TESTS FOR EXPONENTIALITY AGAINST IFRA ALTERNATIVES.

5.1. INTRODUCTION.

The Increasing Failure Rate Average (IFRA) classes are the smallest of life distributions which contains the exponential distribution and is closed under formation of coherent systems. Also these distributions arise as life distributions from useful shock models (See, Barlow and Proschan 1965). This motivated many research workers to obtain tests for exponentiality against IFRA class of life distributions.

Let $X$ be a continuous, non-negative random variable with distribution function $F$ and failure rate $r(x)$. Then our interest is to test the null hypothesis

$$H_0 : F \text{ is exponential}$$

versus

$$H_1 : F \text{ is IFRA but not exponential.}$$

Let $F$ and $G$ be two probability distributions such that $F(0) = G(0) = 0$. It is well known that $F$ is an increasing failure rate average distribution if and only if for $x > 0$

$$-1/t \log F(t) \uparrow t, \quad t > 0$$

$$\Rightarrow [F(t)]^{1/t} \downarrow t, \quad t > 0$$
The above $H_0$ and $H_1$ can now be stated as

$$H_0 : F(bx) = (F(bx))^b \quad , \quad x \geq 0 , \quad 0 < b < 1$$

versus

$$H_1 : F(bx) > (F(x))^b \quad , \quad x > 0 , \quad 0 < b < 1$$

with strict inequality for some $x$.

Several authors have proposed tests of $H_0$ against $H_1$. These include Barlow (1968), Klefsjo (1983), Kochar (1985), Deshpande (1983) and Link (1989). The former three proposed the tests based on linear combination of the normalized spacings $D_1, D_2, ..., D_n$ as given by (3.3.1). Both Deshpande and Link suggested the tests on the basis of U-statistics.

In Sec. 5.2, we will be giving a survey of the available tests. A detailed study of Deshpande's and Link test is given in sections 5.3 and 5.4 respectively.

5.2. SURVEY OF THE AVAILABLE TESTS.

Barlow and Doksum (1975) proved that if $F$ is a life distribution which is IFRA, then $F_r(t)/t$ is decreasing for
Thus, since \( r_F(t)/t \) being decreasing is a necessary (but not sufficient) condition for \( F \) to be IFRA, Klefsjø (1983) proposes a statistics which investigates whether the analogous property tends to hold for Total Time Test (TTT) -plot. If \( F \) is IFRA, we would expect \( T_i/(i/n) > T_j/(j/n) \) for all \( i < j \) and \( i = 1, 2, \ldots, n-1 \), where \( T_j = \sum_{i=1}^{j} D_i \).

This suggests the following statistic

\[
B = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (jT_i - iT_j)
\]

For large values of \( B \) the hypothesis is rejected in favour of \( H_1 \).

Klefsjø shows that (5.2.1) can be written in terms of the normalized spacings as

\[
B = \sum_{j=1}^{n} \beta_j D_j / T_n
\]

where,

\[
\beta_j = \frac{1}{6} \left( 2j^3 - 3j^2 + j(1-3n-3n^2) + 2n + 3n^2 + n^3 \right)
\]

Klefsjø shows that under \( H_0 \) the statistic \( B \cdot (210/n^3)^{1/2} \) can be treated asymptotically as an \( N(0,1) \) variable. Klefsjø also shows that the test that rejects for large values of \( B \) is consistent against the class of continuous IFRA distributions.

Barlow (1968) devices a likelihood ratio statistic, where lower percentiles of which are for testing exponentiality versus
IFRA, and upper percentiles of which are intended for testing IFRA against DFRA. He gives tables for $n = 2(1)10$, and percentiles .01, .05, .10, .90, .95 and .91.

Other tests of exponentiality versus IFRA alternatives, motivated by TTT processes are given by Barlow and Campo (1975) and Bergman (1977). Specifically Barlow and Campo suggest the statistic "$L =$" number of crossings between the TTT-plot and the 45° line.

Deshpande (1983) defines a class of tests based on the U-statistics which we will be studying in detail in Sec. 5.3.

Kochar's test statistic $T_n$ is given by

$$T_n = \sum_{j=1}^{n} e_{jn} \sum_{j=1}^{n} D_j,$$

where

$$e_{jn} = (n-j+1)^{-1} \sum_{i=j}^{n} J(i/(n+1))$$

$$J(u) = 2(1-u)(1 - \log(1-u)) - 1$$ and $D_1, D_2, \ldots, D_n$ are the normalized spacings (See. Kochar. 1985).

5.3. DESHPANDE's TEST (1983).

5.3.1. THE PROPOSED TEST.

It is known that from (5.1.1) $F$ is IFRA if and only if

$$\bar{F}(bx) \geq [\bar{F}(x)]^b$$
Define a parameter
\[ M(C) = \int_{0}^{\infty} F(bx) \, dF(x) \] (5.3.1)

If \( F \) is exponential, then
\[ M(C) = \int_{0}^{\infty} e^{-bx} e^{-x} \, dx = 1/(b+1) \]

and under \( H_1 \)
\[ M(C) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (F(bx) \, dF(x) > F(\infty) \, dF(x)) = 1/(b+1) \]

Hence \( M(C) - (b+1)^{-1} \) may be taken as a measure of deviation of \( F \) from the null hypothesis of exponentiality.

Define a kernel as
\[ h_b(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > b \, x_2 \\ 0 & \text{otherwise} \end{cases} \] (5.3.2)

where \( b \) is a fixed number belonging to \([0,1]\). Then the corresponding U-statistic is given by
\[ J_b = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} h_b(x_i, x_j) \] (5.3.3)

It is seen that
\[ \text{EC}(J_b) = E \left[ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} h_b(x_i, x_j) \right] \]
\[ = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E \left[ h_b(x_i, x_j) \right] \]
\[ = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} P \left[ x_i > b x_j \right] = M(C) \]
and intuitively large values of the statistic $J_b$ indicates the alternative hypothesis. Hence, reject the null hypothesis if $J_b > J_{b,1-\alpha}$, where $J_{b,1-\alpha}$ is the upper $(1-\alpha)^{th}$ quantile of the distribution of the statistic $J_b$ under $H_0$.

The value of the statistic ranges from $.5$ to $1$.

If $X_{(i)} < b X_{(i+1)}$, for $i = 1, 2, \ldots, n-1$, then the statistic is equal $.5$ and if $X_{(i)} > b X_{(i+1)}$, then the statistic is equal to $1$.

For practical situations $J_b$ can be computed as follows.

Let us consider the combined increasing arrangements of $X_i$'s and $Y_i$'s, where $Y_j = b X_j$, $i, j = 1, 2, \ldots, n$. Let $R_{(i)}$ be the rank of $X_{(i)}$ in the combined increasing arrangement. Then total number of $Y_j$'s below $X_i$'s are $R_i - i$ and one of them is $b X_{(i)}$ itself. Therefore total number of pairs $(X_{(i)}, Y_j)$ in which $Y_j$ are below $X_{(i)}$ are $(R_i - i - 1)$ and for every fixed $i$ only these pairs contribute $J_b$.

Therefore total number of points which contribute $J_b$ are

$$\sum_{i=1}^{n} (R_{(i)} - i - 1) = \sum_{i=1}^{n} R_{(i)} - \frac{n(n+1)}{2} - n$$

Since $\sum_{i=1}^{n} R_{(i)} = \sum_{i=1}^{n} R_i$, where $R_i$ is the rank of $X_i$ in the combined ordered arguments. Thus we have

$$J_b = \frac{1}{n(n-1)} \left[ \sum R_i - \frac{n(n+1)}{2} - n \right]$$

(5.3.4)
(5.3.4) is same as the Wilcoxon statistic for the data of X's and bX's.

5.3.2. UNBIASEDNESS AND ESTIMATES OF POWER.

To show the unbiasedness of $J_b$ we have to prove that probability of rejection is not less than $\alpha$ whenever the alternative hypothesis is true.

Let $G$ be an exponential distribution with unit mean, and let $F$ be a distribution in the alternative hypothesis.

Then for any $0 < b < 1$, $x > 0$

$$\log F(bx) \leq b \log F(x).$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from $F$. Let

$$Y_i' = G^{-1}(F(X_i)) = -\log F(X_i).$$

(5.3.5)

Then $Y_1', Y_2', \ldots, Y_n'$ have the same probability distribution as a random sample $Y_1, Y_2, \ldots, Y_n$ from $G$.

Now,

$$X_1 \leq b X_2 \Rightarrow G^{-1}(F(X_2)) \leq G^{-1}(F(bX_2)) \leq b G^{-1}(F(X_2))$$

Thus,

$$X_1 \leq b X_2 \Rightarrow Y_1' \leq b Y_2', \quad \text{using (5.3.5)}.$$

Therefore,

$$h_b(Y_1', Y_2') \leq h_b(X_1, X_2).$$

But $h_b(Y_1, Y_2)$ has the same probability distribution as $h_b(Y_1', Y_2')$. Hence $J_b(Y_1, Y_2, \ldots, Y_n)$ and $J_b(Y_1', Y_2', \ldots, Y_n)$
have the same probability distribution and $J_b(X_1, X_2, \ldots, X_n)$ is stochastically larger than $J_b(Y_1, Y_2, \ldots, Y_n)$.

Hence,

$$P_{F_b}(J_b \geq C_{a, \nu}) \geq P_{g_b}(J_b \geq C_{a, \nu}) = \alpha. \quad (5.3.8)$$

That is the power of the test at a fixed alternative $F$ of $H_1$ is greater than or equal to $\alpha$, where $\alpha = P_{g_b}(J_b \geq C_{a, \nu})$.

Let $g(a_1, a_2)$ be the class of all distributions with support of $[a_1, a_2]$, where $a_1 > b a_2$. Then we have that 'rejection' has power one for any alternative belonging to the class $g(a_1, a_2)$, whenever the level of significance $\alpha \leq P_{H_0}(J_b = 1)$, and it is not necessary that all members of this class are in the increasing failure rate average class.

Table 5.1 provides the estimates of critical values, exact level of significance and power of the $J_5$ and $J_9$ tests corresponding to the significance level $\alpha = 0.05$. The study was done for $n = 5, \ldots, 15$ each value being based on 10,000 samples of required size.
Table 5.1.
Monte Carlo estimates of critical values, exact levels of significance and power of the J₅ and J₀ tests: α = .05.

<table>
<thead>
<tr>
<th>n</th>
<th>Exact α</th>
<th>P(b,a,n,0)</th>
<th>Cₐ</th>
<th>P(b,a,n,1)</th>
<th>P(b,a,n,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b = .5</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td>.130</td>
<td>16</td>
<td>.578</td>
<td>.193</td>
<td></td>
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<tr>
<td></td>
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<td>17</td>
<td>.340</td>
<td>.074</td>
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<td>32</td>
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<td></td>
<td>.035</td>
<td>33</td>
<td>.405</td>
<td>.068</td>
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<tr>
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<td>.033</td>
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<td>.481</td>
<td>.067</td>
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<tr>
<td></td>
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<td>100</td>
<td>.724</td>
<td>.130</td>
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<td></td>
<td>.040</td>
<td>101</td>
<td>.661</td>
<td>.094</td>
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<tr>
<td>15</td>
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<td></td>
<td>.037</td>
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<td>.690</td>
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<tr>
<td></td>
<td>b = .9</td>
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<tr>
<td>5</td>
<td>.073</td>
<td>12</td>
<td>.233</td>
<td>.098</td>
<td></td>
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<tr>
<td></td>
<td>.024</td>
<td>13</td>
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<td>.032</td>
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<tr>
<td>7</td>
<td>.075</td>
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<td></td>
<td>.021</td>
<td>41</td>
<td>.439</td>
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<tr>
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<td>.406</td>
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<td>.452</td>
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<td></td>
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<tr>
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<td>.043</td>
<td>60</td>
<td>.272</td>
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<tr>
<td>13</td>
<td>.380</td>
<td>83</td>
<td>.436</td>
<td>.124</td>
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<tr>
<td></td>
<td>.033</td>
<td>84</td>
<td>.281</td>
<td>.358</td>
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</tr>
<tr>
<td>15</td>
<td>.057</td>
<td>111</td>
<td>.407</td>
<td>.094</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>112</td>
<td>.286</td>
<td>.048</td>
<td></td>
</tr>
</tbody>
</table>
The two alternatives to the null hypothesis $H_0$ of exponentiality with $\theta = 1$ for which the power has been estimated are $H_1$: the Weibull distribution of index 2, with $F(x) = 1 - e^{-x^2}$, and $H_2$: a linear failure rate distribution with $\theta = 1$, with $F(x) = 1 - \exp(-x - (1/2)x^2)$. Both these distributions are IFR, and hence also IFRA. Table 5.1 gives the CR for $n(n-1) J_b^H$ for $\alpha = 0.05$. The probability which are estimated are,

$$P(b, \alpha, n, i) = P_{H_i} (n(n-1) J_b \geq C_\alpha), i = 0, 1, 2; b = 0.8, 0.9,$$

and $C_\alpha$ is the estimated critical point for $\alpha$ near to 0.05.

5.3.3. ASYMPTOTIC NORMALITY AND CONSISTENCY.

We use the results of Hoeffding (1948) (See. The.1.7), To show the asymptotic normality of $J_b^H$, that is, the asymptotic distribution of $n^{1/2}[J_b^H - \text{MCF}]$ is normal with mean zero and variance $4\text{E}_1$, where,

$$\text{E}_1 = \mathbb{E} [\psi_1^2(x_1) - \text{MCF}^2],$$

and $\psi_1(x_1) = \mathbb{E} [h_b^*(x_1, x_2)]$, provided $\text{E}_1 > 0$.

Here $h_b^*(x_1, x_2)$ is the symmetric kernel of $h$.

That is

$$h_b^*(x_1, x_2) = \frac{1}{2} [h(x_1, x_2) + h(x_2, x_1)]$$

where

$$h(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > bX_2 \\ 0 & \text{otherwise} \end{cases}$$

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and
\[ h(x, X) = \begin{cases} 
1, & \text{if } X > bX, \\
0, & \text{otherwise}
\end{cases} \]

Under H, the mean and variances are obtained as follows.

Now,
\[ \psi(x) = \frac{1}{2} \left( E h(x, X) + E h(X, x) \right) \]
\[ = \frac{1}{2} \left( P \{ x > bX \} + P \{ X > bx \} \right) \]
\[ = \frac{1}{2} \left( 1 - e^{\frac{x}{b}} + e^{-bx} \right). \]

Therefore,
\[ E \left[ \psi(x) \right] = \frac{1}{b+1} \]

and
\[ E \left[ \psi^2(x) \right] = \int_0^\infty \left[ 1 + e^{-\frac{x}{b}} + e^{-bx} + e^{-x} \right] e^{\frac{-x}{b}} e^{-(a+b)x} e^{-x} dx \]
\[ = \frac{1}{4} \left[ 1 + \frac{b}{b+2} - \frac{2b}{b+1} + \frac{1}{2b+1} + \frac{2}{b+1} - \frac{2b}{b^2+b+1} \right] \]

Hence,
\[ \psi = \frac{1}{4} \left[ 1 + \frac{b}{b+2} + \frac{1}{2b+1} + \frac{2(1-b)}{b+1} - \frac{2b}{b^2+b+1} - \frac{4}{(b+1)^2} \right] \]

This ensures the consistency of the J_b test whenever \( E[J_b] > (b+1)^{-4} \) and the alternatives are continuous increasing failure rate average distribution.
5.3.4. ASYMPTOTIC RELATIVE EFFICIENCY.

Here the Pitman asymptotic Relative Efficiency of the $J_b$ test for both $b = 0.5$ and $b = 0.9$ have been calculated for three parametric families of distributions. See. Th.1.14 . Eq.(1.8.3).

The distributions are

(a) The Weibull distribution

$$F(x) = \exp(-x^\theta), \quad \theta \geq 1, \quad x > e, \quad \theta = 1$$

(b) The Makeham distribution,

$$F(x) = \exp(-x + e^x + \theta e - 1), \quad \theta > 0, \quad x > 0, \quad \theta = 0$$

(c) The linear failure rate distribution,

$$F(x) = \exp(-x + 1/2 \theta x^2), \quad \theta > 0, \quad \theta = 0$$

Table 5.2 gives the values of the Pitman asymptotic Relative Efficiency of the two tests with respect to Hollander and Proschan (1972) test and to the cumulative total time (Bickel and Doksum, 1969) test for these three parametric families. The table gives a moderate high efficiency when compared with competitors.

<table>
<thead>
<tr>
<th></th>
<th>Weibull</th>
<th>Makeham</th>
<th>Linear F. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A{R}(J, 0.5, \text{HP})$</td>
<td>1.006</td>
<td>0.948</td>
<td>0.931</td>
</tr>
<tr>
<td>$A{R}(J, 0.9, \text{HP})$</td>
<td>1.022</td>
<td>1.020</td>
<td>1.020</td>
</tr>
<tr>
<td>$A{R}(J, 0.5, \text{BD})$</td>
<td>0.937</td>
<td>0.787</td>
<td>0.418</td>
</tr>
<tr>
<td>$A{R}(J, 0.9, \text{BD})$</td>
<td>0.955</td>
<td>0.815</td>
<td>0.459</td>
</tr>
</tbody>
</table>

HP : Hollander and Proschan test
BD : Bickel and Doksum test.
5.4. LINK TEST (1984):

5.4.1. TEST PROCEDURE.

In Sec. 5.3 we have seen that if \( P(X_1 > b X_2) \) is greater than or equal to \((b+1)^{-1}\) then the corresponding distribution function \( F \) is IFRA but not exponential. Based on this Deshpande proposed his test \( J_b \).

Now suppose that \( X_1 \) and \( X_2 \) are independent random variables, each with distribution \( F \). Let \( G \) be the distribution of \( \min(X_1, X_2) / \max(X_1, X_2) \). Then the statistic \( J_b \) can be modified as follows.

Let us define,

\[
h_b(X_i, X_j) = \begin{cases} 1 & \text{if } \min(X_i, X_j) > b \max(X_i, X_j) \\
0 & \text{otherwise.} \end{cases}
\]

(5.4.1)

\( \{ \min(X_i, X_j) > b \max(X_i, X_j) \} \)

\( \Rightarrow \{ X_i > b \max(X_i, X_j) ; X_j > b \max(X_i, X_j) \} \)

\( \Rightarrow \{ X_i > b X_j , X_i > b X_j ; X_j > b X_i , X_j > b X_j \} \)

\( \Rightarrow \{ X_i > b X_j ; X_j > b X_i \} \)

Now,

\( \{ X_i > b X_j \} = \{ X_i > b X_j \} \cap \{ (X_j > b X_i) \text{ or } X_j < b X_i) \} \)

\( = \{ X_i > b X_j , X_j > b X_i \text{ or } X_i > b X_j , X_j < b X_i \} \)

That is,
\[ P \{ X_i > b X_j \} = P \{ X_i > b X_j, X_j > b X_i \} + P \{ X_i > b X_j, X_j < b X_i \} \]

Therefore,

\[ P \{ X_i > b X_j, X_j > b X_i \} = P \{ X_i > b X_j \} - P \{ X_i > b X_j, X_j < b X_i \} \]

\[ = P \{ X_i > b X_j \} - P \{ X_i > b X_j, X_i > X_j / b \} \]

\[ = P \{ X_i > b X_j \} - P \{ X_i > X_j / b \} \]

\[ = P \{ X_i > b X_j \} - P \{ X_j < b X_i \} \]

i.e.

\[ P \{ X_i > b X_j \} = P \{ X_i > b X_j, X_j > b X_i \} + P \{ X_j < b X_i \} \]

From this we can see that

\[ J_b = J_1 + J_2 \quad \text{(5.4.2)} \]

where,

\[ n(n-1) J_b = P \{ X_i > b X_j \} \]

\[ n(n-1) J_1 = P \{ X_i > b X_j, X_j > b X_i \} \]

\[ n(n-1) J_2 = P \{ X_j < b X_i \} \]

we have

\[ \sum_{i \neq j} h(X_j < b X_i) + \sum_{i \neq j} h(X_j > b X_i) = n(n-1) \]

\[ \Rightarrow \sum_{i \neq j} h(X_j < b X_i) = n(n-1) - n(n-1) J_b \]
\[ n(n-1) J_2 = n(n-1) - n(n-1) J_b \]

\[ J_2 = 1 - J_b \]  

(5.4.3)

Using (5.4.3) in (5.4.2) we get

\[ J_b = J_1 + 1 - J_b \]

i.e.,

\[ 2 J_b = 1 + J_1 \]

i.e.,

\[ J_b = 1/2 + J_1/2 \]

Thus the statistic \( J_b \) can be written as

\[ J_b = 1/2 + \left( 2n(n-1) \right)^{-1} \sum_{i \neq j} h(x_i, x_j) \]

(5.4.4)

where \( h(x_i, x_j) \) is as per (5.4.1).

From (5.4.4) it can be seen that Deshpande's test is a test of a percentile of G. Based on this Link proposed his statistic for testing

\( H_0 : \) The \( b^* \) percentile of G is \( b \)

versus

\( H_1 : \) The \( b^* \) percentile of G is not \( b \), where

\[ b^* = 1/2 \left( \frac{1}{b+1} - \frac{1}{2} \right) \]

We found this value as correct after the whole calculations and is noticed that the author's value of \( b^* = (b+1)^{-1} - 1/2 \) as incorrect.
He proposed his test statistic based on the expected value, rather than a percentile of $G$ as

$$\Gamma = \left\{ \frac{n(n-1)}{2} \right\}^{-1} \sum_{i \neq j} \frac{\min(X_i, X_j)}{\max(X_i, X_j)}$$

(5.4.5)

Since the statistic $\Gamma$ is scale invariant, it can be expressed in the simple form as

$$\Gamma = \frac{2}{n(n-1)} \sum \frac{X_{(i)}}{X_{(j)}}$$

(5.4.6)

where $X_{(i)}$'s and $X_{(j)}$'s are the order statistics.

5.4.2. ASYMPTOTIC DISTRIBUTION.

Define a parameter,

$$\mu(F) = \int_0^\infty \int_0^\infty F(t \cdot x) \, dF(x) \, dt$$

(5.4.7)

under $H_0$:

$$\mu(F) = \int_0^\infty \int_0^\infty e^{-tx} e^{-x} \, dx \, dt$$

= \log 2.

Under $H_1$: $\mu(F)$ will be a greater quantity. Thus the parameter $\mu(F)$ can be thought of as a measure of the departure from exponentiality in the IFRA class.

The U-statistics $\Gamma$ can be expressed in its symmetric form as

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\[
\Gamma = \frac{2}{n(n-1)} \sum_{i<j} h^*(X_i, X_j)
\]

where,
\[
h^*(X, Y) = \frac{X}{Y} h(X, Y) + \frac{Y}{X} h(Y, X)
\]

and
\[
h(X, Y) = \begin{cases} 
1 & \text{if } X \leq Y \\
0 & \text{otherwise}
\end{cases}
\]

where \(X\) and \(Y\) are independent random variables with distribution \(F\).

Then using the results of Hoeffding (1948) The.1.7, we have
the asymptotic distribution of \(n^{1/2} (\Gamma - \psi(F))\) has a limiting
normal distribution with mean zero and variance \(4 \Gamma_1\), where \(X\), \(Y\) and \(Z\) are independent random variables, each with distribution \(F\).

\[
\Gamma_1 = E \left[ h^*(X, Y) h^*(X, Z) \right] - \left[ \psi(F) \right]^2
\]

and
\[
\psi(F) = E \left[ h^*(X, Y) \right]
\]

\[
E \left[ h^*(X, Y) \right] = E \left[ \frac{X}{Y} \mid X \leq Y \right]
\]

\[
= \int_{0}^{1} P \left( \frac{X}{Y} > t \mid X \leq Y \right) dt
\]

\[
= \int_{0}^{1} \int_{0}^{\infty} P \left( ty \leq x \leq y \right) dF(y) dt
\]

\[
= \int_{0}^{1} \int_{0}^{\infty} \left[ F(y) - F(ty) \right] dF(y) dt
\]

\[
= 2 \left[ \int_{0}^{\infty} F(y) dF(y) \right] - \psi(F)
\]

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\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{y} \bar{F}(t) \, dt - \int_{-\infty}^{0} \int_{-\infty}^{y} dF(y) \, dt \right) \, dy
\]

\[
= 2 \left[ \frac{1}{2} + \mu(F) - 1 \right]
\]

i.e. \( \psi(F) = 2 \mu(F) - 1 \)

Under \( H_0 \), \( \psi(F) = 2 \log 2 - 1 \) and \( 4 \mu(F) = 0.048225 \).

Therefore, for large \( n \), the null distribution of

\[
\Gamma^* = n^{1/2} \left[ \Gamma - (2\log 2 - 1) \right] / (0.048225)^{1/2}
\]

is approximately normal. Tests of \( H_0 \) against \( H_1 \) can be carried out by comparing \( \Gamma^* \) to the appropriate percentiles of the standard normal distribution.

A Monte Carlo study was carried out to estimate percentiles of the null distribution of \( \Gamma \) for small sample sizes. The study was done for \( n = 3(2)15 \), each value being estimated on the basis of 7999 samples of required sizes. The results are given in the following table.
Table 5.3.

Monte Carlo estimates of percentiles of the distributions of \( r \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0643</td>
<td>0.0939</td>
<td>0.1248</td>
<td>0.7312</td>
<td>0.8000</td>
<td>0.8766</td>
</tr>
<tr>
<td>5</td>
<td>0.1575</td>
<td>0.1821</td>
<td>0.2075</td>
<td>0.6139</td>
<td>0.6700</td>
<td>0.7277</td>
</tr>
<tr>
<td>7</td>
<td>0.2029</td>
<td>0.2230</td>
<td>0.2417</td>
<td>0.5575</td>
<td>0.5936</td>
<td>0.6350</td>
</tr>
<tr>
<td>9</td>
<td>0.2228</td>
<td>0.2448</td>
<td>0.2628</td>
<td>0.5324</td>
<td>0.5662</td>
<td>0.6053</td>
</tr>
<tr>
<td>11</td>
<td>0.2393</td>
<td>0.2577</td>
<td>0.2752</td>
<td>0.5158</td>
<td>0.5428</td>
<td>0.5758</td>
</tr>
<tr>
<td>13</td>
<td>0.2524</td>
<td>0.2717</td>
<td>0.2871</td>
<td>0.5036</td>
<td>0.5266</td>
<td>0.5581</td>
</tr>
<tr>
<td>15</td>
<td>0.2599</td>
<td>0.2763</td>
<td>0.2921</td>
<td>0.4925</td>
<td>0.5187</td>
<td>0.5444</td>
</tr>
</tbody>
</table>

It has been noticed that a large class of tests of \( H_0 \) against \( H_1 \) can be constructed based on U-statistics of the form

\[
\Gamma_q = \frac{2}{n(n-1)} \sum_{i<j} q\left( \frac{X_i}{X_j} \right),
\]

where \( q \) is any non-decreasing function satisfying \( q(0) = 0 \) and \( q(1) = 1 \). The parameter estimated for \( \Gamma_q \) is

\[
\frac{1}{\infty} \int \int F(tx) \, dF(x) \, d\psi(t) - 1
\]

When \( q(t) = t \) we get \( \psi(F) \) as given above.

5.4.3. ASYMPTOTIC RELATIVE EFFICIENCY AND SMALL SAMPLE POWER COMPARISONS.

Here we compare the test with that of (5.2.3) and (5.3.3) for 0.5 and 0.9. The class of distributions considered here is
\[ \mathcal{A} = \{ F \mid F(x) = \exp(- (x + \theta x^\beta)), \theta \geq 0, \beta > 0 \} \]

when \( \beta > 1 \) then \( F \) will be IFRA and for \( \theta = 0 \), the distributions are exponential. Table 5.2. presents the ARE's of \( \Gamma \) relate to \( J_{.5} \), \( J_{.9} \) and \( T_n \) for selected alternatives \( F \) from the class \( \mathcal{A} \).

Table 5.4.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( J_{.5} )</th>
<th>( J_{.9} )</th>
<th>( T_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.7458</td>
<td>2.010</td>
<td>6.0745</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4638</td>
<td>1.6268</td>
<td>3.8737</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2778</td>
<td>1.3749</td>
<td>2.5970</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0570</td>
<td>1.0780</td>
<td>1.2995</td>
</tr>
<tr>
<td>1.05</td>
<td>1.0048</td>
<td>1.0050</td>
<td>1.0308</td>
</tr>
<tr>
<td>1.10</td>
<td>0.9899</td>
<td>0.9847</td>
<td>0.9559</td>
</tr>
<tr>
<td>1.20</td>
<td>0.9633</td>
<td>0.9485</td>
<td>0.8277</td>
</tr>
<tr>
<td>1.50</td>
<td>0.9042</td>
<td>0.8672</td>
<td>0.5505</td>
</tr>
<tr>
<td>2.0</td>
<td>0.8532</td>
<td>0.7954</td>
<td>0.3001</td>
</tr>
<tr>
<td>5.0</td>
<td>1.1096</td>
<td>1.2700</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

A monte carlo study of small sample power of \( \Gamma \), \( J_{.5} \) and \( T_n \) against Weibull alternatives with \( \theta = 2 \) for \( n = 3(2)15 \) has given in the following table.

The results are due to Kochar. The level of significance was set at \( \alpha = 0.05 \).
Table 5.5.
Monte carlo estimates of Power against Weibull alternatives ($\alpha = .5$)

<table>
<thead>
<tr>
<th>$T_n$</th>
<th>$J_{.5}$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.206</td>
<td>0.126</td>
<td>0.186</td>
</tr>
<tr>
<td>0.376</td>
<td>0.350</td>
<td>0.327</td>
</tr>
<tr>
<td>0.490</td>
<td>0.475</td>
<td>0.598</td>
</tr>
<tr>
<td>0.720</td>
<td>0.570</td>
<td>0.694</td>
</tr>
<tr>
<td>0.805</td>
<td>0.638</td>
<td>0.789</td>
</tr>
<tr>
<td>0.859</td>
<td>0.694</td>
<td>0.839</td>
</tr>
<tr>
<td>0.880</td>
<td>0.742</td>
<td>0.879</td>
</tr>
</tbody>
</table>

It is seen that for $\theta = 2$, $\Gamma$ performs much better than $J_{.5}$.

Another Monte carlo study of small sample power against chi-square alternatives with 3 degree of freedom is reported in the table 5.6. The same conclusion can be drawn here also.
Table 3.6.

Rejection of $H_0$: out of 8000 trials, by $\Gamma$ and $\Gamma_n$ for $\chi^2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\Gamma$</th>
<th>$\Gamma_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>.05</td>
<td>654</td>
<td>599</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>348</td>
<td>326</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>130</td>
<td>129</td>
</tr>
<tr>
<td></td>
<td>.05</td>
<td>841</td>
<td>857</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>386</td>
<td>488</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>147</td>
<td>175</td>
</tr>
<tr>
<td>5</td>
<td>.05</td>
<td>1145</td>
<td>940</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>532</td>
<td>456</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>315</td>
<td>176</td>
</tr>
<tr>
<td>7</td>
<td>.05</td>
<td>1309</td>
<td>1199</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>717</td>
<td>703</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>319</td>
<td>323</td>
</tr>
<tr>
<td>9</td>
<td>.05</td>
<td>1631</td>
<td>1410</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>961</td>
<td>812</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>461</td>
<td>400</td>
</tr>
<tr>
<td>11</td>
<td>.05</td>
<td>1784</td>
<td>1478</td>
</tr>
<tr>
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<td>1110</td>
<td>848</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>513</td>
<td>415</td>
</tr>
<tr>
<td>13</td>
<td>.05</td>
<td>2002</td>
<td>1859</td>
</tr>
<tr>
<td></td>
<td>.025</td>
<td>1138</td>
<td>982</td>
</tr>
<tr>
<td></td>
<td>.010</td>
<td>588</td>
<td>448</td>
</tr>
</tbody>
</table>

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