TESTS FOR EXPONENTIALITY AGAINST NBUE ALTERNATIVES.

7.1. INTRODUCTION.

A life distribution \( F \) with finite mean \( \mu \) is said to be New Better than Used in Expectation (NBUE) if and only if

\[
\mu_F(t) \leq \mu_F(0)\]

\[
\Rightarrow \int_0^t \frac{F(u)}{F(t)} \, du \leq \mu_F(0)\]

\[
\Rightarrow \int_0^t \frac{F(u)}{F(t)} \, du \leq \mu_F(0) \frac{F(t)}{t} \quad \text{for all } 0 < t < \infty \quad (7.1.1)
\]

This is equivalent to

\[
\int_0^y \frac{F(u)}{F(y)} \, du \geq \mu_F(y) \quad , \quad 0 < y < \infty \quad (7.1.2)
\]

Let \( H_F^{-1}(t) = \int_0^{F^{-1}(t)} F(x) \, dx \).

Then if \( F \) is continuous then (7.1.2) can be written as (using transformation \( y = F^{-1}(t) \)),

\[
H_F^{-1}(t) \geq H_F^{-1}(1) \quad , \quad 0 \leq t \leq 1 \quad (7.1.3)
\]

If \( F \) is exponential then equality obtains in (7.1.2). Conversely, suppose equality holds in (7.1.2), this is true only if \( F \) has density \( f \) and is given by the relation

\[
\mu f(y) = \bar{F}(y) \quad , \quad 0 < y < \infty.
\]
This differential equation is satisfied by the exponential and the degenerate distribution at 0. If the degeneracy case is dropped then our problem of hypothesis testing is

\[ H_0 : F \text{ is exponential} \]

versus

\[ H_1 : F \text{ is NBUE, not an exponential,} \]
on the basis of random samples \( x_1, x_2, \ldots, x_n \) from \( F \).

The Theorem 2.2 of Marshall and Proschan (1972) shows that the average waiting time between any two consecutive failures when no planned replacement policy is adopted is smaller than a equal to the similar quantity when an age replacement policy is adopted iff the life distribution is NBUE. Now suppose the average waiting time between the consecutive failures is an important criterion in deciding whether to adopt an age replacement policy over the unplanned replacement policy for a given system. One reasonable way to decide would be to test the above hypothesis for the life distribution of the given system on the basis of the life data of the same. Rejection of \( H_0 \) would be interpreted as favouring the adoption of age replacement plan.

The problem of testing \( H_0 \) against \( H_1 \) for continuous \( F \) is equivalent to testing

\[ H_0^* : H_F^{-1}(t) = t \mu_F(0), \quad 0 \leq t \leq 1 \]

versus

\[ H_1^* : H_F^{-1}(t) \geq t \mu_F(0) \text{ with strict inequality} \]

for at least one \( 0 < t < 1 \).
This problem has been considered by many authors. Hollander and Proschan introduced an NBUE test using the Total-Time-on-Test statistic. The TTT test statistic is \( \sum_{i=1}^{n-1} U_i \), where \( U_i = T_i / T \), \( i = 0,1,2,\ldots,n \) and \( T_i = \sum_{j=1}^{i} D_j \) has been considered earlier by Barlow (1968), Bickel and Doksum (1969), Barlow and Proschan (1969), Barlow et al. (1972) and Barlow and Doksum (1972) as a test statistic for testing exponentiality against IFR alternatives. However, Hollander and Proschan (1975), and later Klefsjo (1983) show that \( k \) arises in a natural way as a test against NBUE (or NWUE) alternatives, where

\[
k = n^{-2} \sum_{i=1}^{n} d_i X_i, \tag{7.1.4}
\]

\( d_i = 3n/2 - 2i + 1/2. \)

Hollander and Proschan consider the parameter

\[
\eta(F) = \int_0^\infty \tilde{F}(x) \left( \mu_F(x) - \mu_F(\infty) \right) dF(x) \quad \text{as a measure of deviation for a given } F \text{ from exponential to NBUE. The sample counterpart to } \eta(F) \text{, obtained by replacing } F \text{ by the empirical distribution function } \tilde{F}_n \text{ is } k. \text{ Dividing } k \text{ by } \bar{X} \text{ to make it scale invariant, Hollander and Proschan proposed } k^* = k / \bar{X} \text{ as a statistic for testing exponentiality against NBUE alternatives and pointed out that}
\]
\[ n^{-1} \sum_{i=1}^{n} U_i = n k^* + (n-1)/2 \quad (7.1.5) \]

Significantly large values of \( k^* \) suggest NBUE alternatives and significantly small values suggest NWUE alternatives. Thus the total-time-on-test statistic, originally proposed to detect IFR (DFR) alternatives can be more suitably viewed as a test statistic designed to detect the larger NBUE (NWUE) class. Barlow (1968) has given tables of percentile points of \( \sum_{i=1}^{n-1} U_i \) for \( n = 2(1)10 \) and \( n \) in the lower and upper .01, .05 and .10 regions. The large sample approximation under \( H_0 \) treats asymptotically

\[ k' = (12n)^{1/2} k^* \quad (7.1.6) \]
as an NC(0,1) random variable.

Klefsjo (1983), by considering the TTT-plot, is also led to the derivation of the \( k^* \) statistic as a test statistic for exponentiality versus NBUE alternatives.

Barlow and Doksum (1972) advocated the statistic

\[ D^+ = \max_{1 \leq i \leq n} \left( U_i - i/n \right) \]

for testing \( H_0 \) versus \( H_1 \), large values being significant. Koul (1978b) shows that rejecting \( H_0 \) for large values of \( D^+ \) can be more appropriately viewed as a test against the larger class of alternatives. The null distribution of \( D^+ \) as tabled by Birnbaum and Tingey (1951) is also approximate in this testing context. Asymptotically, under \( H_0 \),

\[ P \left( n^{1/2} D^+ \leq x \right) = 1 - \exp(-2x^2) \]

alternatives using the parameter

\[ \eta^*(F) = \int_0^\infty F(x) (\mu_F(0) - \mu_F(x)) \, dx. \]

Note that \( \eta^*(F) \) has the integrand in common with the parameter \( \eta(F) \) suggested by Hollander and Proschan with the essential difference being that \( \eta^*(F) \) integrates with respect to "\( dx \)" rather than \( dF(x) \).

Now we study the test statistic given by Koul (1978) in the following sections.

7.2. KOUl'S TEST

7.2.1. TEST STATISTIC AND CONSISTENCY

Let \( F \) be an NBUE with \( 0 < \mu < \infty \). Let

\[ D = \operatorname{Sup} \left\{ \mu^{-1} \int_0^y F(x) \, dx - F(y) \right\} \quad (7.2.1) \]

If \( F \) is exponential then (7.2.1) will be zero. On the other hand, if \( F = 0 \) and \( F \) is NBUE then \( F \) must be an exponential. All degenerate distributions have \( D = 1 \). The farther \( D \) is from 0, the more evidence there is for \( F \) belonging to \( H_1 \).

Let \( J \) be a nondecreasing continuous function on \([0,1]\) with

\[ b(J,F) = \int_0^\infty J(\mu^{-1}L(y)) \, dF(y) , \]

\[ L(y) = \int_0^y F(x) \, dx. \]
Clearly if $F$ is exponential then the left equality obtains in (7.2.2) and on the other hand if left equality obtains in (7.2.2) and $F$ is NBUE, since $J$ is nonincreasing and continuous then we must have

$$F(y \mid \mu^{-1} L(y) = F(y)) = 1.$$  

If an NBUE $F$ is absolutely continuous, then we conclude that equality (7.2.2) implies $F$ is exponential everywhere. If $F$ is degenerate at $\mu$, then $b(J,F) = J(1)$. Thus $b(J,F)$ is a measure of exponentiality of $F$. The larger $b(J,F)$ is than $\int_0^\infty J(F(y)) \, dF(y)$, the more there is evidence in favour of NBUE.

The sample estimate is obtained by substituting the empirical distribution function $F_n$ of $F$ in $b(J,F)$ and $\Delta F$.

Let

$$L_n(y) = \int_0^y F(x) \, dx$$

$$= n^{-1} \sum_{i=1}^n I(x_{(i)}, y),$$

where $x_{(i)}$'s are ordered statistics and

$$I(x, y) = \begin{cases} 1, & \text{if } x < y \\ 0, & \text{otherwise} \end{cases}$$

Consider

$$b_n(J) = b(J,F) = \int_0^\infty J(x^{-1} L_n(y)) \, dF_n(y)$$

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Observe that

\[
\bar{X}^{-1} L_n(X_{(i)}) = n^{-1} \sum_{j=1}^{n} (X_{(i)} - X_{(j)}) / \bar{X}
\]

\[
= \left[ \sum_{j=1}^{n} X_{(j)} + (n-1)X_{(i)} \right] / n \bar{X}
\]

\[
= \left[ \sum_{j=1}^{i} (n-j+1) (X_{(j)} - X_{(i)}) \right] / \left[ \sum_{j=1}^{n} (n-j+1) (X_{(j)} - X_{(i)}) \right]
\]

\[
= W_{ni}, \quad 1 \leq i \leq n. \quad (7.2.3)
\]

Therefore,

\[
b_n(j) = n^{-1} \sum_{i=1}^{n} J(W_{ni}) \quad (7.2.4)
\]

If \( b_n(j) \) is large reject \( H_0 \) in favour of \( H_1 \).

Now,

\[
D_n = D(F_n) = \sup_{0 < y < \infty} \left[ \int_0^y \frac{F_n(\infty)}{\mu} dx - F_n(y) \right]
\]

\[
= \sup_{0 < y < \infty} \left[ \frac{L_n(y)}{\bar{X}} - F_n(y) \right]
\]

\[
= \max_{1 \leq i \leq n} \left[ \frac{L_n(X_{(i)})}{\bar{X}} - \frac{1}{n} \right]
\]

\[
= \max_{1 \leq i \leq n} \left[ W_{ni} - 1/n \right] \quad (7.2.5)
\]

Here we reject \( H_0 \) in favour of \( H_1 \) if \( D_n \) is large. Note that the derivations of \( b_n \) and \( D_n \) statistic are valid even if \( F \) is discrete.

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Consistency of the test will be seen from the following lemma.

Lemma 7.1. Let $F$ be a life distribution with $0 < \mu < \infty$ and $F(0) = 0$. Then

$$\sup_{0 < y < \infty} \left| L_n(y) - L(y) \right| \xrightarrow{a.s.} 0, \quad \cdots \quad (7.2.6)$$

where

$$L(y) = \int_0^y F(x) \, dx, \quad 0 < y < \infty.$$  

Proof. Since both $L_n$ and $L$ are nondecreasing and continuous and since $L(0) = 0$ and $L(\infty) = \infty$, it is sufficient to show

(i) $\left| L_n(y) - L(y) \right| \xrightarrow{a.s.} 0$ for each fixed $0 < y < \infty$ and

(ii) for each $\epsilon > 0$ there exists $y_0$ such that

$$\int_{y_0}^\infty F(x) \, dx < \epsilon$$

almost surely.

The case (i) is a direct consequence of the Glivenko-Cantelli lemma.

To prove (ii) choose $y_0$ such that

$$\int_{y_0}^{\infty} F(x) \, dx < \epsilon$$

which is possible since $\mu < \infty$.

Next,

$$\lim_{y_0} \int_{y_0}^\infty F(x) \, dx = \lim_{y_0} \sum_{i=1}^{n} (X_i' - y_0) I(X_i' > y_0)$$
The generalized version of the above lemma for absolutely continuous $F$ is due to Barlow - Vanzwet (1970). The lemma is stated here without proof.

**Lemma.** 7.2. Let $J$ be a nondecreasing uniform continuous function on $[0,1]$, $F(0) = 0$ and $0 < \mu < \infty$. Then

$$\omega = \int_0^\omega F(y) \, dy \leq \epsilon,$$

using the strong law of large numbers.

The importance of this lemma is that the result remains true for any $F$ as long as its mean is finite and $F(0) = 0$.

**Proposition.** 7.1. Consider the problem of testing $H_0$ against $H_1$.

(a) The tests that reject $H_0$ in favour of an $F \in H_1$, where $b_n(J)$ is large are consistent against all those $F \in H_1$ for which

$$b_n(J) - b(J,F) \rightarrow 0.$$  

(7.2.7)

(b) The tests that reject $H_0$ in favour of $H_1$ when $D_n$ is large are consistent against all those $F \in H_1$ for which $DF(0) > 0$.

**Proof.** (a) The first part of (a) follows from (7.2.7). Second part of (a) follows from the fact that $F \in H_1$.
The continuity implies that there exists an open interval $I$ of $F$-measure such that
\[ \mu^{-1} L(y) > F(y) \quad \text{for all } y \in I. \]

Since $J$ is continuous and strictly increasing we have
\[
\lim_{y \to I} b(J, F) - \int_{J} J(F) \, dF \geq \int_{I} [J(\mu^{-1} L(y)) - J(F(y))] \, dF(y) > 0.
\]

(b) This part follows directly from Lemma 7.1.

Remarks. (i). We give an example of an $F \in H_1$ such that (7.2.8) is satisfied but $F$ is discrete. Take $F$ to be the two point distribution giving probability $p$ to $\langle a \rangle$ and $q = 1 - p$ to $\langle b \rangle$, where $0 < a < b$ and $pb < a(1+p)$. This guarantees that
\[ \mu^{-1} L(y) > F(y) \quad \text{for some } y. \]

Assume $J$ strictly increasing and note that
\[ b(J, F) = p J(\mu^{-1} L(a)) + q J(1) \]
and
\[ \int J(F) \, dF = p J(p) + q J(1). \]

Thus (7.2.8) would be satisfied iff $\mu^{-1} L(a) > p$, which is the case iff $bp < a(1+p)$ because $\mu^{-1} L(a) = a/(ap+bq)$.

Hence $b(J)$ is consistent against the above discrete $F$'s in $H_1$. One can also show that
\[ b(F) = \max \{ a/(ap+bq), 1-p \}, \]
which is clearly as large as $p < 1$ or $0 < a$.

(ii) Consider $F(x) = q \lfloor x \rfloor$, $x \geq 0$, $0 < q < 1$, then
\[ L(y) = \begin{cases} y & 0 \leq y \leq 1 \\ \frac{(1-q_j)}{(1-q)} \cdot q^j(y-j) & j \leq y \leq j+1, \ j \geq 1 \end{cases} \]

and

\[ b(J,F) = \sum_{j=1}^{\infty} j(1-q^j) pq^{j-1} = \int F dF. \]

On the other hand,

\[ D(F) = \max \left[ \max_{j \geq 1, j \leq y \leq j+1, p > 0} \sup_{q \geq 0} pq^j(y-j), p \right] = p > 0 \]

as long as \( p > 0 \).

Thus in some sense \( D(F) \) is a better discriminator of exponentiality than \( b(J,F) \). Moreover the \( D_n \) test is certainly consistent.

(iii) If we choose \( J(u) = u \) then we get the Hollander and Proschan test statistic (1975) based on total-time-on-test.

7.2.2. BAHADUR EFFICIENCY COMPARISON.

Here the relative Bahadur efficiency of two statistics \( b_n(J) \) and \( D_n \) are computed. We will be giving the parameters which provides in the computation briefly as below.

The two statistics under considerations are \( b_n(J) \) when \( J(u) = u \). Call it the \( W_n \) statistic.

\[ i.e. \quad W_n = n^{-1} \sum_{i=1}^{n} W_i, \quad \text{where } (W_i) \text{ is defined in (7.2.3)}, \]

and \( D_n \) statistics defined in (7.2.5).
Slope of $W_n$ statistic.

\[ C_W(F) = 2 f(b(F)) \]  

(7.2.9)

where

\[ f(t) = \ln \left\{ e^t \left( e^{u_0} - 1 \right) / u_0 \right\}, \]

$u_0$ is the solution of the equation,

\[ f(t) = -\ln \inf_{u>0} \left( e^u - 1 \right) / u, \quad 1/2 \leq t < 1. \]

where $u_1, u_2, \ldots, u_n$ are random sample of size n-1 from the uniform $[0,1]$.

Slope of the $D_n$ statistic.

\[ C_D(F) = 2 g(D(F)) \quad \text{for all } F \in H^1, \text{ with } 0 < D(F) < 1. \]  

(7.2.10)

where $g(d) = f(d, t_0)$, $t_0$ is the solution of

\[ \ln \frac{(d+t)(1-t)}{t(1-d-t)} = \frac{d}{t(1-t)} \]

Koul has computed Relative Behadur efficiency as

\[ \varepsilon(W, D) = \frac{C_W}{C_D}. \]

(7.2.11)

Let us consider the Weibull distribution given by

\[ F(x) = \exp(-x^\theta), \quad \theta > 1. \]

It is well known that this c.d.f is IFR and hence NBUE. Let
x be an exp(1) random variable, then $x^{1/\theta}$ has the Weibull(\(\theta\)) distribution. It follows that 

$$\mu = \int (1/\theta + 1) = 1/\theta \int 1/\theta.$$

Furthermore,

$$\int_0^\infty F^2(x) \, dx = \int_0^\infty e^{-2x} \, dx = \frac{1/\theta}{e^{2^{1/\theta}}}$$

and hence

$$b(\theta) = \frac{1}{\mu} \int_0^\infty F^2(x) \, dx = (1/\theta)^{1/\theta}, \quad \theta > 1.$$

let

$$k(y) = \mu^{-1} \int_0^y F(x) \, dx - F(y)$$

$$k'(y) = \mu^{-1} F(y) - f(y) = F(y) \left( \mu^{-1} - e^{y^{(\theta-1)}} \right)$$

Hence

$k$ is nondecreasing for $y \leq (1/\theta)^{1/(\theta-1)}$

and

$k$ is nonincreasing for $y \geq (1/\theta)^{1/(\theta-1)}$

Hence supremum of $k$ is attained at $1/\theta = (1/\theta)^{1/(\theta-1)}$

Then,

$$\Xi(\theta) = \Xi(\theta) = \sup k(y) = k(1/\theta)$$

$$= I \left( \frac{[1/\theta]^{-\theta/(\theta-1)}}{1/\theta} , 1/\theta - 1 \right) - \left[ 1 - e^{-(1/\theta)^{\theta/(\theta-1)}} \right] ,$$

$\theta > 1$, where $I(u, p) = \int_0^{(p+1)} z^p e^{-2} \, dz$, $u > 0$, $p > -1$

Note that $1 < \theta < \infty \Rightarrow -1 < 1/\theta - 1 < 0$. The function $I(u, p)$
has been tabulated by K. Pearson (1922). These tables together with
those of David (1963) for the Gamma functions were used in
computing $D(x)$ for selected values of $1/e$.

For the selected values of $1/e$, Tab. 7.1. below gives $u_v(x)$, $t_v(x)$, $b(x)$, $D(x)$, $C_v(x)$, $C_p(x)$ and $e(W, D)$.

Table 7.1.

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