int f_i \otimes b_i d\eta = \sum_{i=1}^{n} \int f_i d(\eta \circ b_i) = 0. \]

Thus \( \eta_f \in (A \hat{\otimes} B)^\perp \) and hence \( \eta_f \in (A \otimes B)^\perp. \)

In view of Remark 4.2.12, (iii) \( \Rightarrow \) (ii) can be proved by the same argument as given for (i) \( \Rightarrow \) (ii).

Remarks 4.2.14. (i) Since \( \mathcal{K}_{\mathcal{FP}}(A) = \mathcal{K}_{\mathcal{FP}}(A \hat{\otimes} B) \) and members of \( \mathcal{K}_{\mathcal{FP}}(A) \) are p-sets for A, by Proposition 4.2.13, members of \( \mathcal{K}_{\mathcal{FP}}(A \hat{\otimes} B) \) are p-sets for A \( \hat{\otimes} \) B.

(ii) We know that \( \mathcal{K}(A \hat{\otimes} B) = \mathcal{K}(A \# B) = \mathcal{K}(N(A)) \) and \( \mathcal{K}(A \# B) = \mathcal{K}(A \# B) = \mathcal{K}(A \hat{\otimes} B) = \mathcal{K}(N(A)) \) (Remark 4.2.4(i)). Also, members of \( \mathcal{K}(N(A)) \) and \( \mathcal{K}(N(A)) \) are p-sets for N(A). Since N(A) = N(A \hat{\otimes} B) = N(A \# B), in view of Lemma 4.1.26, the members of \( \mathcal{K}(A \hat{\otimes} B) \) and \( \mathcal{K}(A \hat{\otimes} B) \) are also p-sets for A \( \hat{\otimes} \) B and A \# B.

We do not know whether a p-set for a complex function space A is a p-set for A \# B. But for a function algebra A, we get (ii) \( \Rightarrow \) (i) in Proposition 4.2.13, by the following corollary and Proposition 4.2.10.
Corollary 4.2.15. Let $A$ be a function algebra on $X$ and $F$ be a closed subset of $X$. Then $F$ is a p-set for $A \# B$ ($A \hat{\otimes} B$) if and only if $F$ is a generalized peak set for $A \# B$ ($A \hat{\otimes} B$).

Proof. Since $A \# B$ is a vector function algebra on $X$, by Corollary 4.1.20, a generalized peak set for $A \# B$ is a p-set for $A \# B$.

Conversely, suppose that $F$ is a p-set for $A \# B$. Then, by Proposition 4.2.13, $F$ is a p-set for $A$ and hence $F$ is a generalized peak set for $A$ (see Remark 0.1.8(ii)). Therefore, by Proposition 4.2.10, $F$ is a generalized peak set for $A \# B$.

Similarly, we can prove that $F$ is a generalized peak set for $A \hat{\otimes} B$ if and only if $F$ is a p-set for $A \hat{\otimes} B$.

From the proof of the above corollary, we can see that even when $A$ is a complex function space on $X$, a p-set for $A \# B$ ($A \hat{\otimes} B$) is a generalized peak set for $A \# B$ ($A \hat{\otimes} B$) (see Remark 4.1.21).

Proposition 4.2.16. Let $A$ be a complex function space on $X$ and $\mathcal{E}$ be a decomposition of $X$.

(i) $\mathcal{E}$ has the (S)-property for $A \hat{\otimes} B \Rightarrow \mathcal{E}$ has the (S)-property for $A$.

(ii) If $A$ is a function algebra on $X$, then $\mathcal{E}$ has the (S)-property for $A \# B \Rightarrow \mathcal{E}$ has the (S)-property for $A$.

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Proof. (i) Suppose that $\mathfrak{s}$ has the (S)-property for $A \hat{\otimes} B$.
Let $F$ be a $p$-set for $A$ which is saturated with $\mathfrak{s}$. Then, by Proposition 4.2.13, $F$ is a $p$-set for $A \hat{\otimes} B$. Hence $\mathfrak{s} \cap F$ has the (D)-property for $A \hat{\otimes} B|_F$. But $A \hat{\otimes} B|_F = (A|_F) \hat{\otimes} B$.
Therefore, $\mathfrak{s} \cap F$ has the (D)-property for $(A|_F) \hat{\otimes} B$. Thus, by Proposition 4.2.8, $\mathfrak{s} \cap F$ has the (D)-property for $A|_F$.
Hence $\mathfrak{s}$ has the (S)-property for $A$.

(ii) Let $A$ be a function algebra on $X$. Assume that $\mathfrak{s}$ has the (S)-property for $A \# B$. Let $F$ be a $p$-set for $A$ which is saturated with $\mathfrak{s}$. Therefore, $\mathfrak{s} \cap F = \{ E \in \mathfrak{s} : E \cap F \neq \emptyset \}$.
Since $A \# B$ is a vector function algebra, $F$ is a $p$-set for $A \# B$. To show that $\mathfrak{s} \cap F$ has the (D)-property for $A|_F$, let $f \in \Omega(F)$ and $f|_E \in ((A|_F)|_E)\overline{\jmath} = (A|_F)\overline{\jmath}$ for each $E \in \mathfrak{s} \cap F$.
Then, for each $E \in \mathfrak{s} \cap F$,
$$f|_E \otimes e = (f \otimes e)|_E \in (A \hat{\otimes} B|_F)\overline{\jmath}$$
$$\subseteq (A \# B|_F)\overline{\jmath}$$
$$= ((A \# B|_F)\overline{\jmath}).$$
Since $\mathfrak{s} \cap F$ has the (D)-property for $A \# B|_F$, $f \otimes e \in A \# B|_F \subseteq (A|_F) \# B$. Hence $f \in A|_F$ and $\mathfrak{s} \cap F$ has the (D)-property for $A|_F$. Thus $\mathfrak{s}$ has the (S)-property for $A$.

We do not know whether a decomposition $\mathfrak{s}$ of $X$ has the (S)-property for $A$ implies that $\mathfrak{s}$ has the (S)-property for $A \hat{\otimes} B$ or $A \# B$. 

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Finally, we discuss the essential sets of \( A \otimes B \) and \( A \# B \) and show that some results analogous to the results for the essential set of a complex function space hold true for them.

**Proposition 4.2.17.** Let \( A \) be a complex function space on \( X \) and \( E(A), E(A \otimes B), E(A \# B) \) denote respectively the essential sets of \( A, A \otimes B \) and \( A \# B \). Then \( E(A) = E(A \otimes B) = E(A \# B) \).

**Proof.** Let \( I \) denote the largest closed ideal of \( \mathcal{C}(X; \mathcal{B}) \) contained in \( A \otimes B \). Then \( \overline{I}_\phi \) is a closed ideal of \( \mathcal{C}(X) \) and \( I_{\phi} \subseteq A \) for each \( \phi \in m(\mathcal{B}) \), where \( I_{\phi} = \{ \phi \circ f : f \in I \} \). If \( I_{\phi} = A \) for some \( \phi \in m(\mathcal{B}) \), then \( \overline{I}_\phi = \mathcal{C}(X) \), since \( 1 \in A \). Hence \( A = \mathcal{C}(X) \) and in this case \( E(A) = E(A \otimes B) = \phi \). Let \( \overline{I}_\phi \not\subseteq A \) for each \( \phi \in m(\mathcal{B}) \). Then \( E(A) \subseteq E_\phi \), where \( E_\phi = \{ x \in X : f(x) = 0 \text{ for all } f \in I_{\phi} \} \), for each \( \phi \in m(\mathcal{B}) \). Hence \( E(A) \subseteq E(A \otimes B) \).

By Proposition 4.1.32(ii), \( E(A \otimes B) \subseteq E(N(A \otimes B)) = E(N(A)) \), as \( N(A \otimes B) = N(A) \). But \( E(N(A)) = E(A) \). Hence \( E(A \otimes B) \subseteq E(A) \).

Similarly, we can prove that \( E(A) = E(A \# B) \).

**Remarks 4.2.18.** (i) Since \( E(M(V)) = E(V) \) for a vector function space \( V \) on \( X \) and \( E(A) = E(N(A)) \), the essential sets of \( M(A \otimes B), M(A \# B), A \otimes B, A \# B, A, N(A), N(A \otimes B) \) and \( N(A) \# B \) are the same. We denote it by \( E \).
(ii) We know that \( E = \bigcup_{\alpha} K_{\alpha} \), where \( K_{\alpha} \)'s are nontrivial members of \( \mathcal{K}(A) \) or \( \mathcal{K}_{FP}(A) \). But \( \mathcal{K}(A) = \mathcal{K}(A \otimes B) \) and \( \mathcal{K}_{FP}(A) = \mathcal{K}_{FP}(A \otimes B) \). Therefore, \( E \) is also the closure of the union of nontrivial members of \( \mathcal{K}(A \otimes B) \) \( (\mathcal{K}_{FP}(A \otimes B)) \). Consequently, \( \mathcal{K}(A \otimes B) \subseteq \mathcal{K} \), where \( \mathcal{K} = \{ E \} \cup \{ \{ x \} : x \in X - E \} \) and hence \( \mathcal{K} \) also has the \( (GA) \)-property for \( A \otimes B \).

(iii) \( E \) is a \( p \)-set for \( N(A) = N(A \otimes B) = N(A \# B) \). Hence, by Lemma 4.1.26, \( E \) is a \( p \)-set for \( A \otimes B \) and \( A \# B \).

For a complex function space \( A \) on \( X \), we know that \( E \) has the following property:

\[
(* \quad \text{If } f \in \mathcal{C}(X) \text{ and } f|_{F} = 0 \text{ for a closed subset } F \text{ of } X \rightarrow f \in A, \text{ then } E \subseteq F.
\]

We prove a similar result for \( A \otimes B \).

**Proposition 4.2.19.** Let \( A \) be a complex function space on \( X \) and \( E \) denote the essential set of \( A \otimes B \). Then \( E \) has the following property:

\[
(** \quad \text{If } f \in \mathcal{C}(X;B) \text{ and } f|_{F} = 0 \text{ for a closed subset } F \text{ of } X \rightarrow f \in A \otimes B, \text{ then } E \subseteq F.
\]

**Proof.** Let \( F \) be a closed subset of \( X \) with the property that, \( f \in \mathcal{C}(X;B) \) and \( f|_{F} = 0 \Rightarrow f \in A \otimes B \). Let \( g \in \mathcal{C}(X) \) and \( g|_{F} = 0 \). Then \( g \circ e \in \mathcal{C}(X;B) \) and \( (g \circ e)|_{F} = 0 \). Hence \( g \circ e \in A \otimes B \). Thus \( g \in A \). That is, \( g \in \mathcal{C}(X) \) and \( g|_{F} = 0 \Rightarrow g \in A \). Therefore, by the property \( (*) \) of \( E \), \( E \subseteq F \). Hence the property \( (** \) is satisfied.
Remark 4.2.20. We note that many results of this chapter (for example, Corollary 4.1.14, Proposition 4.1.24, Theorem 4.1.27) hold good even if we assume that $B$ is a Banach space in place of a Banach algebra. However, to prove some results (for example, Propositions 4.2.3, 4.2.5, 4.2.8, 4.2.13), it is necessary that $B$ is a Banach algebra. Hence we have assumed throughout that $B$ is a Banach algebra for the sake of simplicity and uniformity.