CHAPTER 3  
GOODNESS-OF-FIT-TEST

3.1. INTRODUCTION.

The assumption of exponentiality which is common in Life testing and Reliability studies is analogous to the normality assumption in the analysis of variance of linear models in that both assumptions lead to simplicity of theory and computations. However, the methods based upon exponentiality enjoy less robustness than the normal techniques. The problem of testing exponentiality is therefore important and the literature on the subject is substantial.

In Reliability work it is often of interest to determine whether a certain item has constant failure rate versus either increasing or decreasing failure rate. The type of failure rate will effect such things as replacement policies and inspection schedules.

In this connection many tests have been developed to determine whether a given set of distribution fits the exponential distribution. This includes graphical and analytical procedures. So in this chapter our sole aim is to give a wider exposure to the goodness-of-fit tests sometimes discarding the form of the alternative hypothesis. We tried to obtain as many papers as we can. Due to the nonavailability of papers some of the results may be quoted as such.
Main references like Benjamin Epstein (1960), Lin and Mudholkar (1980), Gail and Gastwirth (1978) and Fercho and Ringer (1972) are widely used for the preparation of this chapter. Other references include Gastwirth (1972), Shapiro and Wilk (1972), Hartly (1950), D.S. Moore (1968) etc.

The entire chapter is divided into six sections. Sec. 3.2. motivates to test whether or not one is justified in assuming that the underlying distribution of life is exponential. A number of procedures including graphical procedure has been described in this section. A test of exponentiality based on bi-variate F-distribution is described in Sec. 3.3. Some modifications are also shown in this section. Sections 3.4. and 3.5. describes respectively, the Scale-free goodness-of-fit test for the exponential distribution based on the Lorenz curve and Gini statistics. Both of the tests have been described in detail. The comparisons based on these tests have been carried out in the last section.

3.2. EPSTEIN 's PROCEDURES.

In this section we will be discussing seven different tests of exponentiality against different alternative hypothesis depicting violation of exponentiality. These tests are due to Epstein (1960). First we begin with a graphical procedure.
1. A GRAPHICAL PROCEDURE.

Suppose the underlying distribution is exponential with mean \( \theta \). Then the c.d.f. is given by

\[
F(x) = \begin{cases} 
0 & , x < 0 \\
1 - \exp(-x/\theta) & , x \geq 0 , \theta > 0.
\end{cases}
\]

Then \( \log \left[ \frac{1}{1-F(x)} \right] = x/\theta \). Thus \( -\log (1-F(x)) \) is a straight line with slope \( 1/\theta \). This suggests the following graphical procedure.

Suppose we place \( n \) items on life test and let \( x_1 < x_2 < \ldots < x_n \) be the times when the items fail.

Since \( E[F(x_{(i)})] = i/(n+1) \), then we can take the value of \( F(x_{(i)}) \) associated with \( x_{(i)} \) as \( F(x_{(i)}) = i/(n+1) \).

If we plot the graph \( -\log \left[ \frac{1}{1-F(x)} \right] \) against \( x_{(i)} \) and if the exponential assumption holds then the plotted points can be fitted well by a straight line passing through the origin. If the experimentation is discontinuous at time \( x_r \), \( r < n \), when the \( r \)th failure occurs, then we expect a good linear fit up to \( x_r \). This procedure seems to be quick way of checking exponentiality, however, it is recommended when we have large number of observations.

Remark. It may happen that the underlying density function of life is the two-parameter exponential with p.d.f.

\[
f(x; \theta, A) = \begin{cases} 
\frac{1}{\theta} \exp \left[ -\frac{(x-A)}{\theta} \right] , & x \geq A \geq 0 \\
0 & , \text{elsewhere.}
\end{cases}
\]

Then if we plot the graph \( -\log \left[ \frac{1}{1-F(x_{(i)})} \right] \) against \( x_{(i)} \), instead...
of passing through the origin, the straight line cuts the x-axis at the point \( x = A \).

2. THE \( \chi^2 \) TEST FOR GOODNESS-OF-FIT TEST.

Let \( \hat{\theta} \) be the maximum likelihood estimate of \( \theta \) based on \( n \) failures and let the time axis be divided into \( k \) intervals by the \((k-1)\) times \( 0 = t_0 < t_1 < t_2 < \ldots < t_{(k-1)} < t_k = \infty \).

Furthermore, let \( o_i \) be the observed number of failures in the interval \( [t_{(i-1)}, t_i) \) and let \( e_i \) be the expected number of failures in the \( i^{th} \) interval, where \( e_i = n \frac{1}{\theta} \int_{t_{(i-1)}}^{t_i} e^{-t/\theta} \, dt \), \( i = 1, 2, \ldots, k \) (3.2.1)

On the basis of this definition we can define the \( \chi^2 \) statistic as

\[
\chi^2 = \sum_{i=1}^{k} \left( \frac{(o_i - e_i)^2}{e_i} \right) \tag{3.2.2}
\]

The statistic on the right hand side of (3.2.2) is distributed as \( \chi^2_{(k-2)} \), if the same size (number of failures) is large.

3. A CRITERION BASED ON THE CONDITIONAL DISTRIBUTION OF TOTAL LIVES.

This test makes use of the basic properties of Poisson
Process. If one observes a Poisson process for a fixed length of time $T$ and if $r$ events occur in $[0,T]$ at times $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)} \leq T$, then these times can be considered as ordered observations on a random variable uniformly distributed over $(0,T)$. The explanation is as follows.

Let $X_{(i)}$ = time when the $i^{th}$ event occurs. Then

$$P(x_{(1)} < X < x_{(1)} + \Delta x_{(1)} , \ldots , x_{(r)} < X < x_{(r)} + \Delta x_{(r)} \, / \, r \text{ events occur in } (0,T))$$

$$= \lambda^r e^{-\lambda T} \prod_{i=1}^{r} \frac{\Delta x_{(i)} e^{-\lambda T}}{r!}$$

$$= \frac{(r! / T^r)}{\prod_{i=1}^{r} \Delta x_{(i)} e^{-\lambda T}} , x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)} .$$

Hence the times $\{x_{(i)}\}$ observations on $X$ which is uniformly distributed in $[0,T]$.

For $r$ even moderately large $\sum_{i=1}^{r} X_{(i)} / r$ is approximately normally distributed with mean $T/2$ and variance $T^2/(12r)$. This can be used to as large sample test to see whether or not the data are drawn from a Poisson Process.

One can also show that if one observes a Poisson Process until exactly $r$ events occur ($r$ is a preassigned integer) and if the events occur at $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$, then the $(r-1)$ random variables $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r-1)}$ can be considered when ordered.
observations from $X$, is uniformly distributed over $[0,x_{(r)}]$ for $X_{(r)} = x_{(r)}$. The explanation for this is given below.

Let $X_{(i)} = \text{time of occurrence of the } i^{th} \text{ event, then}$

\[ PC_{x_{(i)}} < X < x_{(i)} + \Delta x_{(i)} , \quad x_{(2)} < X < x_{(2)} + \Delta x_{(2)}, \quad \ldots, \]

\[ x_{(r-1)} < X_{(r-1)} < x_{(r-1)} + \Delta x_{(r-1)} < x_{(r)} < x_{(r)} + \Delta x_{(r)} \]

\[
= \lambda^r e^{-\lambda x_{(r)}} \prod_{i=1}^{r-1} \frac{\Delta x_{(i)}}{(r-1)!} \lambda x_{(r)} = (r-1)! \prod_{i=1}^{r-1} \frac{\Delta x_{(i)}}{x_{(i)}} , \quad x_{(2)} \leq x \leq \ldots \leq x_{(r)}. \]

(3.2.4)

Hence the times $x_{(1)}, x_{(2)}, \ldots, x_{(r-1)}$ ordered observations from $X$, are uniformly distributed over $[0,x_{(r)}]$.

In the context of life testing if the failed items are not replaced then all we need to do is to use the total lives $S_i$, where

\[ S_i = \sum_{j=1}^{i} D_j \]

(3.2.5)

where, \[ D_i = (n-i+1) (x_{(i)} - x_{(i-1)}) , \quad i = 1, 2, \ldots, n. \]

In literature the statistics $D_i$'s are known as the normalized spacings.

If $S_i$ is the total life observed in getting the $i^{th}$ failure, then $S_1 \leq S_2 \leq \ldots \leq S_r$. In the case where the life starts with $n$ items and is terminated at a preassigned total life $T^*$, then the
number of failures observed will be a random variable \( r \) with

\[
S_1 = n X_1 = D_1 \\
S_2 = X_1 + (n-1) X_2 = D_1 + D_2 \\
\ldots \\
S_r = X_1 + X_2 + \ldots + X_{(r-1)} + (n-r+1) X_r = \sum_{i=1}^{r} D_i.
\]

Here also one can show that the total lives \( S_1, S_2, \ldots, S_r \) can be considered as being drawn from a density function which is uniform over \([0,T^*]\). If the life test ends as soon as the first \( r \) failures occur, then the \((r-1)\) random variables \( S_1, S_2, \ldots, S_{r-1} \) can be considered as ordered statistics corresponding to random sample population which is uniform over \([0,S_r]\).

The fact that the conditional distribution of total lives is uniform over suitable intervals makes it quite evident that one has a good tool for detecting whether the failure rate is indeed constant. Thus, the contamination of a purely exponential distribution by early failures would manifest itself in a pronounced tendency to get too many failures clustering together in the early part of the time or total life, thus violating uniformity. If the failure rate changes, for example, increases with time then this should result in a tendency for failures to cluster together as time goes on, again violating uniformity. If the amount of failure data observed is quite small, then we can expect a large changes from exponentiality. Otherwise one can use
a $\chi^2$ test to detect whether the conditional distribution of times
to failure or total lives deviates excessively from being uniform.

4. A TEST FOR ABNORMALLY EARLY FAILURES.

Suppose that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)}$ are the first $r$
failures. If all the $x_{(l)}$ are drawn from a common exponential,
then $S_1$, the total life in $[0, x_{(1)}]$ and $S_r - S_1$, the total life
in $[x_{(l)}, x_{(r)}]$ are distributed independently of each other, where
$2S_1/\theta$ is distributed as $\chi^2_{(2)}$ and $2(S_r - S_1)/\theta$ is distributed
as $\chi^2_{(2r-2)}$ degrees of freedom each. Therefore, $(r-1)S_1/(S_r - S_1)$
is distributed as $F_{(2, 2r-2)}$. If the ratio is too small then we
assert that $x_{(1)}$ is abnormally small. More precisely, if $\alpha$ is the
significance level of our test, we will say $x_{(1)}$ is abnormally
small if $S_1 < F_{(2, 2r-2)}^{-1}(1-\alpha)$. If we suppose we wanted to
detect both $x_{(1)}$ and $x_{(2)}$ are
abnormally small and if all the $x_{(l)}$ are drawn from a common
exponential distribution, then the total life in $[0, x_{(2)}]$ and the
total life in $[x_{(2)}, x_{(r)}]$ are respectively $S_2$ and $(S_r - S_2)$ are
independently distributed of each other. Then if the ratio $(r-2)S_2/
2(S_r - S_2)$ which is distributed as $F_{(4, 2r-4)}$ is too
small, we may assert that both $x_{(1)}$ and $x_{(2)}$ are abnormally
small.
5. A TEST FOR WHETHER OR NOT THE MEAN LIFE FLUCTUATES DURING THE LIFE TEST.

Consider a group of $k$ life times each of size $r$ as

$$x_{(1)r} < x_{(2)r} < ... < x_{(r)r}.$$  

Assuming that the mean life $\theta$ is constant within each group, then our object is to test whether or not $\theta$ fluctuates from group to group.

The total lives in $[0, x_{(1)r}), [x_{(1)r}, x_{(2)r}), ..., [x_{(k-1)r}, x_{(k)r})$ are respectively $S_{1r}, S_{2r} - S_{1r}, ..., S_{kr} - S_{(k-1)r}$. Under the null hypothesis these observations are independantly and identically distributed exponential with parameter $\theta$. Furthermore,

$$2. \left[ S_{jr} - S_{(j-1)r} \right] / \theta$$

is for each $j = 1, 2, 3, ..., k$ distributed as $\chi^2_{(2kr)}$.

Now let us use a test devised by Neyman and Pearson (1931) for testing equality of variance of $k$ population which may have different unspecified means $\mu_i, i = 1, 2, ..., k$. Let $x_{ij}, i = 1, 2, ..., k$ be a random sample of size $n_i$ from the population $i = 1, 2, ..., k$. Let the $i^{th}$ sample mean and sample variance be respectively

$$\bar{x}_i = \frac{\sum_{j=1}^{n_i} x_{ij}}{n_i}, \text{ and } \quad S_i^2 = \frac{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2}{n_i}.$$
Further the combined sample variances under the null hypothesis of equality of variances is,

\[ S^2_a = \sum_{i=1}^{k} n_i S^2_i / N, \quad \text{where} \quad N = \sum_{i=1}^{k} n_i. \]

then the test proposed by Neyman rejects the null hypothesis for sufficiently small values of \( \lambda \),

\[ \lambda = \prod_{i=1}^{k} \left( \frac{S^2_i}{S^2_a} \right)^{n_i/2}. \]

It can be seen that if the populations are assumed to be normal then \( \lambda \) reduces to the likelihood ratio statistics, \( L_1 \).

Then \( \lambda^{2/N} \), the \( L_1 \) statistic of Neyman and Pearson is given by

\[ L_1 = \prod_{i=1}^{k} \left( \frac{S^2_i}{S^2_a} \right)^{1/k}, \quad \text{since} \quad N = kn. \quad (3.2.6) \]

If suppose \( n = 2r+1 \) are drawn from each of \( k \) normal populations and we wish to test for the equality of variances.

The \( L_1 \) test is thus obtained by replacing \( S^2_i \) by \( (S_{ir} - S_{(i-1)r}) \^2 \)
and replacing \( S^2_a \) by \( S_{kr} / k \).

Thus \( L_1 \) becomes,

\[ L_1 = \prod_{i=1}^{k} \left[ \frac{(S_{ir} - S_{(i-1)r})^{1/k}}{S_{kr} / k} \right]. \]

therefore, \( \log L_1 = (1/k) \sum_{i=1}^{k} \log \left[ \frac{S_{ir} - S_{(i-1)r}}{S_{kr} / k} \right] - \log (s_{rk}) \)

(3.2.7)
An analogous test is given by Bartlett (1937), which is suitable for most practical purposes by defining

\[ \mu^{2/v} = \prod_{i=1}^{k} \left[ \frac{\sum_{i=1}^{k} v_i s_i^2}{n_i v_i} \right]^{v_i/v_n} \text{, where } v_i = n_i - 1 \]

and \( v = \Sigma v_i \), then \((-2\log \mu)/c\), where

\[ c = 1 + \frac{1}{3(k-1)} \left\{ \Sigma_{i=1}^{k} \left( \frac{1}{v_i} - \frac{1}{v} \right) \right\} \]

is approximated well by the \( \chi^2_{(k-1)} \) distribution. In the life testing context, \( n_i = 2r+1 \), \( v_i = 2r \) for each \( i \) and \( v = \Sigma v_i = 2rk \). Hence \( c \) becomes,

\[ c = 1 + \frac{1}{3(k-1)} \left\{ k/2r - 1/2rk \right\} = 1 + \frac{(k+1)}{6rk} \]

Further, \( \mu^{2/v} = \mu^{1/rk} = L_1 \). Hence \((-2\log \mu)/c\) becomes \((-2rk \log L_1)/c\) which is approximated well by the \( \chi^2_{(k-1)} \) distribution. i.e.

\[ -2rk \left( \frac{1}{k} \sum_{i=1}^{k} \log (S_{ir} - S_{(i-1)r}) - \log (S_{rk}/k) \right) / \left[ 1 + \frac{k+1}{6rk} \right] \]

\[ \sim \chi^2_{(k-1)} \]

i.e.

\[ \frac{2rk \left[ \log S_{rk}/k - \left( \frac{1}{k} \right) \sum_{i=1}^{k} \log (S_{ir} - S_{(i-1)r}) \right]}{1 + \frac{k+1}{6rk}} \sim \chi^2_{(k-1)} \]

let us call it by "E"  \[(3.2.8)\]

The hypothesis that \( \theta \) is the same over the entire group is rejected if \( E \) exceeds \( \chi^2_{\alpha, (k-1)} \) is the upper \( \alpha^{th} \) percentile of
chi-square distribution with \(( k-1)\) d.f.

If we put \( r = 1 \) in (3.2.8) we get

\[
E_p = \frac{2k \left\{ \frac{\log S}{k} - \frac{1}{k} \sum \log (S_i - S_{i-1}) \right\}}{1 + \frac{k+1}{5k}} \\
\sim \chi_{(k-1)}^2.
\]  

(3.2.9)

6. A TEST BASED ON THE MAXIMUM F-DISTRIBUTION.

Consider the same situation as described in the above test where the ordered failures are arranged in \( k \) groups of size \( r \). Then a test for homogeneity (equivalently the test for exponentiality) is obtained by

\[
U = \frac{\max \left\{ \sum_1^r (S_i - S_{i-1}), (S_i - S_{i-1}), \ldots, (S_k - S_{(k-1)r}) \right\}}{\min \left\{ \sum_1^r (S_i - S_{i-1}), (S_i - S_{i-1}), \ldots, (S_k - S_{(k-1)r}) \right\}},
\]

(3.2.10)

another most useful and a quick test for homogeneity based on the results of Hartley (1950). Here one should reject the hypothesis of homogeneity if \( U \) is too large. Hartley (1950) has given 5% points for \( U \) for various values of \( k \) and \( r \) and has made it plausible that the test is almost as efficient as the Bartlett homogeneity.

7. A TEST BASED ON CONDITIONAL RATE OF FAILURE.

Here an adhoc test is discussed based on failure rate. A
characteristic property of the exponential distribution is that
the conditional probability of failing in the interval \((x, x+\Delta x)\)
given that it has survived up to time \(x\) is independent of \(x\). Let us consider
\[
 f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \quad \theta > 0 \quad \text{and therefore,}
\]
\[
 F(x) = F(x; \theta) = 1 - e^{-x/\theta}, \quad x > 0.
\]
For this case, the conditional probability of failing in
\((x, x+\Delta x)\) given survival in \((0, x)\) is given by,
\[
\frac{f(x) \Delta x}{1 - F(x)} = \frac{(1/\theta) e^{-x/\theta}}{e^{-x/\theta} \theta} = \frac{\Delta x}{\theta} \quad (3.2.11)
\]
The practical importance of this remark is that if we start
with a large number \(N\) of items on life test, divide the axis into
intervals \((0, x), (x, 2x), (2x, 3x), \ldots\) and if \(n_1, n_2, \ldots, n_k\) are the
number of items failing in these intervals then
\[
\frac{n_1}{N}, \frac{n_2}{N-n_1}, \frac{n_3}{N-n_1-n_2}, \ldots, \frac{n_k}{N-n_1-n_2-\ldots-n_{k-1}}
\]
should fluctuate within reasonable limits about a constant value,
namely the failure rate. If this is not true then there may be
violation of the assumption of exponentiality.

3.3. GNEDENKO'S TEST FOR EXPONENTIALITY AND ITS MODIFICATIONS.

In this section we shall discuss Gnedenko's test for
exponentiality based on normalized spacings and also the
modifications of Gnedenko's test.
As usual let \( x_1, x_2, \ldots, x_n \) be a random sample from a population with density function \( f(x) \) and the null hypothesis under consideration

\[ H_0 : f(x) = e^{-\theta x}, \quad x \geq 0, \quad \theta > 0, \]

where \( \theta \) is unspecified.

Then the normalized spacings are

\[ D_i = (n-i+1) \left( x_{(i)} - x_{(i-1)} \right), \quad i = 1, 2, \ldots, n, \quad (3.3.1) \]

\( x_{(0)} = 0 \) and \( x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)} \) are the order statistics. Now we study the distribution property of these normalized spacings.

Under \( H_0 \), for \( i = 1, 2, \ldots, n \) we get,

\[
\begin{align*}
D_1 &= n x_{(1)} \\
D_2 &= (n-1) \left( x_{(2)} - x_{(1)} \right) \\
D_3 &= (n-2) \left( x_{(3)} - x_{(2)} \right) \\
& \quad \vdots \\
D_n &= (x_{(n)} - x_{(n-1)}) .
\end{align*}
\]

Summing all these equations we get

\[
\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} x_{(i)}.
\]

The inverse transformations are

\[ x_{(i)} = \sum_{j=1}^{i} \frac{D_j}{n-j+1}, \quad i = 1, 2, \ldots, n \]

Therefore the Jacobian transformation is
Hence the joint distribution of $D_1, D_2, \ldots, D_n$ is,

$$L(D_1, D_2, \ldots, D_n ; \theta) = L(x ; \theta) | J |$$

$$= n! \theta^n \exp \left( -\theta \sum_{i=1}^{n} D_i \right) \frac{1}{n!}$$

$$= \prod_{i=1}^{n} \theta^{-D_i / \theta}$$

$$= 1 / \Gamma(n)$$

Which implies that $D_i$'s are i.i.d. exponential distribution with scale parameter $\theta$.

Based on these normalized spacings Gnedenko (1960) has proposed an F-test as follows.

Divide the $n$ normalized spacings into two groups with one containing first $r$ points and the remaining $n-r$ points in the other group. Then the proposed statistics based on these two groups is given by

$$Q(r) = \frac{\sum_{i=1}^{n} D_i / r}{\sum_{i=1}^{n} D_i / (n-r)}$$

where the point $r$ is completely arbitrary.

Since under $H_0 : D_i$'s are i.i.d. exponential with scale parameter $\theta$, $\sum D_i$ is a gamma distribution with parameter $\theta$ and $r$. 

\[ 72 \]
Therefore \( \sum_{i=1}^{r} D_i / \theta \) is gamma distribution with parameter \( r \) and

\[ 2 \sum_{i=1}^{r} D_i / \theta \] is a chi-square distribution with \( 2r \) degrees of freedom.

i.e. \[ 2 \sum_{i=1}^{r} D_i / \theta \sim \chi^2_{2r} \] \hspace{1cm} (3.3.4)

Similarly the distribution of \( 2 \sum_{i=r+1}^{n} D_i / \theta \) is a chi-square with \( 2(n-r) \) degrees of freedom.

i.e. \[ 2 \sum_{i=r+1}^{n} D_i / \theta \sim \chi^2_{2(n-r)} \] \hspace{1cm} (3.3.5)

Thus \( Q(r) \) is the ratio of two independent \( \chi^2 \) variates which are respectively divided by their corresponding degrees of freedom follows an F distribution with \([2r, 2(n-r)]\) degrees of freedom.

The hypothesis is rejected for both small and large values of \( Q(r) \). Fercho and Ringer (1972) recommended setting \( r = \left[ n/2 \right] \) and claim that the test is well suited for Weibull and Gamma alternatives with monotone hazard rates.

Other modifications of the tests are given by Haris (1976) and Fercho and Ringer (1972).

### 3.3.3 Haris' Modifications of Onedenko's F-Test.

This test was proposed by Haris (1975) and was discussed by Lin and Mudholkar (1980).
Here the entire range of observations are divided into three parts. The distributions of the first $r$ points and the last $r$ points are combined together in the numerator and the remaining $n-2r$ points are taken in the denominator. Thus the test statistics is given by

$$Q'(r) = \frac{\left( \sum_{i=1}^{r} D_i + \sum_{i=n-r+1}^{n} D_i \sqrt{2r} \right)}{\sum_{i=r+1}^{n-r} D_i / (n-2r)}$$

and its distribution under $H_0$ is $F$ with $[4r, 2(n-2r)]$ degrees of freedom. The hypothesis is rejected for both small and large values of $Q'(r)$. This procedure is claimed to be powerful against the log-normal distribution and inferior for monotone hazards. Also it has been seen that a good power could be achieved if $r = n/4$.

3.3.2. Bi-Variate F-Test of Exponentiality.

Let $D_i$'s are the normalized spacings defined in (3.3.1). These $D_i$'s are natural occurrences in Reliability and are the times between the successive failures of a system.

Pyke (1965) shows that in large samples the spacings may be regarded as approximately independant exponential random variables with in general different scale parameters.

If $f(x)$ is the density function of $F(x)$ and for $0 < u < v < 1$ $s = F^{-1}(u)$, $t = F^{-1}(v)$, $i/n \rightarrow u$, $j/n \rightarrow v$ as $n \rightarrow \infty$ then
\[ \lim_{n \to \infty} F_{b_i, b_j}^{d_i, d_j}(x, y) = \left[ 1 - \exp \left( -\frac{f(s)}{1-u} x \right) \right] \left[ 1 - \exp \left( -\frac{f(t)}{1-v} y \right) \right] \]  

(3.3.7)

Now observe that \( \frac{f(s)}{(1-u)} = \frac{f(F^{-1}(u))}{(1-F(F^{-1}(u)))} \)  

\[ = r(F^{-1}(u)) \]

where \( r(x) = f(x) / (1-f(x)) \) is the hazard rate function of \( F \). Therefore in large samples \( D_i \) and \( D_j \) are approximately independent and exponentially distributed with hazard rate function evaluated as \( F^{-1}(u) \) and \( F^{-1}(v) \) as the respective scale parameters.

Thus, for \( 0 < u < v < 1 \) and \( i/n \to u, j/n \to v \) as \( n \to \infty \), \( D_i \) is smaller than \( D_j \) if \( r(F^{-1}(u)) \) is smaller than \( r(F^{-1}(v)) \). This relationship between the shape of the hazard rate function and the values of the spacings may be used to understand the nature of both tests of exponentiality.

Let us now consider an alternative method of simultaneously guarding against diverse alternatives by separately comparing the upper sum (say \( S_u \)), lower sum (say \( S_l \)) and the middle sum (say \( S_m \)) respectively as

\[ S_l = \sum_{i=1}^{r} D_i, \quad S_u = \sum_{i=n-r+1}^{n} D_i, \quad S_m = \sum_{i=r+1}^{n-r} D_i \]  

(3.3.8)

Define,

\[ F_l = \frac{S_l}{S_m} \left( \frac{1}{n-2r} \right) \]  

(3.3.9)

\[ F_u = \frac{S_u}{S_m} \left( \frac{1}{n-2r} \right) \]  

(3.3.10)
Then under $H_0: F_l$ and $F_u$ follows a Bi-variate $F$-distribution. The hypothesis is rejected if either $F_l$ or $F_u$ falls outside $(a, b)$ where $a$ and $b$ are the critical constants for bi-variate $F$-test ($BF(r)$) may be determined by using the following theorem due to Hewitt and Bulgren (1971).

**THEOREM. 3.1.** Let $F_l$ and $F_u$ be defined as in (3.3.9) and (3.3.10) respectively, then for any $0 < a < b < \infty$,

$$P\left( a \leq F_l \leq b, a \leq F_u \leq b \mid H_0 \right) \leq \left[ P\left( a \leq BF(r) \leq b \right) \right]^2$$

(3.3.11)

where $BF(r)$ is the Snedecor's $F$ random variable with $2r$ and $2(n-2r)$ degrees of freedom. The cut off points $a$ and $b$ are determined by setting the right hand side of (3.3.11) equal to $1-\alpha$ where $\alpha$ is the level of significance and assuming equal tail probabilities of the $F$-distribution.

The power comparison of the above tests against some of the known alternatives are given at the end of this chapter. The importance of this study shows that some of the alternatives either resembles the exponential distribution or are frequently proposed as alternative to the exponential distribution model.

3.4. A SCALE-FREE GOODNESS OF FIT TEST FOR EXPONENTIAL DISTRIBUTION BASED ON LORENZ CURVE.

3.4.1. INTRODUCTION.

Consider the numbers $x_i$ as a sample drawn from the
distribution function $F(x)$. We assume that the mean $\mu$ of $F(x)$ exists and $F(x)$ is increasing on $[0,1]$. The first assumption implies that $F^{-1}(p)$ is well defined and is the population p\textsuperscript{th} quantile. Given any degree of freedom $F(x)$, the theoretical Lorenz curve corresponding to it is defined by

$$L(p) = \mu^{-1} \int_{0}^{p} F^{-1}(t) \, dt$$

(3.4.1)

This represents the fraction of the total variable measured (e.g., income) that the holders of the smallest p\textsuperscript{th} fraction possess.

Let us put $u = F^{-1}(t)$ in (3.4.1). Then $L(p)$ becomes

$$L(p) = \mu^{-1} \int_{0}^{F^{-1}(p)} u \, dF(u).$$

If we put the estimate of the corresponding parameters then we can derive the sample Lorenz curve. Therefore,

$$L_n(p) = \mu^{-1} \int_{0}^{np/n} u \, dF(u)$$

$$= \mu^{-1} \sum_{i=1}^{[np]} x_{(i)}/n$$

$$= \frac{1}{n} \sum_{i=1}^{[np]} x_{(i)} / \bar{x}$$

$$L_n(p) = \mu^{-1} \int_{0}^{p} F^{-1}(t) \, dt$$

(3.4.2)

where $[np]$ denotes the greatest integer less than or equal to np.

The most common measure of inequality, the Gini index $G$, is the ratio of the area between the Lorenz curve and the 45\textdegree line to the area under the 45\textdegree line (which is $1/2$).

An alternative formula for the Gini index $G$ is based on the mean difference, $\Delta$, of the underlying d.f.$F(x)$ and is given by
\[ \Delta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x - y| \, dF(x) \, dF(y) \]

\[ = 2 \int F(x) \left( 1 - F(y) \right) \, dx \]

\[ = 4 \int x \left( F(x) - 1/2 \right) \, dF(x). \]

To derive the sample Gini statistic we shall substitute the "good" estimates of \( \Delta \) and \( \mu \).

Let \( \phi(x, y) = |x - y| \), so that \( E[\phi(x, y)] = \Delta \), which can be treated as a kernel for U-statistics. Therefore \( \hat{\Delta} \), the estimate of \( \Delta \) can be obtained using the corresponding U-statistics as

\[ \hat{\Delta} = \frac{1}{n(n-1)} \sum_{i<j} |x_i - x_j| \]

and the estimate of \( \hat{\mu} \) is given by \( \bar{x} \). Then the sample Gini statistic can be obtained as

\[ G_n = \frac{1}{2} \sum_{i<j} \frac{|x_i - x_j|}{n^2} \]

(3.4.3)

It can be seen that the distribution of both \( L_n(p) \) and \( G_n \) are scale-free. Therefore it seems reasonable to use them as test statistics for testing

\[ H_0 : F(x) = 1 - e^{-\lambda x}, \quad x > 0, \quad \lambda > 0. \]

This has been done by Gail and Gastwirth (1978a, 1978b) which will be discussed in detail here.

Here we describe a goodness-of-fit test for exponential based on the empirical Lorenz curve, which provides a powerful, easily
computed test which does not depend on the unknown scale parameter.

Let \( x_1, x_2, \ldots, x_n \) be a realization of a sample of observations from a population with support on \([0, \infty)\) with ordered realization \( x_1 \leq x_2 \leq \ldots \leq x_n \). Let \( r = \lfloor np \rfloor \) denote the greatest integer less than or equal to \( np \). Then the sample Lorenz curve is defined in (3.4.2) as

\[
L_n(p) = \sum_{i=1}^{[np]} x_i / \sum_{i=1}^{n} x_i , \quad 0 < p < 1.
\]

Let \( \mu \) be the mean of the underlying population with distribution function \( F \) which is strictly increasing on its support, and let \( G \) be the unique inverse of \( F \). The corresponding population Lorenz curve is defined by,

\[
L(p) = \mu^{-1} \int_0^p x \, dF(x) = \mu^{-1} \int_0^p F(t) \, dt, \quad \text{since } F^{-1}(t) = x
\]

\[
= \mu^{-1} \int_0^p G(t) \, dt = \int_0^p G(t) \, dt = \eta(p) / \mu.
\]

If for an exponential distribution with c.d.f. \( F(x) = 1 - \exp(-\beta x) \), \( x \geq 0 \), we get

\[
L(p) = \beta \int_0^p -\frac{1}{\beta} \log(1-t) \, dt = \beta \int_0^p -\log(1-t) \, dt.
\]

\[ F(x) = x = 1 - e^{-\beta t} \Rightarrow e^{-\beta t} = 1 - t. \text{ i.e. } x = -\beta^{-1} \log(1-t) = G(t) \]
The table below gives the Lorenz curve generated by some of the common distributions.

Table 3.1.
THE LORENZ CURVES GENERATED BY SOME COMMON DISTRIBUTIONS.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>c.d.f.</th>
<th>Lorenz curve</th>
</tr>
</thead>
</table>
| Equal            | \[ F(x) = \begin{cases} 
\frac{x}{\mu}, & x < \mu \\
1, & x \geq \mu 
\end{cases} \] | \( p \)                           |
| Exponential      | \[ F(x) = 1 - e^{-\lambda x}, \ x > 0 \] | \( p + (1-p) \log(1-p) \)       |
| Shifted exponential | \[ F(x) = 1 - e^{-\lambda(x-a)}, \ x > a \] | \( p + \frac{1}{1+\lambda a} (1-p) \log(1-p) \) |
| General Uniform  | \[ F(x) = \frac{x-a}{\lambda}, \ a < x < a+\lambda \] | \( \frac{a p + \lambda p^2/2}{a + \lambda^2/2} \) |
| Pareto           | \[ F(x) = 1 - (a/\lambda)^\alpha, \ x > 0, \alpha \geq 1 \] | \( 1 - (1-p) \)                   |

Since each distribution with a finite mean uniquely defines its Lorenz curve and conversely, goodness-of-fit test can be based either on the sample distribution function or on the sample
Lorenz curve. The curve shown in (3.4.2) is scale-free. So we have to study the exact distribution theory of $L(p)$ under $H_0$. A simple modification of this curve is given as

$$L^*_n(p) = \frac{\sum_{i=1}^{[np]} (x_i - x_{(i)})}{\sum_{i=1}^{n} (x_i - x_{(i)})}, \quad (3.4.4)$$

yields a goodness-of-fit test against the more general null hypothesis $H_0 : 1 - \exp(-\beta(x - \theta))$ for $x \geq \theta$, $\beta > 0$, $-\infty < x < \infty$ and $L^*_n(p)$ does not depend on the nuisance parameters $\theta$ and $\lambda$.

Now, the asymptotic convergence of $L^*_n(p)$ to $\lambda(p)$ is given in the following theorem.

**THEOREM 3.2.** Let $F$ be the c.d.f of a positive random variable $x$ with $E|x| < \infty$ and suppose $F$ has a unique quantile $\xi_p$ satisfying $F(\xi_p) = p$. Then if $F$ is continuous at $\xi_p$, $L^*_n(p)$ converges to $\lambda(p)$ almost surely, where $\lambda(p) = \eta(p)/\mu$.

**Proof.** To prove this theorem we need the following well known results given in Gibbons (1987) and Rao (1973).

**THEOREM 3.3 (Glivenko-Cantelli lemma).** Let $F_n(x)$ be sample cumulative distribution of $F_x(x)$ based on a sample of size $n$. Then $F_n(x)$ converges to $F_x(x)$ with probability one.

i.e. $P \left[ \limsup_{n \to \infty} \sup_{-\infty < x < \infty} |F_n(x) - F_x(x)| = 0 \right] = 1$

81

Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences of random variables such that \( x_n \rightarrow x \) and \( y_n \rightarrow y \). Then \( x_n y_n \rightarrow xy \) provided \( y \neq 0 \).

Let
\[
I_i = \begin{cases} 
1 & \text{for } x_i \leq \xi_p \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
S_n = \sum_{i=1}^{n} x_i I_i / n.
\]

Now
\[
| u_n - \eta(p) | \leq | u_n - S_n | + | S_n - \eta(p) | \tag{3.4.5}
\]

Here if we show \( | u_n - S_n | \) and \( | S_n - \eta(p) | \) both converges almost surely to zero, then it follows that \( | u_n - \eta(p) | \) converges almost surely to zero.

Now,
\[
| u_n - S_n | = \left| \sum_{i=1}^{n} x_i / n - \sum_{i=1}^{n} x_i I_i / n \right|
\]
\[
= \left| \sum_{i=1}^{n} x_i / n - \sum_{i=1}^{n} x_i I_i / n \right|
\]

where,
\[
I_i(\omega) = \begin{cases} 
1 & \text{for } x_i(\omega) \leq \xi_p \\
0 & \text{otherwise}
\end{cases}
\]

When \( x_{[np]} \leq \xi_p \) then,
\[
| u_n - S_n | = \sum_{i=\{np\}+1}^{n} x_i(\omega) = \int_{\xi_p}^{n} t \, dF(x) \tag{3.4.6}
\]

and when \( x_{[np]} > \xi_p \) then,
\[
| u_n - S_n | = \sum_{i=1}^{\{np\}-1} x_i(\omega) / n + \sum_{i=\{np\}}^{n} x_i I_i(\omega) / n
\]

and when \( x_{[np]} > \xi_p \) then,
\[
| u_n - S_n | = \sum_{i=1}^{\{np\}-1} x_i(\omega) / n + \sum_{i=\{np\}}^{n} x_i I_i(\omega) / n
\]
= \sum_{i=1}^{[np]} \left( \frac{x_{(i)} - x_{(i)}}{n} \right) + \frac{x_{[np]} - I}{n}

= \sum_{i=1}^{[np]} \left( \frac{x_{(i)} - x_{(i)}}{n} \right) + 0

= \int_0^1 t \, dF_n(t)

Combining (3.4.6) and (3.4.7) we can write

\max \{ x_{[np]}^*, \xi_p \}

\left| u - S_n \right| = \int_0^1 t \, dF_n(t)

\min \{ x_{[np]}^*, \xi_p \}

= \int_0^1 t \, dF_n(t)

where,

Z_n = \min \{ x_{[np]}^*, \xi_p \}

and

Y_n = \max \{ x_{[np]}^*, \xi_p \}

Since

0 \leq \int_0^1 t \, dF_n(t) \leq y_n \left( F(Y_n) - F(Z_n) \right),

if we show

Y_n \left( F(Y_n) - F(Z_n) \right) \to 0

as

then it follows that

\int_0^1 t \, dF_n(t) \to 0

Consider

\left| F(Y_n) - F(Z_n) \right|

= \left| F(Y_n) - FC(Y_n) + FC(Y_n) - F(Z_n) + FC(Z_n) - FC(Z_n) \right|

= \left| FC(Y_n) - FC(Y_n) + FC(Z_n) - F(Z_n) + FC(Z_n) - FC(Z_n) \right|

\leq \left| F(Y_n) - FC(Y_n) \right| + \left| FC(Z_n) - F(Z_n) \right| + \left| FC(Y_n) - FC(Z_n) \right|
Then by Theorem 3.3, \( |F_n(Y) - F(Z)| \) and \( |F_n(Z) - F(Z)| \) both tend to zero almost surely and since \( F \) is continuous at \( \xi_p \) and both \( Y_n \) and \( Z_n \) converges a.s. to \( \xi_p \) (See. Rao. 1973. pp. 423), \( |F_n(Y) - F(Z)| \) also converges to zero almost surely.

Thus \( |F_n(Y) - F(Z)| \xrightarrow{a.s.} 0 \).

Now using Theorem 3.4, we see that
\[
Y_n \bigg| F_n(Y) - F_n(Z) \bigg| \xrightarrow{a.s.} 0.
\]

Hence,
\[
\int t \ dF_n(t) \xrightarrow{a.s.} 0.
\]

i.e.
\[
|u_n - S_n| \xrightarrow{a.s.} 0. \tag{3.4.9}
\]

Using the strong law of large numbers we can easily see that
\[
\text{A.s. } S_n \xrightarrow{a.s.} \mu \text{ and therefore } |S_n - \mu| \xrightarrow{a.s.} 0. \tag{3.4.10}
\]

Using (3.4.9) and (3.4.10) in (3.4.5) we get
\[
|u_n - \eta(p)| \xrightarrow{a.s.} 0.
\]

i.e. \( u_n \) converges almost surely to \( \eta(p) \).

Furthermore, since \( v_n \) is the sample mean it converges to \( \mu \) in probability. Since converges almost surely implies convergence in probability, we have
\[
\text{A.s. } u_n \xrightarrow{a.s.} \eta(p) \Rightarrow u_n \xrightarrow{p} \eta(p) \quad \text{and} \quad v_n \xrightarrow{p} \mu
\]

This implies \( u_n/v_n \xrightarrow{a.s.} \eta(p)/\mu \).

Thus \( L_n(p) \) converges to \( \lambda(p) \) in probability.
THEOREM 3.5. Let $F$ satisfies the conditions of theorem 3.2. and in addition suppose $E X^2 < \infty$. Then $\{ L_n(p) - \lambda(p) \} [\text{Var} \ L(p)]^{-1}$ converges in distribution to a standard variate.

Proof. To prove this theorem we shall use the following theorems due to Moore (1968) and a theorem given in Tao (1973, pp. 387).

THEOREM 3.6. Let $x_1, x_2, \ldots, x_n$ be a random sample from a continuous distribution function $F(x)$ with finite variance $\sigma^2$. Let $G(u)$ be any inverse of $G$.

Let $T_n = 1/n \sum_{i=1}^{n} J(i/n) x_i$,

$$= \int_{-\infty}^{\infty} x \cdot J(F(x)) \, dF(x) \quad (3.4.11)$$

where $J$ is a continuous function on $[0,1]$ except for jump discontinuities at $p_1, p_2, \ldots, p_m$ and $J'$ is continuous and of bounded variation on $[0,1] - \{p_1, p_2, \ldots, p_m\}$. Also let us assume that

$$\int_{0}^{\infty} |G(u)| \, du < \infty.$$ 

Then

$$n^{1/2} \left[ T_n - \int_{-\infty}^{\infty} J(F(x)) \, dF(x) \right] \xrightarrow{L} N(0, \sigma^2) \quad (3.4.12)$$

where $\sigma^2 = 2 \int \int J(F(s)) J(F(t)) F(s) [1-F(t)] \, ds \, dt$.

which is the Moore's theorem.

THEOREM 3.7. Let $T_n$ be a $k$-dimensional statistic $(T_1, T_2, \ldots, T_k)$ such that the asymptotic distribution of $n^{1/2}(T_1 - \theta_1), n^{1/2}(T_2 - \theta_2), \ldots, n^{1/2}(T_k - \theta_k)$ is $k$-variate normal with mean 0 and dispersion matrix $\Sigma = (\sigma_{ij})$. Further let $g$ be a function of $k$ variables
which is totally differentiable. Then the asymptotic distribution of \( n^{1/2} u_n = n^{1/2} [ g(T_{1n}, T_{2n}, ..., T_{kn}) - g(\theta_1, \theta_2, ..., \theta_k) ] \) is normal with mean 0 and variance,

\[

\nu(\theta) = \sum \sum \alpha_{ij} \frac{\partial g}{\partial \theta_i} \frac{\partial g}{\partial \theta_j},
\]

provided \( \nu(\theta) \neq 0 \).

### Proof of the theorem 3.5.

We have

\[

u_n = \sum_{i=1}^{np} x_i/n
\]

\[

= \sum_{i=1}^{np} J(i/n) x_i
\]

\[

= \int_{-\infty}^{\infty} J(F(x)) dF(x)
\]

where

\[

J(u) = \begin{cases} 1 & \text{if } u \leq p \\ 0 & \text{otherwise} \end{cases}
\]

Clearly \( J \) satisfies the required conditions of Moore's theorem.

Therefore \( n^{1/2} (u_n - \eta(p)) \) converges asymptotically to normal with mean \( \eta \) and variance \( \sigma_1^2 \), where

\[

\sigma_1^2 = 2 \int \int J(F(s)) J(F(t)) F(s) [1 - F(t)] ds dt
\]

Now

\[

J(F(s)) J(F(t)) = \begin{cases} 1 & \text{if } F(s) \leq F(t) \leq p \\ 0 & \text{otherwise.} \end{cases}
\]

\[

= \begin{cases} 1 & \text{if } s < t < F^{-1}(p) = \xi(p) \\ 0 & \text{otherwise.} \end{cases}
\]

Hence, \( \sigma_1^2 = 2 \int \int [J(F(s)) ds] [1 - F(t)] dt \)

(3.4.14)

Again

\[

v = \sum_{i=1}^{n} x_i/n
\]

86
\[
\sum_{i=1}^{n} J_2(i/n) X(i/n) = \int_{-\infty}^{\infty} J_2(F(x)) dF(x)
\]

where

\[
J_2(u) = \begin{cases} 
1 & \text{if } 0 \leq u < 1 \\
0 & \text{otherwise.}
\end{cases}
\] (3.4.15)

clearly \(J_2(u)\) is also satisfies all the conditions of theorem 3.6.

Therefore \(n^{1/2} (\mu - \mu)\) converges in distribution to normal
with mean 0 and variance \(\phi^2\), where,

\[
\phi^2 = 2 \int \int J_2(F(s)) J_2(F(t)) [1 - F(t)] ds dt
\]

with

\[
J_2(F(s)) J_2(F(t)) = \begin{cases} 
1 & \text{if } F(s) \leq F(t) \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Hence

\[
\phi^2 = 2 \int \int [F(s) ds] [1 - F(t)] dt \quad (3.4.16)
\]

Now, we will consider the linear combination of \(u_n\) and \(v_n\).

i.e. \(w_n = a_1 u_n + a_2 v_n\), such that at least one of the
\(a_1\) and \(a_2\) is nonzero.

i.e. \(w_n = \sum_{i=1}^{[n]} J_1(i/n) X(i/n) + \sum_{i=1}^{n} J_2(i/n) X(i/n)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} J(i/n) X(i/n)
\]

\[
= \int_{-\infty}^{\infty} J(F(x)) dF(n)
\]

87
where,

\[ J(u) = \begin{cases} (a_1 + a_2) & \text{if } u \leq p \\ a_2 & \text{otherwise} \end{cases} \]

clearly \( J(u) \) satisfies all the conditions of theorem 3.6.

Therefore \( n^{1/2} \left( w_n - a_1 \eta(p) - a_2 \mu \right) \) converges in distribution to normal with mean 0 and variance \( \sigma^2 \), where,

\[ \sigma^2 = 2 \oint \oint J(F(s))J(F(t)) F(s) [1 - F(t)] ds \, dt \]

with

\[ J(F(s))J(F(t)) = \begin{cases} (a_1 + a_2)^2 & \text{if } s < t \leq \eta(p) \\ (a_1 + a_2)a_2 & \text{if } s < \eta(p) \leq t \\ a_2^2 & \text{if } \eta(p) \leq s < t \end{cases} \]

therefore,

\[
\sigma^2 = 2 \left[ \int_0^{\eta(p)} \int_0^{\eta(p)} (a_1 + a_2)^2 F(s) [1-F(t)] \, ds \, dt \\
+ \int_0^{\eta(p)} \int_0^{1} (a_1 + a_2) a_2 F(s) [1-F(t)] \, ds \, dt \\
+ \int_0^{\eta(p)} \int_0^{\eta(p)} a_2^2 F(s) [1-F(t)] \, ds \, dt \right]
\]

\[
= a_1^2 \sigma_1^2 + 2a_1 a_2 \left[ \int_0^{\eta(p)} \int_0^{\eta(p)} \sigma^2 F(s) \, ds \, \int_0^{\eta(p)} (1-F(t)) \, dt \right] + a_2^2 \sigma_2^2
\]

\[
= a' \Sigma a
\tag{3.4.17}
\]

where

\[ a = (a_1, a_2) \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \]

and \( \sigma_{12} = \sigma_1^2 + \int F(s) \, ds \int (1-F(t)) \, dt \).

This is true for all \( a \neq 0 \).
Therefore \[ n^{1/2} \left( \frac{u_n - \eta(p)}{\nu_n} \right) \] converges in distribution to multivariate normal with mean vector 0 and dispersion matrix \( \Sigma \).

Therefore by theorem 3.7, it follows that \( n^{1/2} \left[ \frac{u_n}{\nu_n} - \frac{\eta(p)}{\mu} \right] \) converges in distribution to normal with mean zero and variance
\[
\sigma_1^2/\mu^2 + \sigma_2^2 \eta^2/\mu^4 - 2 \sigma_{12} \eta/\mu^3.
\]

Equivalently \( [L_n(p) - \lambda(p)]/\text{Var}(L_n(p))^{1/2} \) converges to standard normal.

Here the distribution function \( F \) need not be everywhere continuous for Theorem 3.6. to hold, and is valid only when \( p \) is not a discontinuity point of \( G \).

Here the equation (3.4.14) can be expressed in some other form also. By putting \( F(x) = s \) and \( F(y) = t \) we get
\[
\sigma_1^2 = 2 \int_0^p \int_0^t \int_0^1 s G'(s) ds (1-t) G'(t) dt \quad (3.4.18)
\]
and
\[
\sigma_{12} = \sigma_1^2 \int_0^p \int_0^1 s G'(s) ds \int_0^{1-t} G'(t) dt \quad (3.4.19)
\]
respectively.

As an illustration let us consider \( F(x) = 1 - e^{-x} \) for \( x \geq 0 \),
then \( G(u) = -\log(1-u) \) and \( G'(u) = 1/(1-u) \). Using the above results we can see that \( \mu = \sigma_2^2 = 1 \) and
\[
\eta(p) = \int_0^p G(u) du = p + (1-p) \log(1-p)
\]
= p + q \log q, \text{ where } q = 1 - p

from (3.4.18) and (3.4.19) we obtain

\[ \sigma^2 = 2 \int_0^1 \left( \int_0^s \frac{s}{1-s} \, ds \right) \, dt \]

\[ = 2(1-p) \log(1-p) + p + p(1-p) \]

\[ = 2q \log q + p + pq \]

and

\[ \sigma_{12} = \sigma^2 + \int_0^1 s/(1-s) \, ds \int_0^1 \, dt \]

\[ = \sigma^2 + (1-p) \int_0^1 s/(1-s) \, ds \]

\[ = p + (1-p) \log(1-p) \]

\[ = p + q \log q \]

Thus \( L_n(p) \) has asymptotic mean \( \eta(p)/\mu = p + q \log q \) and from (3.4.14), \( n^{1/2} L_n(p) \) has asymptotic variance

\[ \sigma^2 = 2q \log q + p + pq - (p + q \log q)^2. \quad (3.4.20) \]

Thus the theorem follows.

Now we will study the exact distribution theory of \( L_n(p) \) when the underlying distribution is exponential.

2. DISTRIBUTION THEORY OF \( L_n(p) \)

When \( F(x) = 1 - \exp(-\beta x) \), for \( x \geq 0 \) then the variables

\[ y_i = x_i/\Sigma x_j, \quad i = 1, 2, \ldots, n \]

have the Dirichlet density,

\[ f(y_1, y_2, \ldots, y_{n-1}) = \frac{1}{n}, \text{ for } y_i \geq 0, \quad 0 < \Sigma y_i < 1, \quad (3.4.21) \]

and can be seen as follows.

Let \( x_1, x_2, \ldots, x_n \) be independent random variables having exponential distribution with p.d.f.
\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \beta e^{-\beta x_i} \quad \text{for} \quad 0 < x_i < \infty, \quad \text{otherwise.} \]

Let
\[ y_i = \frac{x_i}{(x_1 + x_2 + \ldots + x_n)} \quad \text{for} \quad i = 1, 2, \ldots, n-1, \]
and
\[ y_n = x_1 + x_2 + \ldots + x_n. \]

Then the associated transformation maps
\[ \mathcal{X} = \{(x_1, x_2, \ldots, x_n) \mid 0 < x_i < \infty, \quad i = 1, 2, \ldots, n\} \]
on to the space
\[ \mathcal{Y} = \{(y_1, y_2, \ldots, y_n) \mid 0 < y_i, \quad i = 1, 2, \ldots, n, \quad \sum y_i < 1, \quad 0 < y_n < \infty\}. \]

Then the inverse functions are
\[ x_1 = y_1 y_n, \quad x_2 = y_2 y_n, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ x_{n-1} = y_{n-1} y_n, \quad x_n = y_n (1 - y_1 - y_2 - \ldots - y_{n-1}). \]

Then the Jacobian transformation is given by
\[
J = \begin{vmatrix}
 y_n & 0 & 0 & \ldots & 0 & \ldots & y_1 \\
 0 & y_n & 0 & \ldots & 0 & \ldots & y_2 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 -y_n & -y_n & -y_n & \ldots & -y_n & (1 - y_1 - y_2 - \ldots - y_{n-1}) \\
\end{vmatrix}
\]

\[ |J| = y_n^{n-1} \]

Hence the joint density of \( y_1, y_2, \ldots, y_n \) is given by
\[
f(y_1, y_2, \ldots, y_{n-1}, y_n) = \frac{\beta^n e^{-\beta \sum y_i y_n}}{y_n^{n-1}} \quad \text{for} \quad 0 < y_i < \infty, \quad i = 1, 2, \ldots, n-1, \quad \sum y_i < 1, \quad 0 < y_n < \infty\].

\[ = \beta^n e^{-\beta y_n} \cdot y_n^{n-1} \]
therefore,

\[
f(y_1, y_2, \ldots, y_{n-1}) = \int_0^\infty f(y_1, y_2, \ldots, y_n) \, dy
\]

\[
= \beta^n \int_0^\infty e^{-\beta y_n} y_n^{n-1} \, dy
\]

\[
f(y_1, y_2, \ldots, y_{n-1}) = \lfloor n \rfloor, \text{ for } 0 \leq \sum_{i=1} y_i < 1
\]

This distribution is also that of spacings into which the unit interval is divided by a random sample of \( n-1 \) uniform \( U(0,1) \) variables (Gibbons 1985). Thus \( L(p) = \sum_{i=1}^{\lfloor np \rfloor} \frac{x_i}{\sum_{i=1}^{n} x_i} \) has the same distribution as the sum of first \( \lfloor np \rfloor \) ordered uniform spacings.

From Mauldon's (1951) result on the sum of the \( k = n - \lfloor np \rfloor \) largest uniform spacings, Gail and Gastwirth (1978a) obtained the following representation by symmetry:

\[
P \left( L_n(p) \leq \ell \right) = \begin{cases} 
1 & , \ell \leq 0 \\
0 & , \ell > \frac{r}{n} \\
\sum_{j=0}^\ell \frac{n!(r-j-\ell)(n-j)^{n-1}}{(n-j)!} \left( -\frac{1}{n} \right)^j & , \ell \geq \frac{r}{n} 
\end{cases}
\]

\[
\left( r = \lfloor np \rfloor \text{ and } \xi = (r-1n)/(1-\ell) \right)
\]

Summation over \( j \) is an nonnegative largest integers strictly less than \( \xi \). Exact percentiles of \( L_n(p) \) computed from the above formula (3.4.22) are given in the table 3.2. below.
### Exact Percentiles of the Lorenz Statistic L^*(.5)

<table>
<thead>
<tr>
<th>Probability Level</th>
<th>Percentile</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0050</td>
<td>0.0050</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0060</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0120</td>
<td>0.0120</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0180</td>
<td>0.0180</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0362</td>
<td>0.0362</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0544</td>
<td>0.0544</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0726</td>
<td>0.0726</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0909</td>
<td>0.0909</td>
</tr>
<tr>
<td>0.60</td>
<td>0.1091</td>
<td>0.1091</td>
</tr>
<tr>
<td>0.70</td>
<td>0.1273</td>
<td>0.1273</td>
</tr>
<tr>
<td>0.80</td>
<td>0.1455</td>
<td>0.1455</td>
</tr>
<tr>
<td>0.90</td>
<td>0.1638</td>
<td>0.1638</td>
</tr>
<tr>
<td>0.95</td>
<td>0.1820</td>
<td>0.1820</td>
</tr>
<tr>
<td>0.99</td>
<td>0.2002</td>
<td>0.2002</td>
</tr>
</tbody>
</table>

Note: The table continues with similar entries for other probability levels.
3. ASYMPTOTIC RELATIVE EFFICIENCY OF $L_n(\theta)$ AGAINST WEIBULL AND GAMMA ALTERNATIVES.

Consider a family of distributions $F(x; \theta)$ such that $\theta = \theta^*$, $F(x; \theta^*)$ is exponential. Let $\hat{\theta}$ be the M.L.E of $\theta$. Then based on $\hat{\theta}$ a test can be proposed for $H_0: \theta = \theta^*$ against $H_1: \theta > \theta^*$ which rejects $H_0$ for $\hat{\theta} - \theta^* > c_\alpha$ where $c_\alpha$ is the upper level $\alpha$ cut off point. We know that under certain regularity conditions (See. Rao. 1973, pp. 363-365) $n^{1/2} (\hat{\theta} - \theta^*)$ tends to normal with mean zero and variance say, $\nu(\theta^*)$, which is the inverse of the Fisher information computed for $\theta^*$.

Now consider the sequence of alternatives $\theta_n = \theta^* + \theta/n^{1/2}$, where $\theta$ is some arbitrary positive constant, so that $\theta_n \to \theta^*$ as $n \to \infty$. Under this sequence of alternatives $n^{1/2} (\hat{\theta} - \theta^*)$ tends to normal with mean $\theta$ and variance $\nu(\theta^*)$. Then efficiency of $\hat{\theta}$ is

$$K_\theta = \left. \frac{(d/d\theta (\hat{\theta}))}{\nu(\theta^*)} \right|_{\theta = 0}^2 = 1/\nu(\theta^*)$$

(3.4.23)

Now suppose under $F(x; \theta)$ alternatives $n^{1/2} [L_n(\theta) - \lambda(\theta)]$ converges in distribution to normal with mean $\lambda(\theta, \delta)$ and $\sigma^2(\theta, \delta)$. And also suppose that $\sigma^2(\theta, \delta) \to \sigma^2(\theta, \delta^*)$ as $\delta \to \delta^*$ then efficiency of $L_n(\theta)$ is given by

$$K^2_{L_n(\theta)} = \left. \frac{(d/d\delta (\lambda(\theta, \delta)))}{\sigma^2(\theta)} \right|_{\delta = \delta^*}^2$$

(3.4.24)

Therefore efficiency of $L_n(\theta)$ with respect to the likelihood...
statistic $\hat{\delta}$ is given by

$$\text{A.R.E.}(p) = \frac{K_{L(p)}^2}{K_0^2}$$

$$= [\nu/\sigma^2(p)] [\frac{d}{d\delta}\lambda(p,\delta)|_{\delta=\delta^*}]^2$$

(3.4.25)

where $\delta^*$ is that value of $\delta$ for which $F_\delta$ is exponential and $\sigma^2(p)$ is as in (3.4.20).

Now we will be computing A.R.E under the Gamma alternatives.

For the Gamma alternatives Moran (1951) showed that $\nu = .645$. Since $\lambda(p,\delta) = \eta(p,\delta)/\mu(p,\delta)$ then we obtain,

$$\frac{d}{d\delta} \frac{\lambda(p,\delta)}{\mu(\delta)} = \frac{\mu(\delta) \eta'(\delta) - \eta(\delta) \mu'(\delta)}{(\mu(\delta))^2}$$

$$= \eta'(\delta)/\mu(\delta) - \lambda(\delta) \mu'(\delta)/\mu(\delta)$$

(3.4.26)

In (3.4.26) $p$ is regarded as fixed and is suppressed.

Also $\eta(\delta) = \int G(u,\delta) \, du$ and $\mu(\delta) = \int G(u,\delta) \, du$, and

$G(u,\delta)$ is the inverse of

$$F_\delta(y) = \int x^{(1+\delta)-1} e^{-x}/[1+\delta] \, dx$$

Since $L_{n}(p)$ is independent of scale in $F_\delta$ scale parameter is taken as one. Here $\delta = \delta^* = 0$, $F_0(y)$ will become exponential.

Since $\mu(\delta) = 1 + \delta$ we have $\mu(0) = \mu'(0) = 1$ and $\lambda(0) = p + q \log q$ is the exponential population curve.

$$\eta(\delta) = \int x^{(1+\delta)} e^{-x}/[1+\delta] \, dx$$

94
\[ \eta'(\delta=0) = \int_0^{\ln(q)} (\ln x + \gamma) x e^{-x} dx - q \ln q \frac{dG}{d\delta} \bigg|_{\delta=0} \]

(3.4.27)

where \( \gamma = 0.5772 \), is the Euler's constant.

Differentiating, \( p = \int_0^{\infty} x e^{-x} / (1+\delta) \), we find

\[ \frac{dG}{d\delta} \bigg|_{\delta=0} = p + \ln(\ln(q))^{-1} + q^{-1} \int_0^{\infty} \frac{1}{x} e^{-x} dx - \ln q \]

(3.4.28)

Thus \( \text{ARE}(p) \) is computed from (3.4.25), (3.4.28), (3.4.27) and (3.4.28).

Gail and Gastwirht obtained the following figure showing the Lorenz statistic \( \text{ARE} \) versus \( p \) for Gamma and Weibull alternatives.
The Lorenz statistic ARE is computed with respect to maximized likelihood statistics.

95.A
The ARE's corresponding to \( L_n(.2), L_n(.3), L_n(.4) \) and \( L_n(.5) \) are .81, .81, .78 and .73 against Gamma alternatives and that the ARE decreases rapidly for \( p > .6 \).

The ARE against Weibull alternatives which are in (1977) have been plotted in figure 3.1. The statistics \( L_n(.2), L_n(.3), L_n(.4), L_n(.5), L_n(.6) \) and \( L_n(.7) \) have ARE's .76, .73, .80, .81, .79 and .75 respectively.

The above figure shows that \( L_n(p) \) have good efficiency for a range of \( p \) values.

The power comparisons are given at the end of this chapter with that of ten other goodness-of-fit tests.

To summarize \( L_n(.4) \) has ARE .78 and .80 against Gamma and Weibull alternatives respectively and \( L_n(.5) \) has corresponding ARE's .73 and .81. Both statistics seen promising against weibull and Gamma alternatives.

3.5. A SCALE-FREE GOODNESS-OF-FIT TEST BASED ON GINI'S STATISTICS.

We have seen the Gini’s index and the Gini statistic given by (3.4.3). Since Gini statistic shares many advantages of \( L_n(.5) \) including ease of computation, availability of exact distribution theory of robustness to truncation and rounding measurement error, we will give the test based on Gini statistic in detail. The test procedures is to reject the \( H_0 \) for large values of \( G_n \). First we obtain the exact distribution of \( G_n \) under exponentiality.
1. DISTRIBUTION THEORY OF $G_n$ UNDER EXPONENTIALITY.

Since $G_n$ is scale-free we assume without loss of generality, that the $x_i$ are unit exponentials.

Then we write (3.4.3) as

$$G_n = \frac{\sum_{i=1}^{n} (i-1)D_i}{(n-1)\sum_{i=1}^{n} x_i}$$

(3.5.1)

Denote $(n-i+1) (x_{(i)} - x_{(i-1)})$ by $D_i$ as usual. Then $D_i$'s are identically and independently distributed unit exponentials. Thus (3.5.1) can be written as

$$G_n = \frac{\sum_{i=1}^{n} (i-1)D_i}{(n-1)\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} (i-1)D_i}{(n-1)\sum_{i=1}^{n} D_i}$$

(3.5.2)

Let $y_i = D_i / \sum_{i=1}^{n} D_i = D_i / (D_1 + D_2 + \ldots + D_n)$, $i = 1, 2, \ldots, n$, denote the $n$ exchangeable Dirichlet variates.

Then as earlier the joint density of $y_1, y_2, \ldots, y_n$ is seen that

$$g(y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} y_i^{(i-1)/(n-1)}$$

if $y_i > 0$ and $\sum_{i=1}^{n} y_i = 1$.

Thus we obtain the representation for $G_n$ as

$$G_n = \sum_{i=1}^{n} \frac{(i-1)/n-1}{y_i}$$

$$= \sum_{j=1}^{n} \frac{(n-j)/n-1}{y_j}$$

97
Now using a theorem due to Dempster and Kleyle (1969), the exact distribution of $G_n$ can be obtained.

The theorem states that if $Y_1, Y_2, ..., Y_n$ are random variables uniformly distributed over the simplex of points $y_1, y_2, ..., y_n$ such that $y_i \geq 0$, $i = 1, 2, ..., n$ and $\sum y_i = 1$. Then the distribution of $G_n = \sum_{j=1}^{n} c_j y_j$ for constants $c_j$ satisfying $c_1 > c_2 > \ldots > c_n > 0$ is easily seen to be given by

$$\mathbb{P}(G_n \leq x) = x^n \left[ \prod_{i=1}^{n} c_i \right]^{-1} - \sum_{j=r+1}^{n} (x - c_j)^n \left[ c_j \prod_{i \neq j} (c_j - c_i) \right]^{-1}$$

for $0 \leq x \leq c_j$. If $x$ is large (for small $r$) and $r$ is the largest index such that $x \leq c_r$, a more convenient form is given by

$$\mathbb{P}(G_n \leq x) = 1 - \sum_{j=1}^{n} (c_j - x)^n \left[ c_j \prod_{i \neq j} (c_j - c_i) \right]^{-1}$$

The following table gives the exact percentiles of $G_n$ for $n = 3, 4, \ldots, 20$. 

\begin{table}
\end{table}
Table 3.3.
Percentile points of Gini statistic $G_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0.950</th>
<th>0.975</th>
<th>0.990</th>
<th>$n$</th>
<th>0.950</th>
<th>0.975</th>
<th>0.990</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.81499</td>
<td>0.88818</td>
<td>0.92932</td>
<td>12</td>
<td>0.64337</td>
<td>0.55992</td>
<td>0.70020</td>
</tr>
<tr>
<td>4</td>
<td>0.77686</td>
<td>0.82288</td>
<td>0.86951</td>
<td>13</td>
<td>0.63725</td>
<td>0.56275</td>
<td>0.69183</td>
</tr>
<tr>
<td>5</td>
<td>0.73834</td>
<td>0.77997</td>
<td>0.82501</td>
<td>14</td>
<td>0.63185</td>
<td>0.55641</td>
<td>0.68448</td>
</tr>
<tr>
<td>6</td>
<td>0.71307</td>
<td>0.75079</td>
<td>0.79260</td>
<td>15</td>
<td>0.62704</td>
<td>0.55076</td>
<td>0.67792</td>
</tr>
<tr>
<td>7</td>
<td>0.69439</td>
<td>0.72931</td>
<td>0.76831</td>
<td>16</td>
<td>0.62273</td>
<td>0.54567</td>
<td>0.67197</td>
</tr>
<tr>
<td>8</td>
<td>0.67988</td>
<td>0.71252</td>
<td>0.74921</td>
<td>17</td>
<td>0.61882</td>
<td>0.54107</td>
<td>0.66559</td>
</tr>
<tr>
<td>9</td>
<td>0.66821</td>
<td>0.69896</td>
<td>0.73370</td>
<td>18</td>
<td>0.61527</td>
<td>0.53688</td>
<td>0.66168</td>
</tr>
<tr>
<td>10</td>
<td>0.65855</td>
<td>0.68768</td>
<td>0.72070</td>
<td>19</td>
<td>0.61201</td>
<td>0.53304</td>
<td>0.65723</td>
</tr>
<tr>
<td>11</td>
<td>0.65039</td>
<td>0.67816</td>
<td>0.70972</td>
<td>20</td>
<td>0.60902</td>
<td>0.52952</td>
<td>0.65308</td>
</tr>
</tbody>
</table>

Larger values of $n$ are needed as the normal approximation discussed below is excellent, even for small $n$. Moreover, the corresponding lower tail percentiles $x_{1-p}$ as $G_n$ is symmetrically distributed about 0.5.

To see that $G_n$ is symmetric about 0.5 consider $Z = G_n - 1/2$.

\[ Z = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)} D_j - \frac{1}{2} \sum_{j=1}^{n} D_j, \text{ since } \sum_{j=1}^{n} D_j = 1 \]

\[ Z = \sum_{j=1}^{n} \frac{(n-2j+1)}{(2n-2)} D_j \quad (3.5.6) \]
when \( n = 2m \).

\[
Z = \sum_{j=1}^{m} \frac{(2m-2j+1)}{(2n-2)} D_j + \sum_{k=m+1}^{2m} \frac{(2m-2k+1)}{(2n-2)} D_k
\]

In the second sum let \( j = 2m-k-1 \) we get

\[
\sum_{j=2m-k-1}^{m} \frac{-(2m-2j+1)}{(2n-2)} D_k
\]

by pairing \( k = 2m-2j+1 \) with index for \( j = 1, 2, \ldots, m \) we recognize \( Z \) as the sum of random variables,

\[
W_j = \frac{(2m-2j+1)}{(2n-2)} D_j - \frac{(2m-2j+1)}{(2n-2)} D_k
\]

\[
= \frac{(2m-2j+1)}{(2n-2)} (D_j - D_k)
\]

Since \( D_j \) and \( D_k \) have the same distribution \( D_j - D_k \) is symmetric about 0 and hence \( W_j \) is symmetric about 0. Since \( Z \) is a linear function symmetric random variables, it is symmetric about 0. Because \( G_n = Z + 1/2 \), \( G_n \) is symmetric about 1/2.

If \( n = 2m + 1 \), then

\[
Z = \sum_{j=1}^{m} \frac{(2m+1-2j+1)}{(2n-2)} D_j + \sum_{j=1}^{(2m+1)-2(n+1)+1} \frac{(2m+1)-2(n+1)+1}{(2n-2)} D_{m+1}
\]

\[
+ \sum_{k=m+2}^{2m+1} \frac{(2m+1)-2k+1}{(2n-2)} D_k
\]

\[
= \sum_{j=1}^{m} \frac{2m-2j+2}{2n-2} D_j + \sum_{k=m+2}^{2m+1} \frac{2m-2k+2}{2n-2} D_k
\]

Now, using the argument made for the case \( n = 2m \), it follows that \( Z \) is symmetric about 0 and \( G_n \) is symmetric about 1/2.
The asymptotic distribution of $G_n$ has been obtained by Hoeffding (1948), which we will reproduce below for completeness.

To obtain the limiting distribution, Hoeffding used the following theorem.

**Theorem 3.8.** Let $X, X_2, \ldots, X_n$ be $n$ independent identically distributed random vectors, $X_\alpha = (X_\alpha^{(1)}, X_\alpha^{(2)}, \ldots, X_\alpha^{(r)})$, $\alpha = 1, 2, \ldots, n$. Let $\phi^{(r)}(X_1, X_2, \ldots, X_m(y))$, $r = 1, 2, \ldots, g$ be $g$ real-valued functions not involving $n$, $\phi^{(r)}$ being symmetric in its $m(y) \leq n$ vector arguments $X_\alpha = (X^{(1)}_\alpha, X^{(2)}_\alpha, \ldots, X^{(r)}_\alpha)$; $\gamma = 1, 2, \ldots, g$. Define,

$$U^{(r)} = \left(\sum_{m(y)}^{n(y)}\right) \sum_{i} \phi^{(r)}(X^{(1)}_{\alpha_1}, X^{(2)}_{\alpha_2}, \ldots, X^{(r)}_{\alpha_{m(y)}}),$$

where summation is over all subscripts such that $1 \leq \alpha_1 \leq \ldots \leq \alpha_{m(y)} \leq n$.

Let

$$U^{(r)}' = U^{(r)} + b_n^{(r)}/n^{1/2}, \quad r = 1, 2, \ldots, g, \quad \text{where}$$

$b_n^{(r)}$ is a random variable.

If $U' = (U^{(1)}', U^{(2)}', \ldots, U^{(g)}')$, a random vector and $E(b_n^{(r)})^2 = 0$, $r = 1, 2, \ldots, g$ and if the function

$h(y) = h(y^{(1)}), y^{(2)}, \ldots, y^{(g)})$ does not involve $n$ and is continuous together with its second order partial derivatives in some neighbourhood of the point $(y) = (\theta) = (e^{(1)}, e^{(2)}, \ldots, e^{(g)})'$, then the distribution of the random variable $n^{1/2}(h(U') - h(\theta))$ tends to normal distribution with mean zero and variance

101
Based on the above theorem, \( n^{1/2} (G - \Delta/2\mu) \) tends to be normally distributed with mean 0 and variance,

\[
\frac{\Delta^2}{4\mu^4} \cdot \xi_4(\mu) = \frac{\Delta^2}{\mu^3} \cdot \xi_4(\mu, \Delta) + 1/\mu^2 \cdot \xi_4(\Delta) \quad \text{(3.5.7)}
\]

where

\[
\Delta = \int \int |y_1 - y_2| \, d\mathcal{F}(y_1) \, d\mathcal{F}(y_2),
\]

\[
\xi_4(\mu) = \int y^2 \, d\mathcal{F}(y) - \mu^2 = \sigma^2(\mu),
\]

\[
\xi_4(\Delta) = \int \left( \int |y_1 - y_2| \, d\mathcal{F}(y_2) \right)^2 \, d\mathcal{F}(y_1) - \Delta^2,
\]

\[
\xi_4(\mu, \Delta) = \int \int |y_1 - y_2| \, d\mathcal{F}(y_1) \, d\mathcal{F}(y_2) - \mu \Delta.
\]

Under the exponentiality these values can be computed as follows.

\[
\Delta = \int \int |y_1 - y_2| \, d\mathcal{F}(y_1) \, d\mathcal{F}(y_2)
\]

\[
= \int_0^\infty \int_0^\infty y_2 \left( \int_0^{y_2} (y_2 - y_1) \, d\mathcal{F}(y_1) + \int_y^{\infty} (y_1 - y_2) \, d\mathcal{F}(y_1) \right) \, d\mathcal{F}(y_2)
\]

\[
= \int_0^\infty \left( \int_0^{y_2} e^{-y_2} \, dy_1 + \int_y^{\infty} e^{-y_1} \, dy_1 \right) e^{-y_2} \, dy_2
\]

\[
= \int_0^\infty \left( 2e^{-y_2} + y_2 - 1 \right) \, d\mathcal{F}(y_2)
\]

\[
= 1.
\]
\[ \mu = \int_{0}^{\infty} x e^{-x} \, dx = 1 \]

\[ \xi_1(\mu) = \int_{0}^{\infty} y^2 dF(y) - \mu^2 \]

\[ = \int_{0}^{\infty} y^2 e^{-y} \, dy - 1 = 1 \]

\[ \xi_1(\Delta) = \int_{0}^{\infty} \left\{ \int_{0}^{\infty} |y_1 - y_2| \, dF(y_2) \right\} dF(y_1) - \Delta^2 \]

Now

\[ \int_{0}^{\infty} |y_1 - y_2| \, dF(y_2) = \int_{0}^{\infty} \left\{ \int_{0}^{\infty} (y_1 - y_2) \, dF(y_2) + \int_{0}^{\infty} (y_2 - y_1) \, dF(y_2) \right\} \]

\[ = \int_{0}^{\infty} (y_1 - y_2) e^{-y_2} \, dy_2 + \int_{0}^{\infty} (y_2 - y_1) e^{-y_2} \, dy_2 \]

\[ = 2 e^{-y_1} + y_1 - 1. \]

Therefore,

\[ \xi_1(\Delta) = \int_{0}^{\infty} (2 e^{-y_1} + y_1 - 1)^2 e^{-y_1} \, dy_1 - \Delta^2 \]

\[ = \int_{0}^{\infty} (4e^{-y_1} + y_1^2 + 2y_1 e^{-y_1} - 4e^{-y_1} - 2y_1 e^{-y_1} - 1) \, dy_1 - \Delta^2 \]

\[ = \frac{8}{6} - 1 \]

\[ = 1/3. \]

\[ \xi_1(\mu, \Delta) = \int_{0}^{\infty} \int_{0}^{\infty} y_1 |y_1 - y_2| \, dF(y_1) \, dF(y_2) - \mu \Delta \]

103
\[
\int_0^\infty \int_0^\infty y_1 |y_1 - y_2| f(y_1) f(y_2) dy_1 dy_2 \\
= \int_0^\infty \left\{ \int_0^\infty y_1^2 f(y_1) dy_1 \right\} f(y_2) dy_2 \\
= \int_0^\infty \left\{ \int_0^\infty y_1 f(y_1) dy_1 \right\} f(y_2) dy_2 \\
= \int_0^\infty \left( y - 2e^{-y} + 2e^{-2y} \right) f(y_2) dy_2 \\
= 4/2 + 2/4 + 1 - 2 \\
= 3/2 \\
\]
Therefore,
\[
\xi_1(\mu, \Delta) = 3/2 - 1 = 1/2.
\]
and hence
\[
\text{Variance} = 1/12.
\]
Thus under \( H_0 : \Delta/2\mu = 1/2 \) and \( \lim \text{Var}(n^{1/2}G_n) = 1/12 \).

Therefore by theorem 3.8, the limiting distribution of \((12n)^{1/2}(G_n - 1/2)\) is a standard normal distribution. If the observations are from exponential population then the exact mean and variance of \( G_n \) are 1/2 and 1/(12(n-1)).

It has been shown that the distribution of \( \frac{G_n - .5}{\sqrt{\text{Var}(G_n)}} \)
can be approximated by standard normal even for small values of $n$. 

3. ASYMPTOTIC RELATIVE EFFICIENCY OF $G_n$.

Now, suppose that under $F(x, \delta)$ alternatives $n^{1/2}(G_n - \Delta/2\mu)$ converges in distribution with mean $G(\delta)$ and variance $\sigma^2(\delta)$ which tends to $\delta^2$ as $\delta \to \delta^*$ then the efficiency of $G_n$ is given by

$$K^2_{\delta_n} = \left[ \frac{d}{d\delta} G(\delta) \bigg|_{\delta=\delta^*} \right]^2 / \sigma^2$$  \hspace{1cm} (3.5.9)

Therefore the efficiency of $G_n$ with respect to the likelihood statistic $\hat{\delta}$ is given by

$$\text{ARE} = K^2_{\delta_n} / K^2_{\hat{\delta}}$$  \hspace{1cm} (3.5.10)

we have

$$G_n = \sum_{i,j} |x_i - x_j| / [2n(n-1)\bar{x}]$$

$$= U_n / \bar{x}$$, where the $U$-statistics $U_n$ can be regarded as the linear combination of order statistics as

$$U_n = \sum_{i=1}^{n} (2i-n-1) x_{(i)} / [n(n-1)]$$  \hspace{1cm} (3.5.11)

Then the quantity $U_n$ converges almost surely to

$$\eta(\delta) = \int_{0}^{1} (2u-1) G(u, \delta) \, du$$  \hspace{1cm} (3.5.12)

where $G(u, \delta)$ is the inverse of $F(x, \delta)$, provided $E \delta < \infty$. (See, Kendall and Stuart 1969, pp. 50). Almost sure convergence
follows from the strong law of large numbers for U-statistics obtained by Hoeffding (1961) and Kingman (1969). And since $\bar{x}$ converges to $\mu(\delta) = \int G(u,\delta) \, du$, $G_n$ converges to $G(\delta) = \eta(\delta)/\mu(\delta)$, and if the family $F(x,\delta)$ is sufficiently regular

$$d/d\delta [ G(\delta) ]|_{\delta=0} = \eta'(0) - .5 \mu'(0) \quad (3.5.13)$$

as $\mu(0) = 1$ and $\eta(0) = .5$. Since Goefgging's result (1948) proves the asymptotic normality of $G_n$ under alternatives $F(x,\delta)$ as well as under exponentiality,

$$k^2_n = 12 \{ \eta'(0) - .5 \mu'(0) \}.$$  

Therefore ARE of $G_n$ with respect to maximum likelihood statistic is

$$\text{ARE} = \frac{k^2_n}{k^2} = 12 \nu \{ \eta'(0) - .5 \mu'(0) \}^2 \quad (3.5.14)$$

To compute $\nu$ for Gamma alternatives consider the density

$$f(x,\delta) = x^\delta e^{-x}/\Gamma(1+\delta).$$

To compute $\eta'(0)$ against Gamma alternatives, we rewrite $\eta(\delta)$ as

$$\eta(\delta) = \int_{-\infty}^{\infty} \left( 2F(x,\delta) - 1 \right) x f(x,\delta) \, dx.$$  

$$= 2 \int_{-\infty}^{\infty} \left( \int_{0}^{x} f(t,\delta) \, dt \right) x f(x,\delta) \, dx - (1+\delta). \quad (3.5.15)$$

Differentiating (3.5.15) with respect to $\delta$, evaluating the two resulting integrals at $\delta = 0$ after changing the order of
integration and recognizing integrals such as \( \int_0^\infty x \log x e^{-x} \, dx \) as derivative of Gamma function one can get \( \eta'(0) = 1 - \ln 2 = 0.30685 \). Since \( \mu(0) = 1 + \delta \), the ARE can be shown to be 0.694. In the similar way ARE can be calculated against Weibull alternatives as 0.876.

Comparing the ARE's for \( g_n \) and \( L_n(.5) \) against Gamma and Weibull alternatives one can see that both the ARE's are quite close, being respectively 0.69 and 0.88 for \( g_n \) and 0.73 and 0.81 for \( L_n(.5) \) [Gail and Gastwirth 1978].

3.6. COMPARISONS OF DIFFERENT TESTS.

In this section we are giving different power series on the tests for goodness-of-fit test for exponentiality discussed in the earlier sections. Many of these power studies are done by the respective authors themselves or by some others.

3.6.1. POWER STUDY BY FERCHO AND RINGER.

Fercho and Ringer have done the power study of tests based on the statistics \( E_p \), \( E \), \( U \) and \( Q(r) \) respectively given in (3.2.8), (3.2.9), (3.2.10) and (3.3.3).

In all these tests, wherever grouping has been done is purely arbitrary, and obviously the way in which the grouping was done had an effect on the power of the test. Following grouping scheme was adopted for the simulation of small sample powers.
Table 3.4.

Grouping scheme

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<th>r₂</th>
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A partial listing of the results of the simulation program for different combinations of n and Beta are given in Tables 3.5 to 3.8. These tables are set up so that power behaviour of the tests for a certain Beta can be readily observed. The results for each combination are given at both the $\alpha = .05$ and $\alpha = .01$ confidence levels.
Table 3.5

Power at $\beta = .50$

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<th>Hartley</th>
<th>Epstein</th>
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</thead>
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Table 3.8.
Power at $\beta = 2.5$

$\alpha = .05$

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$\alpha = .01$

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When $\beta < 1$, two trends developed. First, for any given set of values for $\beta$ and $n$, the powers of the four tests followed the pattern of highest power to lowest power in the order of Gnedenko, E, Hartley and Epstein ($E_p$). However, the powers at $\beta = .05$ of Gnedenko and $E$ shows a close similarity. Secondly, for a given values of $\beta$, for any particular test, the power usually increased as $n$ increased, except when $\beta = .85$.

For values of $\beta > 1$, the above two trends became definite patterns. Except in Tab.3.7. and Tab.3.8. the power of a particular test increased as the sample size increased.

The Graph for Gnedenko's test for $\alpha = .05$ is given below to show just what effect the size of a sample had on the power of a test. See Fig 3.2. PP 113-A.

Based on the figures, it is possible to determine what size sample is necessary to achieve a given power for a particular test. For example consider the above graph, suppose we wanted to achieve a power of .60 at $\beta = 1.75$. Then a sample of 20 is sufficient for this. Similarly at $\beta = .75$ a sample of $n = 50$ is needed. The manner in which the failure times were grouped was given in Table.3.4. It can be seen that for the case of $n = 30$, the data was split into six groups of size five ($k=5$, $r=5$). The powers of Gnedenko and $E_p$ are not affected by different groupings.
Fig. 3.2

Figure 2: Power Curves for Cylindrical, $\alpha = 45$. 

113-A
It was seen that in case of few groups with more number of observations Hartley and Gnedenko test behave equally. Using the grouping scheme as outlined in Tab.3.4, a definite pattern of hierarchy of tests resulted, ranging from highest power to lowest power, the tests ranked as $Q(r)$, $E$, $U$, and $E_p$.

3.6.2. A MONTE CARLO POWER STUDY OF $Q(r)$ AND ITS MODIFICATIONS.

From the above discussion it is obvious that Gnedenko's $Q$ test is found to be superior against the monotone hazard rate alternatives. A simulation experiment was conducted to calculate the powers of the bivariate $F$ test, Gnedenko's $Q(r)$ and its modified version $Q'(r)$ test considered by Harris (See Sec. 3.3). The following alternatives are chosen.

(a) $\chi^2_v$, $v = 1,3,4,8$
(b) Lognormal($\sigma$), $\sigma = .6,.8,1.0,1.2$
(c) Weibull($\eta$), $\eta = .5,2.0$
(d) Beta($\eta,2$)

The simulated powers for some values of $r$ are as tabulated below.
Table 3.9.

Power comparisons of tests for exponentiality; α = .10, n=20, 30.

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<th>$B_{4}^2$</th>
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<td>$\chi^2_4$</td>
<td>.388</td>
<td>.191</td>
<td>.404</td>
<td>.369</td>
<td>.553</td>
<td>.258</td>
<td>.555</td>
<td>.538</td>
</tr>
<tr>
<td>$\chi^2_8$</td>
<td>.924</td>
<td>.554</td>
<td>.927</td>
<td>.878</td>
<td>.983</td>
<td>.788</td>
<td>.995</td>
<td>.990</td>
</tr>
<tr>
<td>Lognormal (o)</td>
<td>.519</td>
<td>.465</td>
<td>.876</td>
<td>.788</td>
<td>.771</td>
<td>.604</td>
<td>.988</td>
<td>.970</td>
</tr>
<tr>
<td>Lognormal (8)</td>
<td>.164</td>
<td>.264</td>
<td>.481</td>
<td>.358</td>
<td>.215</td>
<td>.386</td>
<td>.082</td>
<td>.542</td>
</tr>
<tr>
<td>Lognormal (1, 0)</td>
<td>.112</td>
<td>.221</td>
<td>.271</td>
<td>.232</td>
<td>.121</td>
<td>.209</td>
<td>.977</td>
<td>.324</td>
</tr>
<tr>
<td>Lognormal (1, 2)</td>
<td>.319</td>
<td>.259</td>
<td>.331</td>
<td>.322</td>
<td>.394</td>
<td>.352</td>
<td>.413</td>
<td>.444</td>
</tr>
<tr>
<td>Weibull (5)</td>
<td>.901</td>
<td>.381</td>
<td>.870</td>
<td>.886</td>
<td>.904</td>
<td>.449</td>
<td>.900</td>
<td>.978</td>
</tr>
<tr>
<td>Weibull (2, 0)</td>
<td>.853</td>
<td>.339</td>
<td>.765</td>
<td>.749</td>
<td>.973</td>
<td>.520</td>
<td>.889</td>
<td>.920</td>
</tr>
<tr>
<td>Weibull (1, 2)</td>
<td>.346</td>
<td>.131</td>
<td>.220</td>
<td>.256</td>
<td>.461</td>
<td>.115</td>
<td>.338</td>
<td>.406</td>
</tr>
</tbody>
</table>

The above alternatives are selected because they either resemble the exponential distribution or are frequently proposed as alternatives to the exponential model. For each of the alternative distributions 1000 samples of size $n = 20$ and $n = 30$ are obtained and then used to estimate the power functions of the above tests for $H_0$. As seen in the paper by Fercho and Ringer (1972), the manner in which the grouping is set up has an impact.
on the powers of these tests. The power of the Bi-variate test BF(r), is therefore estimated for various values of r. The study motivated that the performance of BF(r) with different values of r (r ≤ [n/4]) is quite in line with the values reported in Table 3.9. In general, a recommended value for r is given by [n/10]. The values of Q(r) and Q'(r) are r = [n/2] and r = [n/4] as taken in Fercho and Ringer (1972). Furthermore, as is reasonable in the absence of prior knowledge about the alternative, very large as well as very small values of Q(r) and Q'(r) are considered significant.

Thus it was concluded that Gnedenko's statistic Q(r) provides a good test of exponentiality against monotone hazard rate alternatives. The Bi-variate F test offers a good protection against the nonmonotone failure rate alternative, viz. the lognormal distribution, without a significant hazard rate distributions. The two tailed version of Q'(r) considered by Harris appeared to lack any advantage over the Gnedenko Q(r) or the Bi-variate BF(r).

3.6.3. MONTE CARLO ESTIMATES OF POWER FOR \( L_n(p) \)

In this section we compare the power of the Lorenz statistic with that of ten other goodness-of-fit tests of exponentiality against seven different distributions under the alternatives. One
thousand samples of size 20 were generated. The random samples for each alternative except the Gamma were obtained from the same underlying sets of uniform random variables by applying the appropriate inverse probability transformation. The Gamma samples were generated independently as the sum of normalized exponential variates.

Table 3.10 gives the goodness-of-fit tests studied. Table 3.11 gives the different alternatives considered and Table 3.12 Monte Carlo power estimates for significance level $\alpha = .05$.

Table 3.10.
Goodness-of-fit tests studied

<table>
<thead>
<tr>
<th>Name and/or symbol</th>
<th>Definition or description</th>
<th>Equal tail acceptance region and one-tail rejection regions for $n = 20$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorenz statistic</td>
<td>Equ. (3.4.2)</td>
<td>(.09009, .25050) $\geq .23636$, $&lt; .10055$</td>
<td>Gail (1977)</td>
</tr>
<tr>
<td>Kolmogorov-Smirnov tests</td>
<td>$</td>
<td>c</td>
<td>$, $C^+$, $C^-$</td>
</tr>
<tr>
<td>IFR tests</td>
<td>Nonparametric test for increasing hazard</td>
<td>$&gt;-1.96$, $&lt;1.96$, $&gt;-1.645$, $&lt;1.645$</td>
<td>Proschran and Pyke (1967)</td>
</tr>
<tr>
<td>Durbin KS</td>
<td>Two-sided KS-test with exp. mean as $\bar{x}$</td>
<td>Durbin Ks $&lt; .2345$</td>
<td>Lilliefors (1969), Durbin (1975)</td>
</tr>
<tr>
<td>Shapiro-Wilk, $W$</td>
<td>$n\frac{(\bar{x} - X_{(1)})^2}{(n-1)} / \left( \frac{\sum (X_i - \bar{x})^2}{n} \right)^{-1}$</td>
<td>$&lt; .0264$, $&gt;.1121$</td>
<td>Shapiro and Wilk (1972)</td>
</tr>
<tr>
<td>Moran statistic</td>
<td>$2n(1+n)/(6n)^{-1} \log R (8.306, 32.852) \geq 30.143$, $&lt; 10.117$</td>
<td>where $R$ is the ratio of AM to GM</td>
<td></td>
</tr>
</tbody>
</table>

117
Table 3.11.

Alternative distributions studied

<table>
<thead>
<tr>
<th>Name</th>
<th>Distribution F or density f</th>
<th>Support</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weibull</td>
<td>$F(x; k) = 1 - \exp\left(-\beta x^k\right)$</td>
<td>$x \geq 0$</td>
</tr>
<tr>
<td></td>
<td>where, $\beta = \left[\frac{1+k}{k}\right]$</td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>$f(x) = .5$</td>
<td>$[0,2]$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$F(x; k) = 1 - \left{\frac{k-2}{k-1}\right}^{k-1}x^{-k}$</td>
<td>$x \geq \frac{k-2}{k-1}$</td>
</tr>
<tr>
<td>Shifted Pareto</td>
<td>$F(x; k) = 1 - \left{1+x/(k-2)\right}^{1-k}$</td>
<td>$x \geq 0$</td>
</tr>
<tr>
<td>Shifted Exponential</td>
<td>$F(x; \beta, \theta) = 1 - \exp(-x/\theta)\beta$</td>
<td>$x \geq \theta$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$f(x; \alpha) = \alpha x^{\alpha-1}\exp(-\alpha x)/\Gamma(\alpha)$</td>
<td>$x \geq 0$</td>
</tr>
</tbody>
</table>
Table 3.12.

Monte Carlo power estimates based on 1,000 samples of size n = 20

<table>
<thead>
<tr>
<th>Goodness-of-fit test</th>
<th>Weibull $k = .8$</th>
<th>Weibull $k = 1.5$</th>
<th>Uniform $(0, 2)$</th>
<th>Pareto $k = 3$</th>
<th>Pareto $k = 3$</th>
<th>Exponential $\beta = 1, \theta = .2$</th>
<th>Gamma $\alpha = 2$</th>
<th>Null unit exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{20}^{(5)}$ two-tailed</td>
<td>.216</td>
<td>.459</td>
<td>.598</td>
<td>.880</td>
<td>.387</td>
<td>.221</td>
<td>.483</td>
<td>.050</td>
</tr>
<tr>
<td>$L_{20}^{(5)}$ one-tailed</td>
<td>.320</td>
<td>.629</td>
<td>.698</td>
<td>.913</td>
<td>.468</td>
<td>.351</td>
<td>.610</td>
<td>.048(L)</td>
</tr>
<tr>
<td>$C^+$</td>
<td>.297</td>
<td>.000</td>
<td>.000</td>
<td>.068</td>
<td>.530</td>
<td>.003</td>
<td>.000</td>
<td>.046</td>
</tr>
<tr>
<td>$C^-$</td>
<td>.007</td>
<td>.542</td>
<td>.770</td>
<td>.983</td>
<td>.006</td>
<td>.321</td>
<td>.548</td>
<td>.058</td>
</tr>
<tr>
<td>$</td>
<td>C</td>
<td>$</td>
<td>.210</td>
<td>.386</td>
<td>.612</td>
<td>1.000</td>
<td>.445</td>
<td>.198</td>
</tr>
<tr>
<td>IFR two-tailed</td>
<td>.194</td>
<td>.302</td>
<td>.117</td>
<td>.060</td>
<td>.265</td>
<td>.060</td>
<td>.247</td>
<td>.044</td>
</tr>
<tr>
<td>IFR one-tailed</td>
<td>.300</td>
<td>.467</td>
<td>.244</td>
<td>.131</td>
<td>.378</td>
<td>.125</td>
<td>.371</td>
<td>.049(L)</td>
</tr>
<tr>
<td>Moran two-tailed</td>
<td>.238</td>
<td>.519</td>
<td>.462</td>
<td>.870</td>
<td>.330</td>
<td>.432</td>
<td>.558</td>
<td>.052</td>
</tr>
<tr>
<td>Moran one-tailed</td>
<td>.330</td>
<td>.670</td>
<td>.564</td>
<td>.907</td>
<td>.429</td>
<td>.603</td>
<td>.714</td>
<td>.058(L)</td>
</tr>
<tr>
<td>Durbin KS</td>
<td>.173</td>
<td>.385</td>
<td>.521</td>
<td>1.000</td>
<td>.362</td>
<td>.258</td>
<td>.418</td>
<td>.061</td>
</tr>
<tr>
<td>Shapiro-Wilk two-tailed</td>
<td>.179</td>
<td>.235</td>
<td>.681</td>
<td>.463</td>
<td>.463</td>
<td>.047</td>
<td>.163</td>
<td>.047</td>
</tr>
<tr>
<td>Shapiro-Wilk one-tailed</td>
<td>.282</td>
<td>.357</td>
<td>.803</td>
<td>.557</td>
<td>.557</td>
<td>.062(L)</td>
<td>.267</td>
<td>.062(L)</td>
</tr>
</tbody>
</table>

119
The last column of Tab. 3.12 shows that each test rejected approximately 5% of the exponential samples as theory predicts.

Tab. 3.12 shows that the $L_n(0.5)$ statistic compares favourably with the Durbin KS statistic against all alternatives except the Pareto. The Durbin KS is powerful in this case because no values of $x$ lie in the interval $0 \leq x \leq 0.5$. The Moran statistic is more powerful against Gamma, Weibull and Shifted exponential alternatives and less powerful against uniform, Pareto and Shifted Pareto alternatives than $L_n(0.5)$; however the differences are not great. The Lorenz statistic substantially outperforms IFR against all alternatives and it outperforms corresponding Kolmogorov Smirnov tests $c^+$, $c^-$ and $|c|$ against all but uniform, Pareto and Shifted Pareto alternatives. Thus Tab. 3.10 shows that $L_n(0.5)$ has acceptable power against a broad range of alternatives when compared with other tests of exponentiality.

3.6.4. Monte Carlo Power Estimates of $g_n$

In this section the power of the sample Gini statistic given by (3.4.3) has been compared that of Lorenz curve (3.4.2), Pietra statistic, $P_n$ and a scale-free statistic $R_n$, where

$$P_n = \sum | x_i - \bar{x} | / 2n\bar{x} \quad (3.6.3)$$

and

$$R_n = \bar{x} \left( \frac{n-1}{\sum (x_i - \bar{x})^2} \right)^{1/2} \quad (3.6.4)$$

which is a similar version of location-free Shapiro-Wilk (1972) test.
The following table gives the power for $G_{20}$, $L_{20} (.5)$, $P_{20}$ and $R_{20}$ with respect to the Weibull, Uniform, Pareto, Exponential and Gamma alternatives as in Tab. 3.11.

Table 3.13.

Monte Carlo power estimates (based on 1,000 samples of size $n = 20$)

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$G_{20}$</th>
<th>$L_{20} (.5)$</th>
<th>$P_{20}$</th>
<th>$R_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>One- sided sided</td>
<td>.336</td>
<td>.320</td>
<td>.330</td>
<td>.277</td>
</tr>
<tr>
<td>Two- sided sided</td>
<td>.239</td>
<td>.216</td>
<td>.237</td>
<td>.207</td>
</tr>
</tbody>
</table>

Weibull, shape=1.8

<table>
<thead>
<tr>
<th>Weibull, shape=1.8</th>
<th>.643</th>
<th>.629</th>
<th>.596</th>
<th>.596</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weibull, shape=1.5</td>
<td>.829</td>
<td>.698</td>
<td>.718</td>
<td>.857</td>
</tr>
<tr>
<td>Uniform on (0,2)</td>
<td>.829</td>
<td>.698</td>
<td>.718</td>
<td>.857</td>
</tr>
<tr>
<td>Shifted Pareto</td>
<td>.553</td>
<td>.470</td>
<td>.468</td>
<td>.458</td>
</tr>
<tr>
<td>DF $1 - 1/(1+x)^2$</td>
<td>.362</td>
<td>.351</td>
<td>.291</td>
<td>.196</td>
</tr>
<tr>
<td>Shifted exp.</td>
<td>.622</td>
<td>.483</td>
<td>.483</td>
<td>.465</td>
</tr>
<tr>
<td>DF $1 - \exp(-x^2)$</td>
<td>.048</td>
<td>.050</td>
<td>.050</td>
<td>.050</td>
</tr>
<tr>
<td>Gamma, shape=2</td>
<td>.037</td>
<td>.050</td>
<td>.050</td>
<td>.050</td>
</tr>
<tr>
<td>Null case</td>
<td>.048</td>
<td>.050</td>
<td>.050</td>
<td>.050</td>
</tr>
<tr>
<td>Unit exp.</td>
<td>.036</td>
<td>.053</td>
<td>.050</td>
<td>.050</td>
</tr>
</tbody>
</table>

It is obvious from the table that $G_{20}$ has greater power than $P_{20}$, $L_{20} (.5)$ and $R_{20}$ against most of the alternatives studied. The two sided equal tail acceptance regions are $(0.37048, 0.62952)$ for $G_{20}$, $(0.09009, 0.25050)$ for $L_{20} (.5)$, $(0.2578, 0.4628)$ for $P_{20}$ and $(0.7388, 1.5176)$ for $R_{20}$. Corresponding one sided critical regions obtained were $G_{20} > 0.60902$ or $G_{20} < 0.39098$, $L_{20} > 0.60902$ or $L_{20} < 0.39098$, $P_{20} > 0.60902$ or $P_{20} < 0.39098$, $R_{20} > 0.60902$ or $R_{20} < 0.39098$. 

121
The Tab. 3.11 shows that the one-sided $G_{20}$ test is slightly more sensitive than $L_{20} (0.5)$ against all alternatives investigated except the Pareto which is peculiar in having no probability in the interval [0, 0.5]. Note that the two-sided $L_{20} (0.5)$ test is as sensitive as $G_{20}$ against the Gamma alternative and the one-sided tests differ a little. The statistic $G_{20}$ is more sensitive than $P_{20}$ in all cases except the one-sided test against Shifted Pareto alternative. The Gini statistic is more powerful than $R_{20}$ against all alternatives except the uniform and Shifted Pareto for which differences are slight.