CHAPTER 6
TESTING WHETHER NEW IS BETTER THAN USED

6.1. INTRODUCTION.

Amongst the classes of positive aging distributions introduced in chapter 2, New Better than Used is a class larger immediate to IFRA class of life distributions. Therefore one may like to have tests for exponentiality against NBU alternatives. The tests are based on measures of deviations of $F$ from exponentiality towards NBU alternatives.

A life distribution $F$ is New Better than Used if

$$
\bar{F}(x+y) \leq \bar{F}(x) \bar{F}(y) \text{ for all } x, y \geq 0, \text{ where } \bar{F} = 1 - F.
$$

This may be interpreted as stating that the chance $\bar{F}(x)$ that a new unit will survive to age $x$ is greater than the chance $\bar{F}(x+y) / \bar{F}(y)$ that an unfailed unit of age $y$ will survive an additional time $x$. That is, a new unit has stochastically greater life than a used unit of any age.

So, in this chapter our interest is to test a hypothesis of exponentiality against NBU but not exponential.

That is to test

$$
H_0 : F(x) \text{ is exponential}
$$

versus

$$
H_1 : F \text{ is NBU and not exponential.}
$$
on the basis of random sample $x_1, x_2, \ldots, x_n$ from the distribution $F$.

In this connection tests are designed by Hollander and Proschan (1972) and Koul (1977). A detailed study of the former test will be done in the next section and a brief discussion of Koul's test will be given at the end of this chapter.

As an example of a practical problem motivating the choice of the null hypothesis $H_0$ above, consider a unit subject to shocks occurring successively in time according to a poisson process. Since the occurrence of shocks and their effects cannot be directly observed, it is not known whether shocks already experienced by the unit make it more likely to fail under the impact of future shocks or not. However, if $P$ is the probability that the unit survives the first $k$ shocks, then it is believed that either

$$
\begin{align*}
\text{(a)} & \quad P_k = \frac{P_{l+k}}{P_l} \quad \text{for all } k, l \geq 0 \\
\text{(b)} & \quad P_k \geq \frac{P_{l+k}}{P_l} \quad \text{for all } k, l \geq 0.
\end{align*}
$$

Since under Hypothesis (a), the life length is exponential and under hypothesis (b), it is NBU (Esary, Marshall, Proschan (1970 b) The.3.1)), a reasonable way to test (a) vs (b) would be to test $H_0$ vs $H_1$ above from the life length observation.
6.2. HOLLANDER AND PROSCHAN TEST

6.2.1. TEST PROCEDURE.

It is well known that \( F \) is NBU

\[
\Rightarrow F(t+x) \leq F(t) \quad \text{for all } t, x > 0
\]

\[
\Rightarrow \frac{F(t+x)}{F(t)} \leq F(x) \quad \text{for all } t, x > 0
\]

\[
\Rightarrow F(t+x) \leq F(x) F(t) \quad \text{for all } x, t > 0
\]

\[
\Rightarrow F(x) F(t) - F(t+x) \geq 0 \quad \text{for all } x, t > 0
\]

\[
\Rightarrow \int_0^\infty \int_0^\infty [F(x) F(t) - F(t+x)] F(x) F(t) \geq 0
\]

\[
\Rightarrow 1/4 - \int_0^\infty \int_0^\infty F(x+t) F(x) F(t) \geq 0.
\]

Define a parameter

\[
\Delta(F) = \int_0^\infty \int_0^\infty F(x+t) F(x) F(t)
\]

Under \( H_0 \) :

\[
\Delta(F) = \int_0^\infty \int_0^\infty e^{-(x+t)} e^{-x} e^{-t} dx dt
\]

\[
= 1/4
\]

and under \( H_1 \), \( \Delta(F) < 1/4 \)

Let \( \psi(F) = 1/4 - \Delta(F) \),

which is a measure of the deviation of \( F \) from \( H_0 \).
Then the classical non-parametric approach of replacing \( F \) by
the empirical distribution function \( F_n \) suggests rejecting \( H_0 \) in
favour of \( H_1 \) if \( \Delta(C(F)) \) is "too small".

Now define a kernel as

\[
\phi_{HP}(X_i, X_j, X_k) = \begin{cases} 
1 & \text{if } X_i > X_j + X_k \\
0 & \text{otherwise.}
\end{cases}
\]  
(6.2.3)

then the corresponding symmetric kernel is given by

\[
\phi^*_{HP} = 1/3 \left[ \phi_{HP}(X_i, X_j, X_k) + \phi_{HP}(X_j, X_i, X_k) + \phi_{HP}(X_k, X_i, X_j) \right]
\]

Using this a U-statistic can be defined as

\[
HP = \left( \begin{array}{c} n \\ 3 \end{array} \right)^{-1} \sum_{i < j < k} \phi^*_{HP}(X_i, X_j, X_k)
\]

\[
= \frac{2}{n(n-1)(n-2)} \sum_{i < j < k} \phi^*_{HP}(X_i, X_j, X_k)
\]  
(6.2.4)

This statistics was proposed by Hollander and Proschan
(1972).

Again, \( E(HP) = E \left[ \phi_{HP}(X_i, X_j, X_k) \right] \)

\[
= P \left[ X_i > X_j + X_k \right]
\]

\[
= \int \int FC(x+t) dFC(x) dFC(t)
\]

\[
= \Delta(C(F)).
\]
Hence Hollander and Proschan (1972, 1975) have suggested the test for \( H_0 \) as reject \( H_0 \) if \( HP < HP_\alpha \), where \( HP_\alpha \) is the \( \alpha \)th quantile of the null distribution of the statistic HP.

6.2.2. UNBIASEDNESS, ASYMPTOTIC NORMALITY AND CONSISTENCY.

In this section we first show that the test rejects \( H_0 \) if \( HP \leq HP_\alpha \), where \( HP_\alpha \) satisfies \( P_0[HP \leq HP_\alpha] = \alpha \) is unbiased. That is, \( P_1[HP \leq HP_\alpha] \geq \alpha \), where \( P_0 \) and \( P_1 \) are respectively the probabilities computed for an \( F \) belonging to \( H_1 \) and \( H_0 \).

Definition. 6.1. A function \( f(x) \geq 0 \) defined on \([0, \infty)\) is superadditive if and only if
\[
f(x+y) \geq f(x) + f(y) \quad \text{for all } x, y \geq 0.
\]

Definition. 6.2. Let \( F \) and \( G \) be continuous distributions, \( G \) be strictly increasing on its support and \( F(0) = 0 = G(0) \). Then \( F \) is said to be superadditive with respect to \( G \) if \( G^{-1}(F) \) is superadditive, that is
\[
G^{-1}F(x_1 + x_2) \geq G^{-1}F(x_1) + G^{-1}F(x_2), \quad (6.2.5)
\]
for all \( x_1, x_2 \geq 0 \).

When the inequality is reversed, \( F \) is said to be subadditive with respect to \( G \).
THEOREM 6.1. Let $F$ be superadditive with respect to $G$ then

$$HPC(X) \leq HPC(Y),$$

where $X = (X_1, X_2, \ldots, X_n)$ is a random sample from $F$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ is a random sample from $G$.

Proof. Let $Y'_i = G^{-1}(F(X_i))$, $i = 1, 2, \ldots, n$. Then

$$st\quad (Y'_1, Y'_2, \ldots, Y'_n) = (Y_1, Y_2, \ldots, Y_n).$$

Now we show,

$$X_3 \geq X_1 + X_2 \quad \Rightarrow \quad F(X_3) \geq F(X_1 + X_2)$$

$$\Rightarrow \quad G^{-1} F(X_3) \geq G^{-1} F(X_1 + X_2)$$

$$\Rightarrow \quad G^{-1} F(X_3) \geq G^{-1} F(X_1) + G^{-1} F(X_2)$$

due to (6.2.5).

Equivalently

$$Y'_3 \geq Y'_1 + Y'_2$$

From (6.2.6) we have

$$\phi(Y'_3, Y'_1 + Y'_2) \geq \phi(X_3, X_1 + X_2)$$

and thus,

$$HPC(X) \leq HPC(Y') = HPC(Y).$$

Corollary 6.1. The Hollander and Proschan test is unbiased for NBU alternatives.

Proof. It can be easily shown that $F$ is NBU if and only if $F$ is superadditive with respect to the exponential distribution.

Take $G$ to be unit exponential.
\[ i.e. \; e^{-x}, \; x \geq 0. \]

From the Theorem we have \( \text{HP}(X) \leq \text{HP}(Y') \) where,
\[ Y' = [-\log F(X)] \], \( X \) follows an NBU distribution.

We have \( P \{ \text{HP}(X) \leq \text{HP}_\alpha \} = \alpha \)

Now, from Theorem 6.1.
\[ \text{HP}(Y') \leq \text{HP}_\alpha \Rightarrow \text{HP}(X) \leq \text{HP}_\alpha \]

Therefore,
\[ P_0 \{ \text{HP}(X) \leq \text{HP}_\alpha \} = P \{ \text{HP}(Y') \leq \text{HP}_\alpha \} \leq P_1 \{ \text{HP}(X) \leq \text{HP}_\alpha \} \]

i.e. \( \alpha \leq P_1 \{ \text{HP}(X) \leq \text{HP}_\alpha \} \)

i.e. Power of the test is at least \( \alpha \)

which implies that the test is unbiased.

Some examples of parametric families of the distributions which are increasingly superadditive as the parameter \( \theta \) increases are

(a) Weibull, \( F_\theta(t) = 1 - \exp(-\lambda t^\theta), \; t \geq 0, \; \lambda \geq 0. \)

(b) Gamma, \( F_\theta(t) = \frac{1}{\theta} \int_0^t \lambda \theta x^{\theta-1} \exp(-\lambda x) \; dx, \; t \geq 0, \; \lambda \geq 0. \)

In each case for fixed \( \lambda \geq 0 \) and \( 0 < \theta_1 < \theta_2 < \infty \), \( F_{\theta_2} \) is superadditive with respect to \( F_{\theta_1} \), it follows that the power function is an increasing function of the parameter \( \theta \).

For the asymptotic normality we can use Hoeffding's (1948) U-statistic theory (See Theorem 1.8.2).
Let \( \psi(x) = E \left[ \phi^*_\Lambda (x, x', x) \right] \)

\[ = \frac{1}{3} E \left[ \phi_{\Lambda} (x, x_2, x_3) + \phi_{\Lambda} (x_2, x, x_3) + \phi_{\Lambda} (x_3, x_2, x) \right] \]

(6.2.7)

\[ E \left[ \phi_{\Lambda} (x, x_2, x_3) \right] = P (x > x_2 + x_3) \]

\[ = E \left[ P (x > x_2 + x_3) \right] \]

\[ = E \left[ P (x_3 < x_2 - x) \right] \text{, if } x_1 < x_2 \]

\[ = E \left[ 1 - e^{-x_1} \right] \]

Therefore,

\[ E \left[ \phi_{\Lambda} (x, x_2, x_3) \right] = \int_0^x (1 - e^{-x_2}) e^{-x_3} dx_2 \]

\[ = 1 - e^{-x_1} - x_1 e^{-x_1} \] (6.2.8)

\[ E \left[ \phi_{\Lambda} (x_1, x_2, x_3) \right] = P (x_1 > x_2 + x_3) \]

\[ = E \left[ P (x_1 > x_2 + x_3) \right] \]

\[ = E \left[ e^{-x_2} \right] \]

\[ = \int_0^{x_3} e^{-x_3} dx_3 \]

\[ = \int_0^{x_3} \frac{1}{x_3} e^{-x_3} dx_3 \]
Using (6.2.8), (6.2.9) and (6.2.10) in (6.2.7) we get

\[
\psi_1(x) = \frac{1}{3} \left[ 1 - e^{-x} - x e^{-x} + \frac{1}{2} e^{-x} + \frac{1}{2} e^{-2x} \right]
\]

\[
\psi_1^2(x) = \frac{1}{9} \left[ 1 + x^2 e^{-x} - 2x e^{-x} \right]
\]

\[
E[\psi_1^2(x)] = \frac{1}{9} \int_0^\infty (1 + x^2 e^{-x} - 2x e^{-x}) e^x \, dx
\]

\[
= \frac{1}{9} \left[ 1 + \frac{2}{27} - \frac{1}{2} \right]
\]

\[
= \frac{31}{486}
\]

Therefore,

\[
\zeta_1(F) = E[\psi_1^2(x)] - [\Delta F]^2 = \frac{31}{486} - \frac{1}{15}
\]
\[ \psi(x_1, x_2) = E \left[ \phi_{HP}^*(x_1, x_2, X) \right] \]

\[ = \frac{1}{3} E \left[ \phi_{HP}^*(x_1, x_2, X) + \phi_{HP}^*(x_2, x_1, X) + \phi_{HP}^*(X, x_2, x_1) \right] \]

(6.2.11)

\[ E \left[ \phi_{HP}^*(x_1, x_2, X) \right] = P \left[ x_1 > x_2 + x_3 \right] \]

\[ = P \left[ x_3 < x_1 - x_2 \right] \]

\[ = 1 - e^{-(x_1 - x_2)} \text{, if } x_1 > x_2. \]

(6.2.12)

\[ E \left[ \phi_{HP}^*(x_2, x_1, X) \right] = P \left[ x_2 > x_1 + x_3 \right] \]

\[ = P \left[ x_3 < x_2 - x_1 \right] \]

\[ = 1 - e^{-(x_2 - x_1)} \text{, if } x_2 > x_1. \]

\[ E \left[ \phi_{HP}^*(X, x_2, x_1) \right] = P \left[ x_3 > x_1 + x_2 \right] \]

\[ = e^{- (x_1 + x_2)} \text{.} \]

Therefore,

\[ E \left[ \psi(x_1, x_2) \right] = \int \int \psi(x_1, x_2) \, dF(x_1) \, dF(x_2). \]

\[ = \frac{1}{9} \left[ \int \int_{x_1 < x_2} \left( 1 - e^{- (x_2 - x_1)} + e^{- (x_1 + x_2)} \right)^2 \, e^{- (x_1 + x_2)} \, dx_1 \, dx_2 \right] \]
\[ + \int_{x_1}^{x_2} \int_{x_2}^{x_1} \left\{ 1 - e^{-(x - x_1)^2} + e^{-(x + x_1)^2} \right\} e^{x_1^2 + x_2^2} \, dx_1 \, dx_2 \]

Now,

\[ \int_{x_1}^{x_2} \left( 1 - e^{-(x - x_1)^2} + e^{-(x + x_1)^2} \right) e^{x_1^2 + x_2^2} \, dx_1 \, dx_2 \]

\[ = \int_{-\infty}^{\infty} x^2 e^{-3x^2} \, dx \, dx_2 - 2 \int_{\infty}^{\infty} x^2 e^{-3x^2} \, dx \, dx_2 \]

\[ = 1/36 \]

Similarly, the other integral is also obtained as 11/36.

Therefore,

\[ E \left[ \psi_2^2 \left( x_1, x_2 \right) \right] = \frac{1}{9} \left[ 11/36 + 11/36 \right] \]

\[ = 11/162 \]

\[ \zeta_2(F) = E \left[ \psi_2^2 \left( x_1, x_2 \right) \right] - \left[ \Delta(F) \right]^2 \]

\[ = 11/162 - 1/16 = 7/1296 \]  \hspace{1cm} (6.2.13)

\[ \psi_3(x_1, x_2, x_3) = E \left[ \phi_{HP}^* \left( x_1, x_2, x_3 \right) \right] \]

\[ \zeta_3(F) = E \left[ HP^2 \right] - E \left[ HP \right]^2 \]  \hspace{1cm} (6.2.14)
\[= E \left\{ \psi_9^2 (x_1, x_2, x_3) \right\} - (\Delta(F))^2 \]

\[= \frac{1}{3} P \{ X_1 > X_2 + X_3 \} - (p \{ X_1 > X_2 + X_3 \})^2 \]

\[= \frac{1}{12} - \frac{1}{16} = \frac{1}{48} \]

Then

\[
\text{Var}(HP) = \left[ \sum_{k=1}^{3} \left( \frac{n-3}{n-k} \right) \right] \left( \frac{3(n-4)(n-3)}{2} \right) \xi_1(F) + 3(n-3) \xi_2(F) + \xi_3(F) \] (6.2.16)

\[= \frac{6}{n(n-1)(n-2)} \left[ -\frac{3}{3888} \left( n^2 - 7n + 12 \right) + \frac{5}{1291} \right] + \frac{1}{48} \]

\[= \frac{5n^2 + 7n - 12}{432(n^3 - 2n^2 + 2n)} \]

Therefore, \( \lim n \text{Var}(HP) = 9 \xi_1 \), and thus we have the following theorem.

**THEOREM 6.2.** If \( F \) is such that \( \xi_1(F) > 0 \), then the limiting distribution of \( [HP - \Delta(F)] \) \( n^{1/2} \) is normal with mean 0 and variance \( 9 \xi_1(F) \). Under \( H_0 \), this limiting variance is given by \( 5/432 \).

From this theorem it is obvious that the NBU test is consistent if and if \( \Delta(F) < 1/4 \).
THEOREM. 6.3. Let $F$ be superadditive with respect to $G$. Then $\Delta(F) \leq \Delta(G)$.

Proof. Make the transformation $F(x_i,3) = G(y_i,3), i = 1, 2$.

Then

$$\Delta(G) = \int \int G(y_1 + y_2) \, dG(y_1) \, dG(y_2)$$

$$= \int \int \left( G^{-1}(F(x_1) + G^{-1}(F(x_2)) \right) \, dF(x_1) \, dF(x_2)$$

Since $G^{-1}F$ is superadditive, then

$$G^{-1}F(x_1) + G^{-1}F(x_2) \leq G^{-1}(F(x_1) + x_2)$$

Combining (6.2.17) and (6.2.18) gives

$$\Delta(G) \geq \int \int \left( G^{-1}(F(x_1) + x_2) \right) \, dF(x_1) \, dF(x_2)$$

$$\geq \Delta(F)$$

THEOREM. 6.4. If $F$ is continuous NBU, and not exponential, then the NBU test is consistent.

Proof. Here we have to show only that $\Delta(F) < 1/4$.

Since $F$ is continuous $\psi(F) = 1/4 - \Delta(F)$. So in order to show $\Delta(F) < 1/4$, it is sufficient to prove $\psi(F) > 0$. 

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Set \( DC(x_1', x_2') = F(x_1') F(x_2') - F(x_1' + x_2') \).

Then \( DC(x_1', x_2') \geq 0 \) for all \( x_1, x_2 \geq 0 \), since \( F \) is NBU and \( DC(x_1, x_2) \neq 0 \) since \( F \) is not exponential.

Assume that \( x_1^0, x_2^0 \) are such that \( DC(x_1^0, x_2^0) > 0 \).

Let \( x_i' = \text{Sup} \{ x \mid x \geq x_i^0 \text{ and } F(x) = F(x_i^0) \}, i = 1, 2 \).

Then
\[
DC(x_1', x_2') \geq F(x_1') F(x_2') - F(x_1' + x_2')
\]
\[
\geq F(x_1^0) F(x_2^0) - F(x_1^0 + x_2^0)
\]
\[
\geq DC(x_1^0, x_2^0)
\]
\[
> 0.
\]

Since \( F \) is continuous, \( D \) is continuous and there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \), such that \( DC(x_1' + \delta_1, x_2' + \delta_2) > 0 \).

Also \( F(x_i' + \delta_i) - F(x_i') > 0 \), for \( i = 1, 2 \). Since \( x_1' \) and \( x_2' \) are points of increase of \( F \). Thus \( \psi(F) > 0 \).

Now we will be giving some examples, where NBU test is consistent against its alternatives.

**Example 6.1.** Let \( F(x) = \exp(-[x]) \), for \( x \geq 0 \), where \([x]\) denotes the largest integer less than or equal to \( x \).
Since \( \psi(F) \) and \( \Delta(F) \) are scale invariant, then since
\[
\bar{F}(x) \bar{F}(y) - \bar{F}(x + y) = e^{-x} e^{-y} - e^{-(x+y)} = 0,
\]
we have,
\[
\psi(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[ \bar{F}(i) \bar{F}(j) - \bar{F}(i+j) \right] dF(i) dF(j) = 0.
\]
and
\[
\Delta(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{F}(i) \bar{F}(j) dF(i) dF(j)
\]
\[
= \left[ \sum_{i=1}^{\infty} \bar{F}(i) dF(i) \right]^2.
\]
But
\[
\sum_{i=1}^{\infty} \bar{F}(i) dF(i) = \sum_{i=1}^{\infty} e^{-i} \left( e^{-(i-1)} - e^{-i} \right)
\]
\[
= (e - 1) \sum_{i=1}^{\infty} e^{-2i}
\]
\[
= (e - 1) / (e^2-1)
\]
\[
= (e + 1)^{-2}
\]
Therefore,
\[
\Delta(F) = (e + 1)^{-2} = 0.072.
\]
and since \( \Delta(F) < 1/4 \), the NBU test is consistent against this alternative even though \( \psi(F) = 0 \).

The following example provides a class of NBU alternatives for which the NBU test is not only consistent, but for which the NBU test has power identically equal to 1 for every n.
Define
\[ T_n = \frac{n(n-1)(n-2)}{6} HP/2 \]
\[ = \sum_i \phi_{HP}^*(X_{\alpha_1}, X_{\alpha_2} + X_{\alpha_3}) \]  
(6.2.19)

Let \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)} \) denote the order statistics. Since \( a < b < c \) implies \( \phi_{HP}^*(X_a, X_b + X_c) = 0 \), we can write (6.2.19) as
\[ T_n = \sum_{i > j} \phi_{HP}^*(X_{(i)}, X_{(j)} + X_{(k)}) \]  
(6.2.20)

Note that \( T_n \) has possible values \( 0,1,2,\ldots, n(n-1)(n-2)/6 \).

Exact percentile points for the NBU test can be obtained from the distribution of \( T_n \), calculated under the assumption that \( x \)'s are exponential. For even moderate \( n \), these calculations are prohibitive. Some exact probabilities for some special cases are given below. Hollander and Proschan shows that these exact probabilities and the Monte Carlo values (Tab.6.1) are giving an excellent matching.

Define the pacings,
\[ A_i = X_{(i)} - X_{(i-1)} \quad , \quad i = 1,2,\ldots,n \]
then for \( n \geq 3 \),
\[ P[T_0 = 0] = P[X_{(1)} < X_{(2)} + X_{(2)}] \]
\[ = P\left[ \sum_{i=3}^{n} A_i < A_1 \right] \]  
(6.2.22)
\[ = n! \prod_{i=1}^{n} \exp\left(-a_i \right) da_i \]
\[ = \prod_{i=1}^{n} \exp\left(-a_i \right) da_i \]
So that when $H_P = 0$ we reject $H_0$ with probability 1.

Example. 6.2. Let $F_{a,b}$ denote the class of distributions with support $[a,b]$ where $b < 2a$. Then, by considering the three cases

(i) $x < a; y < a$, (ii) $x \geq a; y \geq a$, (iii) $x < a; y \geq a$, and the relation $F(x+y) \leq F(x) F(y)$, for all $x, y \geq 0$ one directly verifies that every $F \in F_{a,b}$ is NBU. But for every $F \in F$

$P_F[H_P = 0] = P_F[X^{(n)} < X^{(1)} + X^{(2)}] = 1$, and

thus $P_F[\text{RE} H_0] \geq P_F[H_n = 0] = 1$.

6.2.3. ASYMPTOTIC RELATIVE EFFICIENCY.

In this section the NBU test has been compared with that of IFR classes. The IFR tests includes $V_n$ of Proschan and Pyke (1967), $K_n$ of Epstein (1960), Barlow (1968), W Bickel and Doksum (1969), Bickel (1969), Barlow and Proschan (1969), and Barlow (1970).

Let $(F_{\theta_n})$ be a sequence of alternatives with $\theta_n = \theta_0 + kn^{-1/2}$ where $k$ is an arbitrary positive constant and $F_{\theta_0}$ is exponential.

From the results of Proschan and Pyke (1967), Bickel and Doksum (1969) and The.6.2., we find the Pittman asymptotic relative efficiency of the NBU test with respect to the Proschan and Pyke test to be
\[ e_{F}(HP, V) = \left( \frac{12}{5} \right) \left( \frac{\Delta'(\theta_{0})}{\mu'(\theta_{0})} \right)^{2} \]  

(6.2.23)

where,

\[ \Delta(\theta) = \int_{0}^{x} \int_{0}^{y} \frac{f_{\theta}(x+y)}{f_{\theta}(x)f_{\theta}(y)} \, dy \, dx \]  

(6.2.24)

and

\[ \mu(\theta) = \int_{0}^{x} \int_{0}^{y} \left[ q_{\theta}(x) + q_{\theta}(y) \right] f_{\theta}(x)f_{\theta}(y) \, dy \, dx \]  

(6.2.25)

are the asymptotic means of HP and V respectively for the alternative \( F_{\theta} \), the factor \( 12/5 \) in (6.2.23) equals \( \left( V(V) / V(HP) \right) \)

and \( \Delta'(\theta_{0}) \), \( \mu'(\theta_{0}) \) is the derivative of \( \Delta(\theta) \), \( \mu(\theta) \) with respect to \( \theta \), evaluated at \( \theta = \theta_{0} \). Bickel and Doksum (1969) have shown \( e(V, W) = 1 \) for all \( F \), and thus \( e_{F}(HP, k) = e_{F}(HP, W) = 3/4 \, e(HP, V) \).

Consider the IFR Weibull and Linear failure rate alternatives respectively as

\[ F_{1}(x) = 1 - \exp(-x^{\theta}) \quad \theta \geq 1 \quad x \geq 0 \quad \text{and} \]

\[ F_{2}(x) = 1 - \exp(-(x + \theta x^{2}/2)) \quad \theta \geq 0 \quad x \geq 0. \]

For \( F_{1} \), \( H_{0} \) is achieved at \( \theta = \theta_{0} = 1 \) and ofr

\( F_{2} \), \( H_{0} \) is achieved at \( \theta = \theta_{0} = 0. \)

The efficiency calculations yields

\[ e_{F_{1}}(HP, V) = 1.25 \]

\[ e_{F_{1}}(HP, W) = 0.937 \]

\[ e_{F_{2}}(HP, V) = 0.60 \]

\[ e_{F_{2}}(HP, W) = 0.45 \]

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Next we compare the power of the NBU test, HP with the power of the IFR tests $V_n$ and $W$ for the class $\mathcal{F}_{a,b}$ of NBU distributions introduced in example 6.2. There we have seen that the HP test has power 1 for every $n \geq 3$, for every $F \in \mathcal{F}_{a,b}$, as long as $\alpha \geq \left(\frac{2n-2}{n}\right)^{-1}$.

Consider the $V_n$ and $W_n$ tests based on the normalized spacings, which we have introduced in Ch. 4. For simplicity, take $n = 3$ and $\alpha = 1/6$.

Then for both $V_n$ and $W_n$, reject $H_0$ when $A = [D_{1} > D_{2} > D_{3}]$ occurs. It is easily seen that for every $F \in \mathcal{F}_{a,b}$, $P[D_{1} > D_{2} > D_{3}] = 1$ implying that for these distributions the power of $V_n$ and $W_n$ is less than 1. Here the $\alpha = 1/4$ test based on HP has power 1. So it is clear that for large $n$ we can exhibit $F's \in \mathcal{F}_{a,b}$ for which the powers of the $V_n$ and $W_n$ tests are less than 1.

Remark. 6.1. Hollander and Proschan has demonstrated that the exact distribution of $T_n$ can be computed under $H_0$. However, even for small values of $n$, say 5, obtaining the exact distribution is very tedious. Therefore they have computed the exact lower and upper critical points in the $\alpha = .01, .025, .05, .075$ and .10 regions for $n = 4(1)20(5)50$ based on Monte Carlo techniques which we have reproduced in Tab. 6.1.
### Table 6.1.

Critical values of T statistics.

<table>
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<tr>
<th>n</th>
<th>Lower tail</th>
<th>Upper tail</th>
</tr>
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<tbody>
<tr>
<td></td>
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</tr>
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</tr>
<tr>
<td>20</td>
<td>.01</td>
<td>0</td>
</tr>
</tbody>
</table>
For \( n \leq 19 \), each value is based on 100,000 replications, for \( n > 19 \), an 10,000 replications. In the table, the lower tail should be used for tests of \( H_0 \) versus \( F_{NBu} \), the upper tail for tests of \( H_0 \) versus \( F_{NWU} \) (New Worse than Used): The lower tail values are integers \( C^L_\alpha \) for which the estimated probabilities \( \hat{P}_n(T = C^L_\alpha) \) are closest to \( \alpha \), and similarly the upper tail values are integers \( C^U_\alpha \) for which the estimated probabilities \( \hat{P}_n(T \geq C^U_\alpha) \) are closest to \( \alpha \). Parenthetical entries adjacent to critical points give the Monte Carlo estimated tail probabilities for those estimated probabilities that are not within .002 of the nominal \( \alpha \). For \( n \geq 25 \), all estimated probabilities agree with the nominal \( \alpha \). For \( n > 50 \), use the normal approximation keeping in mind that

(i) lower tail probabilities of events of the form \( [T_n \leq a] \) are underestimated using the normal approximation and upper tail probabilities of events \( [T_n \geq b] \) are overestimated, and

<table>
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<th>1351</th>
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<td>1051P</td>
<td>10310</td>
<td>10085</td>
<td>15937</td>
<td>15823</td>
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</tbody>
</table>
(ii) for fixed \( n \), the approximation improved as \( \alpha \) increases, \( 0 \leq \alpha \leq 1/2 \).

6.3. OTHER NBU TESTS.

Koul (1977) suggests rejecting \( H_0 \) in favour of \( H_1 \) for significantly small values of \( S = \min_{1 \leq k \leq j \leq n} T_{kj} \), where for \( 1 \leq k \leq j \leq n \),

\[
T_{kj} = n S_{kj} - (n-k)(n-j), \quad \text{and} \quad S_{kj} = \sum_{i=1}^{n} \phi(X_i^k, X_{i+k}^j, X_{i+j}^k).
\]

The modification for the \( S \) statistic is that \( n^{-2} S \) estimates the parameter \( \alpha(F) = \inf_{x,y \geq 0} (\bar{F}(x+y) - \bar{F}(x) \bar{F}(y)) \), and \( \alpha(F) \) is a measure of deviation of \( F \) from \( H_0 \) towards \( H_1 \), being 0 when \( F \) is exponential and negative when \( F \) is NBU. Koul gives critical values of \( S \) for \( \alpha = .005, .01, .025, .05, .10, .20 \), for \( n = 3(1)30(5)50 \).

Koul's test is not as readily implementable as the Hollander and Proschan test as Koul does not provide a convenient large sample approximation for critical values of \( S \) and he also does not provide a dual test of \( H_0 \) versus NBU alternatives. Koul (1978a) suggests a class of tests of \( H_0 \) versus \( H_1 \) based on

\[
\int \int \phi(F_n(x+y), F_n(x), F_n(y), \hat{F}_n(x), \hat{F}_n(y)), \quad \text{where} \quad \hat{F}_n \text{ is the empirical distribution function and} \quad \phi \text{ is a nondecreasing continous function from} \ [0,1] \text{ to} \ [0,\infty] \text{ with} \ \phi(0) = 0. \quad \text{The Hollander and Proschan statistic corresponds to the choice} \ \phi(u) = u. \quad \text{Koul's study advocates the choice} \ \phi(u) = u^{1/2}.
\]