Let $X$ be a compact Hausdorff space and let $C(X)$ ($C_\mathbb{R}(X)$) denote the set of all complex-valued (real-valued) continuous functions on $X$. With usual operations of addition, multiplication and scalar multiplication and with the norm defined by

$$||f|| = \sup \left\{ |f(x)| : x \in X \right\}$$

for $f \in C(X)$ ($C_\mathbb{R}(X)$), $C(X)$ ($C_\mathbb{R}(X)$) is a complex (real) Banach algebra with identity. A function algebra on $X$ is a closed subalgebra of $C(X)$ which contains constants and separates the points of $X$.

A decomposition of $X$ is a collection of disjoint closed subsets of $X$ whose union is $X$. Subalgebras of $C(X)$ ($C_\mathbb{R}(X)$) and the decompositions of $X$ are closely related. For example, a closed ideal of $C(X)$ is determined by a closed subset of $X$. Now, if $F$ is a closed subset of $X$, then we can associate with it the decomposition $D = \{ F \} \cup \left\{ \{ x \} : x \in X - F \right\}$. Thus every closed ideal is associated with a decomposition of $X$ consisting of a closed set and singletons outside the closed set. If $A$ is a closed subalgebra of $C_\mathbb{R}(X)$ (a self-conjugate closed subalgebra of $C(X)$) containing
constants, then the sets of constancy of \( A \) gives a decomposition which is upper semicontinuous. Conversely, if \( \mathcal{D} \) is an upper semicontinuous decomposition of \( X \), then there exists a unique closed subalgebra of \( C(X) \) containing constants whose sets of constancy are precisely the members of \( \mathcal{D} \) [50, Ex. 7.5.7(F)]. This association of decompositions of \( X \) and subalgebras of \( C(X) \) has been found very useful in the study of \( C(X) \) as a direct sum of two subalgebras ([16], [39], [40], [41]).

The role of decompositions in the study of function algebras was highlighted by Šilov [52] and more so by Bishop [5]. The Šilov decomposition for a function algebra \( A \) on \( X \) consists of sets of constancy of \( A = A \cap C(X) \). The Bishop decomposition for \( A \) consists of maximal sets of antisymmetry. Both these decompositions have the following crucial property:

If \( f \in C(X) \) and \( f|_E \in (A|_E)^{-} \) for every member \( E \) in the decomposition, then \( f \in A \).

The above property is known as the (D)-property in the literature [20].

Once the importance of decompositions is recognized, it is natural to ask further questions. Some of the questions are:
(1) Are there decompositions, other than Šilov and Bishop, associated with a function algebra which also have the (D)-property?

(2) Does a Bishop (Šilov) decomposition have a stronger property than the (D)-property?

(3) How are Bishop, Šilov and other decompositions related to each other? Do some of these decompositions determine the others?

(4) Does every member of a decomposition satisfying property such as (D)-property have any special property in relation to a function algebra? (For example, every member of Bishop decomposition of a function algebra is an intersection of peak sets).

(5) How are the decompositions of A and \( \hat{A} \) related, where \( \hat{A} \) is the algebra of Gelfand transforms of A?

(6) Can the decompositions analogous to Šilov and Bishop for a function algebra be defined for a function space? What are their properties?

(7) How about the decompositions for a real function algebra? For an algebra of vector-valued continuous functions?

Some of these and related questions have been discussed in the literature ([9], [10], [20], [26], [33]).
This thesis deals with further investigations of these questions.

Before we give the chapterwise summary of the results proved in the thesis, it will be convenient to set up notations and give definitions and other preliminaries.

1. Preliminaries

Let $X$ be a compact Hausdorff space. A decomposition of $X$ is a collection of disjoint closed subsets of $X$ whose union is $X$. We shall denote decompositions by $\mathcal{S}$, $\mathcal{F}$, $\mathcal{J}$ etc. First we define certain notions related to decomposition of $X$.

Definitions 0.1.1 [20, p.4]. (i) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two decompositions of $X$. If for every $E_1 \in \mathcal{S}_1$, there exists $E_2 \in \mathcal{S}_2$ such that $E_1 \subset E_2$, then $\mathcal{S}_1$ is said to be finer than $\mathcal{S}_2$ and we write $\mathcal{S}_1 < \mathcal{S}_2$.

It is clear that if $\mathcal{S}_1 < \mathcal{S}_2$ and $\mathcal{S}_2 < \mathcal{S}_1$, then $\mathcal{S}_1 = \mathcal{S}_2$. If $\mathcal{S}_1 < \mathcal{S}_2$ and $\mathcal{S}_1 \not< \mathcal{S}_2$, then we write $\mathcal{S}_1 \not< \mathcal{S}_2$.

(ii) Let $\mathcal{S}$ be a decomposition of $X$ and $F$ be a closed subset of $X$. Then $\mathcal{S} \cap F = \{ E \cap F : E \cap F \neq \emptyset, E \in \mathcal{S} \}$.

(iii) A set $F$ is said to be saturated with the decomposition $\mathcal{S}$ of $X$ if, whenever $E \in \mathcal{S}$ and $E \cap F \neq \emptyset$, then $E \subset F$. 

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Definition 0.1.2. A decomposition $\mathcal{S}$ of $X$ is said to be **upper semicontinuous** (u.s.c.) if for each $E \in \mathcal{S}$ and each open set $U$ containing $E$, there is an open set $V$ such that $E \subseteq V \subseteq U$ and if $E \cap V \neq \emptyset$ for $E' \in \mathcal{S}$, then $E' \subseteq U$.

Note that if $\mathcal{S}$ is an u.s.c. decomposition of $X$, then the quotient space $X/\mathcal{S}$ is Hausdorff.

In chapter 2, we define the concept of *Hausdorff decomposable decomposition* (Definition 2.2.4) which is weaker than upper semicontinuity.

Now, we define some well known ideas related to a function algebra. For details, we refer to [6], [17], [29] and [55].

Let $A$ denote a function algebra on $X$, i.e., $A$ is a closed subalgebra of $C(X)$ which contains constants and separates the points of $X$. The **maximal ideal space** $m(A)$ of $A$ is the set of all nonzero complex homomorphisms on $X$. $\hat{A} = \{ \hat{f} : f \in A \}$ denotes the Gelfand transform of $A$. For a subset $E$ of $X$, the $A$-hull of $E$ is the set $\tilde{E} = \{ \phi \in m(A) : |\hat{f}(\phi)| \leq ||f||_E \text{ for all } f \in A \}$, where $||f||_E = \sup \{ |f(x)| : x \in E \}$. A probability measure $\mu$ on $X$ is said to be a representing measure for $\phi \in m(A)$ with respect to $A$ if $\int f d\mu = \phi(f)$ for every $f$ in $A$. The **essential set** of $A$ is the $X$ hull of the largest closed ideal of $C(X)$ contained in $A$. It
is denoted by $E(A)$. If $E(A) = X$, then $A$ is called an essential algebra.

Examples 0.1.3. Let $X$ denote a compact subset of the complex plane $\mathbb{C}$.

(i) Define $A(X) = \{ f \in \mathcal{O}(X) : f \text{ is analytic in the interior of } X \}$. Then $A(X)$ is a function algebra on $X$. If $X = D = \{ z \in \mathbb{C} : |z| \leq 1 \}$, then $A(D)$ is called the disk algebra on the unit disk $D$. The restriction of $A(D)$ to the unit circle $T$ is called the disk algebra on the unit circle $T$.

(ii) Let $P(X)$ denote the uniform closure of all polynomials in $z$. Then $P(X)$ is a function algebra on $X$ and $P(X) \subseteq A(X)$.

If $A$ and $B$ are function algebras on compact Hausdorff spaces $X$ and $Y$ respectively, then we can construct function algebras on $X \times Y$, naturally associated with $A$ and $B$ namely the tensor product $A \otimes B$ and the slice product $A \# B$ of $A$ and $B$, which are defined as follows.

For $f \in A$ and $g \in B$, define $f \otimes g$ on $X \times Y$ by $(f \otimes g)(x,y) = f(x)g(y)$. Then $f \otimes g \in \mathcal{O}(X \times Y)$. The space of all finite linear combinations of functions of the type $f \otimes g$, $f \in A$, $g \in B$ is called the algebraic tensor product of $A$ and $B$ and is denoted by $A \otimes B$. In fact, $A \otimes B$ is a subalgebra of $\mathcal{O}(X \times Y)$ which separates the points of $X \times Y$ and contains the constant functions. The uniform closure of
A \otimes B in C(X \times Y) is called the **tensor product** of the function algebras A and B and is denoted by A \otimes B.

For function algebras A and B on X and Y, take

\[ A \# B = \{ f \in C(X \times Y) : f_y \in A, f_x \in B \text{ for all } x \in X \text{ and } y \in Y \}, \]

where for a fixed \( y \in Y, f_y(x) = f(x, y) \) for each \( x \in X \) and for a fixed \( x \in X, f_x(y) = f(x, y) \) for each \( y \in Y \).

It can be easily verified that \( A \# B \) is a closed subalgebra of \( C(X \times Y) \) and \( A \otimes B \subseteq A \# B \). Hence \( A \# B \) is a function algebra on \( X \times Y \). It is called the **slice product** of A and B.

We shall have several occasions to use the lemma given below in the chapters that follow.

**Lemma 0.1.4.** Let A and B be function algebras on X and Y respectively. Let E and F be closed subsets of X and Y. Then

1. \[ \left( A \otimes B \right|_{E \times F} = (A|_E \otimes B|_F) \right|_{E \times F} \]
2. \[ \left( A \# B \right|_{E \times F} \subseteq (A|_E \otimes B|_F) \right|_{E \times F}. \]

**Proof.** (i) It is clear that \( A \otimes B\|_{ExF} = A|_E \otimes B|_F \). Let \( f \in A \otimes B\|_{ExF} \). Then \( f = g\|_{ExF} \) for some \( g \in A \otimes B \). Therefore, there exists a sequence \( \{ g_n \} \) in \( A \otimes B \) such that \( g_n \rightarrow g \) uniformly on \( X \times Y \). Hence \( g_n\|_{ExF} \rightarrow f \). But \( g_n\|_{ExF} \subseteq A \otimes B\|_{ExF} = A|_E \otimes B|_F \subseteq (A|_E \otimes B|_F) \|_{E \times F} \) for each \( n \).

Hence \( f \in (A|_E \otimes B|_F) \|_{E \times F} \). Thus \( A \otimes B\|_{ExF} \subseteq (A|_E \otimes B|_F) \|_{E \times F} \).
Since the latter is closed, \( (A \otimes B)_{EF} \subseteq (A|_E \otimes B|_F)_S \).

Conversely, let \( f \in (A|_E \otimes B|_F)_S \). Then \( f = \sum_{i=1}^{n} g_i \otimes h_i \), where \( g_i \in (A|_E)_S \) and \( h_i \in (B|_F)_S \) for \( i = 1, 2, \ldots, n \). Thus for each \( i \), there exist sequences \( \{ \phi_{ik} \} \) in \( A \) and \( \{ \psi_{ik} \} \) in \( B \) such that \( \phi_{ik}|_E \rightarrow g_i \) and \( \psi_{ik}|_F \rightarrow h_i \) uniformly, as \( k \rightarrow \infty \). Therefore, for each \( i \), \( \phi_{ik} \otimes \psi_{ik}|_{EF} \rightarrow g_i \otimes h_i \) uniformly. Thus \( \sum_{i=1}^{n} (\phi_{ik} \otimes \psi_{ik})|_{EF} \rightarrow \sum_{i=1}^{n} g_i \otimes h_i = f \). But

\[
\sum_{i=1}^{n} (\phi_{ik} \otimes \psi_{ik})|_{EF} \in A \otimes B|_{EF} \subseteq A \otimes B|_{EF}.
\]

Hence \( f \in (A \otimes B|_{EF})_S \) and \( (A|_E)_S \otimes (B|_F)_S \subseteq (A \otimes B|_{EF})_S \).

Consequently, \( (A|_E)_S \otimes (B|_F)_S \subseteq (A \otimes B|_{EF})_S \).

(ii) Let \( f \in A \# B|_{EF} \). Then there is \( g \in A \# B \) such that \( f = g|_{EF} \). Fix \( x \in E \). Then \( g_x \in B \) and \( g_x|_F \in B|_F \). Thus \( f_x \in B|_F \subseteq (B|_F)_S \). This is true for each \( x \in E \). Similarly, for each \( y \in F \), \( f_y \in (A|_E)_S \). Hence \( f \in (A|_E)_S \# (B|_F)_S \) and \( A \# B|_{EF} \subseteq (A|_E)_S \# (B|_F)_S \). Thus we get

\[
(A \# B|_{EF})_S \subseteq (A|_E)_S \# (B|_F)_S.
\]

We do not know whether \( (A \# B|_{EF})_S = (A|_E)_S \# (B|_F)_S \) is true or not.

Our study is mainly concentrated on the Bishop and Šilov decompositions. For function algebras, these
decompositions appear in literature at many places. See, for example, [20], [29], [33], [48] and [51].

Definitions 0.1.5. (i) A subset $K$ of $X$ is said to be a set of antisymmetry or an antisymmetric set for a function algebra $A$ if whenever $f \in A$ and $f|_K$ is real-valued, then $f|_K$ is constant.

The collection of all maximal sets of antisymmetry for $A$ forms a decomposition of $X$ [29, Lemma 3, p.38]. It is called the Bishop decomposition for $A$ and is denoted by $\mathcal{K}(A)$.

(ii) A set of constancy of $A_r$ is called a Šilov set for $A$, where $A_r = A \cap C_r(X)$.

The collection of all maximal Šilov sets for $A$ is clearly a decomposition of $X$, called the Šilov decomposition for $A$. We denote it by $\mathcal{F}(A)$.

It is clear from the definitions that $\mathcal{K}(A) \subset \mathcal{F}(A)$.

Next, we define certain ideas for a subspace of $C(X)$.

Definition 0.1.6. A closed subspace $A$ of $C(X)$ which contains constants is called a function space on $X$.

Now onwards, $A$ denotes a function space on $X$. 
A closed subset $E$ of $X$ is called a **closed restriction set** (CR set) for $A$ if $A|_E$ is closed in $\mathcal{C}(E)$. $E$ is called an **interpolation set** for $A$ if $A|_E = \mathcal{C}(E)$. Let $\mathcal{M}_{\mathcal{X}}$ denote the set of all regular Borel measures on $X$. Then the annihilator of $A$ is the set $A^\perp = \{ \mu \in \mathcal{M}_{\mathcal{X}} : \int_X fd\mu = 0 \text{ for all } f \in A \}$.

**Definitions 0.1.7.** Let $A$ be a function space on $X$ and $F$ be a closed subset of $X$.

(i) $F$ is called a **peak set** for $A$ if there exists $f \in A$ such that $f|_F = 1$ and $|f(x)| < 1$ for every $x \in X-F$. The intersection of peak sets is called a **generalized peak set** for $A$.

(ii) $F$ is called a **$p$-set** for $A$, if $\mu \in A^\perp \Rightarrow \mu |_F \in A^\perp$, where $\mu |_F (G) = \mu (F \cap G)$ for every Borel subset $G$ of $X$.

**Remarks 0.1.8.** (i) If $F$ is a $p$-set for $A$, then $F$ is a CR set for $A$ [29, p.188].

(ii) It is proved in [13, Proposition 1.5] that a $p$-set for function space $A$ is a generalized peak set for $A$. But a generalized peak set may not be a $p$-set for a function space [13, p.12].

(iii) If $A$ is a function algebra on $X$, then $F$ is a $p$-set for $A$ if and only if $F$ is a generalized peak set for $A$ [29, Theorem 40, p.190].
Remark 0.1.9. If $A$ and $B$ are function spaces on $X$ and $Y$ respectively, then we can define $A \ast B$ and $A \cdot B$ exactly as we have defined for function algebras. Also, it can be checked that Lemma 0.1.4 remains true in this case.

Finally, we define some properties of a decomposition of $X$ which are associated with $A$.

Definitions 0.1.10 [20, p.2]. Let $A$ be a function space on $X$ and $\mathcal{S}$ be a decomposition of $X$.

(i) We say that $\mathcal{S}$ has the (D)-property for $A$ if $f \in \mathcal{C}(X)$ and $f|_E \in (A|_E)^\sigma$ for every $E \in \mathcal{S}$ implies that $f \in A$, where $(A|_E)^\sigma$ denotes the uniform closure of $A|_E$ in $\mathcal{C}(E)$.

(ii) We say that $\mathcal{S}$ has the (E)-property for $A$ if, whenever $F$ is a $p$-set for $A$ and is saturated with $\mathcal{S}$, then $\mathcal{S} \cap F$ has the (D)-property for $A|_F$.

(iii) We say that $\mathcal{S}$ has the (GA)-property for $A$ if, whenever $\mu \in \mathcal{B}(A^\perp)^e$, then $\text{supp} \mu \subset E$ for some $E \in \mathcal{S}$, where $\mathcal{B}(A^\perp)^e$ denotes the set of extreme points of the unit ball $\mathcal{B}(A^\perp)$ of $A^\perp$.

Remarks 0.1.11. (i) For a decomposition $\mathcal{S}$ of $X$, we have (GA)-property $\Rightarrow$ (S)-property $\Rightarrow$ (D)-property [20, Theorem 1.2 and p.14].
(ii) If \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are two decompositions of \( X \) such that \( \mathcal{S}_1 \prec \mathcal{S}_2 \) and if \( \mathcal{S}_1 \) has any one of the above properties for \( A \), then \( \mathcal{S}_2 \) also has the same property for \( A \).

Finally, we define a real function algebra and a vector function space on \( X \), since we shall be discussing decompositions for them.

Let \( X \) be a compact Hausdorff space and \( \tau : X \to X \) be a homeomorphism on \( X \) such that \( \tau \circ \tau \) is the identity map on \( X \). Then \( \mathcal{C}(X, \tau) = \{ f \in \mathcal{C}(X) : f(x) = \overline{f(\tau(x))} \text{ for all } x \in X \} \) is a commutative real Banach algebra with identity.

**Definition 0.1.12.** A subalgebra \( A \) of \( \mathcal{C}(X, \tau) \) is called a **real function algebra on** \((X, \tau)\) if

(i) \( A \) is uniformly closed;

(ii) \( A \) contains (real) constants and

(iii) \( A \) separates the points of \( X \).

Let \( X \) be a compact Hausdorff space and \( B \) be a commutative Banach algebra with identity. Let \( \mathcal{C}(X; B) \) denote the algebra of continuous, \( B \)-valued functions on \( X \). Then \( \mathcal{C}(X; B) \) is a commutative Banach algebra with identity under pointwise operations and the norm given by

\[
||f|| = \sup \left\{ ||f(x)||_B : x \in X \right\}, \ f \in \mathcal{C}(X; B).
\]
Definition 0.1.13. (i) A vector function space on \( X \) is a closed subspace of \( C(X; \mathbb{B}) \) which contains vector constants.

(ii) A vector function algebra on \( X \) is a closed subalgebra of \( C(X; \mathbb{B}) \) which contains vector constants and separates the points of \( X \).

2. Summary

We study those decompositions of \( X \) which are related to subalgebras and subspaces of \( C(X) \). The most well known decomposition of \( X \) in this direction is the Bishop decomposition for function algebras. One important question in the theory of function algebras is: 'When is a function algebra \( A \) on \( X \) equal to \( C(X) \) ?' ([7], [21], [47], [57]). Bishop has given one characterization of \( A \) to be equal to \( C(X) \) in terms of the Bishop decomposition (i.e., the (D)-property) which is a generalization of the Stone-Weierstass theorem. Since then the Bishop decomposition has become an indispensable tool for proving many results for function algebras. Earlier, in 1951, Šilov has introduced a decomposition of \( X \), the Šilov decomposition for a function algebra. Certain properties of these two decompositions are studied by different authors, [5], [18], [20], [33], [48], [51], and [56]. We also concentrate on the Bishop and Šilov decompositions and study them for function algebras, function spaces, real function algebras and vector-valued function spaces.
In the first chapter, we begin by proving certain basic properties of the Šilov decomposition which are known for the Bishop decomposition for function algebras. In most of the well known function algebras, these two decompositions coincide. However, there are function algebras for which these decompositions differ ([17], [49]). We discuss certain conditions under which these two decompositions are the same. The main result, we have proved, is that the Šilov decomposition is the finest u.s.c. decomposition with the (D)-property for a function algebra. Then we discuss the Bishop and Šilov decompositions for restriction algebras. We have also introduced other decompositions of $X$ corresponding to the ideas of integral domain and analytic algebra. The decompositions corresponding to weakly analytic sets and weakly prime sets were defined earlier by Arenson [2] and Ellis [12]. We compare and study all these decompositions in the last section. In particular, we discuss these decompositions for the tensor product.

In 1973, Ellis [10] has defined and discussed the Bishop and Šilov decompositions for $A(K)$, the space of real-valued affine functions on a compact convex set $K$. Also, Păltineanu [44] has defined the decomposition of $X$ into maximal antialgebraic sets for a subspace of $\mathbb{C}_r(X)$. Later, in 1984, Feyel and Pradelle [15] have defined the Bishop decomposition for a subspace of $C(X)$. Then Edwards [9]

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has considered the decomposition of $X$ into maximal antisymmetric sets which is defined with the help of the multiplier of a function space. Recently, Yamaguchi and Wada [58] have defined the Bishop decomposition for a point separating function space on $X$ which is in fact the same as defined by Feyel and Pradelle. In the second chapter, we compare and study, in detail, the Bishop decompositions defined by several authors ([9], [10], [15], [44], [58]). We have defined the Šilov decomposition for a function space and generalized some of the results of section 1 of the first chapter. The principal result states that the Bishop decomposition defined by Feyel and Pradelle is the finest Hausdorff decomposable decomposition of $X$ with the (S)-property for a function space. This generalizes a result of Hayashi [20, Corollary 4.2].

In chapter 3, the decompositions of $X$ for real function algebras have been discussed. The basic ideas of real function algebras can be found in [25], [30] and [31]. It is possible to associate a complex function algebra with a given real function algebra by complexifying it. This technique of complexification is often employed to study the properties of a real function algebra. A good deal of work has been done in the field of real function algebras ([24], [25], [26], [27], [28], [32]). In [26], Kulkarni and Srinivasan have defined the Bishop decomposition for real function algebras. We introduce the Šilov decomposition for
real function algebras and prove some basic properties of it. We also give a characterization of generalized peak sets in terms of annihilating measures analogous to the known one for complex function algebras [29, Theorem 40, p.190]. Finally, we extend the ideas of the Bishop and Šilov decompositions for real function spaces.

The last chapter has been devoted to the study of the decompositions for vector function spaces on X. We define the Bishop and Šilov decompositions in several ways for vector function spaces. If A denotes a complex function space on X then the tensor product $A \hat{\otimes} B$ of A and Banach algebra B can be regarded as a vector function space on X. The concept of the slice product $A \# B$ of a function algebra A with a Banach algebra B has been defined earlier [37]. We extend the idea of $A \# B$ for a complex function space A on X. Then $A \# B$ also can be considered as a vector function space on X and $A \hat{\otimes} B \subset A \# B$. Mainly, we concentrate our study to the decompositions and their properties for vector function spaces of the types $A \hat{\otimes} B$ and $A \# B$, where A denotes a complex function space on X. In particular, we show that the Bishop (Šilov) decompositions for a complex function space A and the vector function space $A \hat{\otimes} B$ are the same.

The results of section 1 of chapter 1 have appeared in [34]. The results in chapter 2 have been communicated for publication [36] while most of the results of chapter 3 will appear in [35].