CHAPTER 3

CONDITIONAL TESTS FOR THE SHAPE PARAMETER IN TWO PARAMETER GAMMA DISTRIBUTION

3.1. INTRODUCTION.

The Gamma family provides a widely applicable class of distributions for Life-testing, because it has a Decreasing Failure Rate (DFR), a Constant Failure Rate, or an Increasing Failure Rate (IFR). The Gamma model is useful as a Life model when the items in the population are systems in a regular maintenance program, where the failure rate may increase initially, but after sometime the system would reach a stable condition due to maintenance and from then on would be as likely to fail in one time interval as in another. Similarly, if failed parts are replaced with new parts when they fail, then the time between failures of a system may reasonably follow a gamma distribution. Indeed after several failures have occurred, the parts will be in scattered states of wear and the time to next failures of the system may be exponentially distributed. As another possibility, the Gamma distribution resulted when considering the time to k th occurrence of a Poisson process. It may be that a product or system would fail after it had received k shocks of some sort.
The two parameter Gamma density is given by

\[ f(x; \theta, \beta) = e^{-x/\theta} x^{\beta-1}/(\theta^\beta \Gamma(\beta)) \quad , x > 0, \theta > 0, \beta > 0, \]  

(3.1.1)

where \( \theta \) is the scale parameter and \( \beta \) is the shape parameter. The shape parameter \( \beta \) is of interest, because the Gamma distribution is DFR, IFR or constant failure rate according as \( \beta < 1, \beta > 1 \) and \( \beta = 1 \) respectively. One disadvantage of the Gamma distribution is that the hazard function and more specifically the reliability cannot be evaluated in closed form. The Gamma distribution is a member of the exponential family, and it follows that the arithmetic mean \( \bar{X} = \frac{\sum X_i}{n} \) and the geometric mean \( \tilde{X} = (\prod X_i)^{1/n} \), are a set of complete sufficient statistics for \( \theta \) and \( \beta \). This fact reveals that, the statistical procedures should be based on these statistics.

Using the ratio \( W = \bar{X}/\tilde{X} \) or \( S = \ln(\bar{X}/\tilde{X}) \) and making use of the principle of invariance, several authors have suggested test statistics for testing hypotheses about the shape parameter \( \beta \). The exact distribution of \( W \) and \( S \) are quite complicated. Linhart (1965) shows that for \( \beta \geq 2 \), \( kS \), where \( k = 2n\beta \), is approximately distributed as \( \chi^2_{(n-1)} \). Bain and Engelhardt (1975) have shown that \( [2n\beta_0 C(\beta_0, n)]S \) is approximately distributed as \( \chi^2 \) with \( \nu(\beta_0, n) \) d.f., where the values of \( C(\beta_0, n) \) and \( \nu(\beta_0, n) \) may be obtained from the tables provided by them. Lawless (1982) and Engelhardt and Bain (1977) have derived most powerful invariant tests for testing
simple null hypothesis versus simple alternative using chi-square approximations. Glaser (1973, 1976a, 1976b) have obtained the exact distribution of \( W \) and showed how the distribution of \( W \) was related to Bartlett's (1937) test statistic for homogeneity for variances among independent random samples from the Normal populations. Their expression for the density function of \( W \) is too complicated and is given in powers of \(-\ln(W)\) and yields a conservative radius of convergence for the series. Recently, Keating et al. (1990) have simplified the expression for the density of \( W \) and provided the tables for the critical values useful for testing the hypothesis about \( \beta \). He has also given the power curves for testing exponentiality against Gamma IFR alternatives.

As seen earlier, the cdf \( F_\theta(x;\beta) \) of the Gamma density (3.1.1) is not expressable in closed form and hence, as in the case of Weibull distribution, here there does not exist a closed form of transformation \( Y=-\ln F_\theta(x;\beta) \). So, in this chapter, we propose a test procedure for testing \( H_0: \beta=1 \) against \( H_1: \beta>1 \) (\( H_2: \beta<1 \)) treating \( \theta \) as the nuisance parameter. In section 3.2, we derive the test based on the observations \( X_i \) given the complete sufficient statistic. The properties of the test statistic, simulation of the percentile points of the distribution of the test statistic and the powers are also the parts of the study in the following sections.
3.2. A TEST BASED ON THE CONDITIONAL DISTRIBUTION OF $X_i$.

3.2.1. Derivation of the test.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Gamma distribution having the pdf (3.1.1). Then the joint density of $X_1, X_2, \ldots, X_n$ is given by

$$f_X(x; \theta, \beta) = e^{-\sum_{i=1}^{n} x_i/\theta} \prod_{i=1}^{n} x_i^{\beta-1}/(\theta^\beta \Gamma(\beta))^n.$$  \hspace{1cm} (3.2.1)

$0 < x_i < \infty, i = 1, 2, \ldots, n, \theta, \beta > 0$. Here $T = \sum_{i=1}^{n} X_i$, is the complete sufficient statistic for $\theta$, when $\beta$ is known, having the pdf

$$f_T(t, \theta) = e^{-t/\theta} t^{n\beta-1}/(\theta^n \Gamma(\beta)), t > 0.$$  \hspace{1cm} (3.2.2)

Then the conditional distribution of $X_1$ given $T=t$ is

$$f_{X_1/T}(x_1/t, \beta) = [B(\beta, (n-1)\beta)]^{-1} t^{-\beta} x_1^{\beta-1} (1-x_1/t)^{(n-1)\beta-1},$$  \hspace{1cm} (3.2.3)

$0 < x_1/t, \beta > 0$. It is seen that the conditional distribution (3.2.3) does not contain the nuisance parameter $\theta$. Moreover, it is the uniformly minimum variance unbiased estimate of the density (3.1.1). We shall derive the test statistic $Q$ defined in (1.1) for testing $H_0: \beta = 1$ against $H_1: \beta > 1$ ($H_2: \beta < 1$) considering that the data have come from the estimate (3.2.3) of (3.1.1) for known value of $T$. First, we obtain conditional mean $\mu_i$ and conditional
covariance $\sigma_{ij}$ as follows:

$$E(X^r_i|T=t) = \int_0^t x_i^rf(x_i|t)dt$$

which yields,

$$\mu_i = E_{H_0}(X_i|T=t) = t/n$$

and

$$\sigma_{ii} = V_{H_0}(X_i|T=t) = \frac{(n-1)t^2}{n^2(n+1)}.$$ (3.2.5) (3.2.6)

Similarly, the conditional pdf of $(X_i, X_j)$ given $T=t$ can be obtained as

$$f_{X_i, X_j|T}(x_i, x_j|t) = \frac{\Gamma(n\beta)}{(\Gamma\beta)^2 \Gamma[(n-2)\beta]} \frac{(x_i x_j)^{\beta-1}}{t^{n\beta-1}} (t-x_i-x_j)^{(n-2)\beta-1}.$$ (3.2.7)

$0<x_i, x_j<\infty$, $x_i+x_j\leq t$. Using (3.2.7), we get,

$$E(X_i^r X_j^s|T=t) = \frac{\Gamma(n\beta)B[\beta+s, (n-2)\beta]B[r+\beta, (n-1)\beta+s]t^{r+s}}{(\Gamma\beta)^2 \Gamma[(n-2)\beta]}$$

which gives

$$E_{H_0}(X_i X_j|T=t) = \frac{t^2}{n(n+1))}$$

and

$$\sigma_{ij} = COV_{H_0}(X_i X_j|T=t) = -\frac{t^2}{n^2(n+1))}.$$ (3.2.9)
From (3.2.6) and (3.2.9), the variance-covariance matrix $\Sigma_0$ of $X|T=t$, under $H_0$, can be written as

$$
\Sigma_0 = \frac{t^2}{n(n+1)} \begin{bmatrix}
1-1/n & -1/n & \ldots & -1/n \\
-1/n & 1-1/n & \ldots & -1/n \\
\ldots & \ldots & \ldots & \ldots \\
-1/n & -1/n & \ldots & 1-1/n
\end{bmatrix}
$$

$$
= \frac{t^2}{n(n+1)} [I_n - \frac{1}{n} E_{nn}],
$$

(3.2.10)

where $I_n$ and $E_{nn}$ have the same meaning as in (2.2.10). Then, the $g$-inverse of $\Sigma_0$ is,

$$
\Sigma_0^{-} = \frac{n(n+1)}{t^2} [I_n + \frac{1}{n} E_{nn}].
$$

Hence the expression for the test statistic

$$
Q_4 = (X_0 - \mu_0)' \Sigma_0^{-}(X_0 - \mu_0)
$$

$$
= n(n+1) \sum_{i=1}^{n} (X_i - t/n)^2 / t^2,
$$

which after replacing the value $t$ by its corresponding random variable $T$ modifies to

$$
Q^*_4 = n(n+1) \sum_{i=1}^{n} X_i^2 / T^2 - 1/n, \quad T = \Sigma X_i.
$$

(3.2.11)

Referring (2.2.13) we have

$$
E_{H_0}(Q^*_4) = n-1
$$
and

\[ V_{H_0}(Q_4^*) = 4n^2(n-1)/(n+2)(n+3). \]  

(3.2.12)

Thus, the standardized form of the test statistic \( Q \) is

\[ Q_{4}(sd) = (Q_4^* - E_{H_0}(Q_4^*) ) / V_{H_0}(Q_4^*)^{1/2} \]

\[ = \sqrt{(n+2)(n+3)/(n-1)} \left[ \frac{(n+1)\sum_{i=1}^{n} X_i^2}{2(\sum_{i=1}^{n} X_i^2)^2} - 1 \right]. \]  

(3.2.13)

To determine the direction of the test, we compute

\[ E_{H_1}\left[ \frac{\sum_{i=1}^{n} X_i^2}{(\sum_{i=1}^{n} X_i^2)^2} \right] = E_{H_1}\left[ \frac{\sum_{i=1}^{n} X_i^2}{(\sum_{i=1}^{n} X_i^2)^2} | T=t \right] \]

\[ = (\beta+1)/(n\beta+1) \]

and therefore,

\[ E_{H_1}(Q_4^*) = (n^2-1)/(n\beta+1). \]

Now,

\[ E_{H_1}(Q_4^*) - E_{H_0}(Q_4^*) = n(n-1)(1-\beta)/(n\beta+1), \]

which shows that \( E_{H_1}(Q_{4}(sd)) \leq 0 \) for \( \beta \geq 1 \) and \( > 0 \) for \( 0 < \beta < 1 \) irrespective of the value of the nuisance parameter \( \theta \). Hence, the test is to reject \( H_0 \) in favour of \( H_1: \beta > 1 \) (\( H_2: \beta < 1 \)) for small(large) values of \( Q_4^*(sd) \).
Remark: On the similar lines as in section 2.2.2, here also we can show that $Q_4^{*}(sd) \sim \chi^2 \rightarrow N(0,1)$.

3.2.2. Simulation study for the percentiles and the powers.

Since under the respective null hypotheses, the distribution of $Q_4^{*}(sd)$ and $Q_1^{*}(sd)$ are same, the simulated percentile points for $Q_4^{*}(sd)$ be referred from Table 2.2.

Because of not having the closed form of the transformation $Y = -\ln(F_\theta(x,\beta))$, it is difficult to derive the test statistic for the general problem of testing $H_0: \beta = \beta_0$ based on the observations $Y_i$. An easy alternative way to tackle this general problem of testing is first to derive statistic $Q_4^{*}(sd)$ for $H_0: \beta = 1$ as done in (3.2.13) and then to use it as follows: For given $X_i$'s assuming the Gamma distribution for $\beta = \beta_0$ (and $\theta = 1$), compute $Y_i = -\ln(F_\theta(x_i,\beta_0))$, where $F_\theta(x,\beta)$ is the cdf of (3.1.1). Use the test statistic $Q_4^{*}(sd)$ replacing $X_i$ by $Y_i$. Note that the same percentile points of Table 2.2 be used for taking decision on acceptance/rejection of $H_0: \beta = \beta_0$.

The simulated powers of the test for different values of $n$ and $\beta$ are given in Table 3.1 and Table 3.2.
Table 3.1.

Power of $Q^{*}_{4(sd)}$ test for values of $\beta < 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta = 0.2$</th>
<th>$\beta = 0.4$</th>
<th>$\beta = 0.6$</th>
<th>$\beta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>5</td>
<td>0.3734</td>
<td>0.5565</td>
<td>0.1469</td>
<td>0.3031</td>
</tr>
<tr>
<td>10</td>
<td>0.5909</td>
<td>0.8123</td>
<td>0.2245</td>
<td>0.4565</td>
</tr>
<tr>
<td>20</td>
<td>0.8620</td>
<td>0.9688</td>
<td>0.3920</td>
<td>0.6650</td>
</tr>
<tr>
<td>30</td>
<td>0.9560</td>
<td>0.9945</td>
<td>0.5381</td>
<td>0.8093</td>
</tr>
<tr>
<td>40</td>
<td>0.9894</td>
<td>0.9993</td>
<td>0.6621</td>
<td>0.8874</td>
</tr>
<tr>
<td>50</td>
<td>0.9978</td>
<td>1.0000</td>
<td>0.7551</td>
<td>0.9404</td>
</tr>
<tr>
<td>60</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.8220</td>
<td>0.9661</td>
</tr>
<tr>
<td>70</td>
<td>0.9999</td>
<td>1.0000</td>
<td>0.8888</td>
<td>0.9819</td>
</tr>
<tr>
<td>80</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9294</td>
<td>0.9895</td>
</tr>
<tr>
<td>90</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9560</td>
<td>0.9944</td>
</tr>
<tr>
<td>100</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9736</td>
<td>0.9978</td>
</tr>
<tr>
<td>120</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9896</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

Table 3.2.

Power of $Q^{*}_{4(sd)}$ test for values of $\beta > 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta = 1.4$</th>
<th>$\beta = 1.6$</th>
<th>$\beta = 2$</th>
<th>$\beta = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>5</td>
<td>0.0188</td>
<td>0.0944</td>
<td>0.0218</td>
<td>0.1245</td>
</tr>
<tr>
<td>10</td>
<td>0.0356</td>
<td>0.1443</td>
<td>0.0560</td>
<td>0.2005</td>
</tr>
<tr>
<td>20</td>
<td>0.0635</td>
<td>0.2043</td>
<td>0.1325</td>
<td>0.3324</td>
</tr>
<tr>
<td>30</td>
<td>0.0953</td>
<td>0.2813</td>
<td>0.1894</td>
<td>0.4386</td>
</tr>
<tr>
<td>40</td>
<td>0.1246</td>
<td>0.3264</td>
<td>0.2423</td>
<td>0.4965</td>
</tr>
<tr>
<td>50</td>
<td>0.1581</td>
<td>0.3803</td>
<td>0.3233</td>
<td>0.5920</td>
</tr>
<tr>
<td>60</td>
<td>0.1888</td>
<td>0.4159</td>
<td>0.3618</td>
<td>0.6535</td>
</tr>
<tr>
<td>70</td>
<td>0.2320</td>
<td>0.4681</td>
<td>0.4720</td>
<td>0.7110</td>
</tr>
<tr>
<td>80</td>
<td>0.2529</td>
<td>0.5111</td>
<td>0.4985</td>
<td>0.7574</td>
</tr>
<tr>
<td>90</td>
<td>0.2768</td>
<td>0.5386</td>
<td>0.5529</td>
<td>0.7989</td>
</tr>
<tr>
<td>100</td>
<td>0.3121</td>
<td>0.5776</td>
<td>0.6081</td>
<td>0.8244</td>
</tr>
<tr>
<td>120</td>
<td>0.3813</td>
<td>0.6540</td>
<td>0.6953</td>
<td>0.8804</td>
</tr>
</tbody>
</table>

From the tables we see that, the powers of the test are good even for small samples for small and large values of $\beta$. 

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For comparison purpose, we give in Table 3.3, the values of the power for $\beta=1.2(.2)2.0$ and for $n=5,10,15,20,30$, which are read approximately from the power curves given in Keating et al. (1990). It can be seen that the powers of our test are at par with Keating et al. even for small samples.

Table 3.3.

Power of the test suggested by Keating et al. (1990). for $\alpha = 0.05$.

<table>
<thead>
<tr>
<th>n</th>
<th>$\beta=1.2$</th>
<th>$\beta=1.4$</th>
<th>$\beta=1.6$</th>
<th>$\beta=1.8$</th>
<th>$\beta=2.0$</th>
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<tbody>
<tr>
<td>05</td>
<td>.06</td>
<td>.08</td>
<td>.11</td>
<td>.15</td>
<td>.17</td>
</tr>
<tr>
<td>10</td>
<td>.09</td>
<td>.15</td>
<td>.21</td>
<td>.30</td>
<td>.39</td>
</tr>
<tr>
<td>15</td>
<td>.10</td>
<td>.19</td>
<td>.31</td>
<td>.48</td>
<td>.58</td>
</tr>
<tr>
<td>20</td>
<td>.12</td>
<td>.23</td>
<td>.42</td>
<td>.58</td>
<td>.69</td>
</tr>
<tr>
<td>30</td>
<td>.18</td>
<td>.32</td>
<td>.58</td>
<td>.72</td>
<td>.88</td>
</tr>
</tbody>
</table>

We compute the powers of Linhart's (1965) approximate $\chi^2$-test($\beta \geq 2$), for $\beta=2,3$ and $n=5,10,20,40$ and give in Table 3.4. We see that our test shows more power than the Linhart test.

Table 3.4.

Power of the test suggested by Linhart (1965).

<table>
<thead>
<tr>
<th>n</th>
<th>$\beta=2$</th>
<th>$\beta=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>5</td>
<td>.0205</td>
<td>.1365</td>
</tr>
<tr>
<td>10</td>
<td>.0740</td>
<td>.2580</td>
</tr>
<tr>
<td>20</td>
<td>.1930</td>
<td>.5040</td>
</tr>
<tr>
<td>40</td>
<td>.5850</td>
<td>.8500</td>
</tr>
</tbody>
</table>

55
We also compute the powers of the test given in Bain and Engelhardt (1975) and present in Table 3.5 and Table 3.6.

Table 3.5.

Power of the test suggested by Bain et al. (1975) for \( \beta > 1 \)

<table>
<thead>
<tr>
<th>n</th>
<th>( \beta=1.5 )</th>
<th>( \beta=2 )</th>
<th>( \beta=3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>5</td>
<td>.0262</td>
<td>.0960</td>
<td>.0401</td>
</tr>
<tr>
<td>10</td>
<td>.0395</td>
<td>.1687</td>
<td>.1125</td>
</tr>
<tr>
<td>20</td>
<td>.1308</td>
<td>.3696</td>
<td>.4170</td>
</tr>
<tr>
<td>40</td>
<td>.2738</td>
<td>.5861</td>
<td>.7846</td>
</tr>
</tbody>
</table>

Table 3.6.

Power of the test suggested by Bain et al. (1975) for \( \beta < 1 \)

<table>
<thead>
<tr>
<th>n</th>
<th>( \beta=0.2 )</th>
<th>( \beta=0.4 )</th>
<th>( \beta=0.6 )</th>
<th>( \beta=0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>5</td>
<td>.3762</td>
<td>.5725</td>
<td>.1446</td>
<td>.2756</td>
</tr>
<tr>
<td>10</td>
<td>.6802</td>
<td>.8876</td>
<td>.2661</td>
<td>.5242</td>
</tr>
<tr>
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<td>.9204</td>
<td>.9862</td>
<td>.5800</td>
<td>.8504</td>
</tr>
<tr>
<td>40</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.7456</td>
<td>.9524</td>
</tr>
</tbody>
</table>

To attain the powers of Bain et al. test, our test requires slightly more number of observations.
3.3. A TEST BASED ON THE LOGARITHMIC OBSERVATIONS.

3.3.1. Derivation of the test.

In this section, we derive the test statistic $Q$, defined in (1.1) for testing $H_0: \beta = 1$ against $H_1: \beta > 1$ ($H_2: \beta < 1$) considering the data $Z = (Z_1, Z_2, \ldots, Z_n)'$, where $Z_i = \ln(X_i)$. From (3.2.3), the conditional distribution of $Z_i$ given $T=t$ is

$$f_{Z_i|T}(z_i|t) = \frac{B(\beta, (n-1)\beta)^{-1}}{(e^{z_i/t} - 1)} (1 - e^{-z_i/t})^{(n-1)\beta - 1}, \quad (3.3.1)$$

$0 < z_i < t$, which does not contain the nuisance parameter $\theta$. In order to compute $Q$, we require the conditional mean and conditional variances and covariances, which can be obtained as follows:

$$\mu_i = E_{H_0}(Z_i|T=t)$$

$$= (n-1) \int_{-\infty}^{\ln(t)} z_i (e^{z_i/t} - 1) (1 - e^{z_i/t})^{n-2} \, dz_i$$

$$= (n-1) \int_0^1 (1-y)^{n-2} \ln(yt) \, dy$$

$$= \ln(t) + (n-1) \int_0^1 (1-y)^{n-2} \ln(y) \, dy$$

$$= \ln(t) + (n-1) I_{n-2}, \quad (3.3.2)$$

where
In the image, there is a mathematical derivation involving integrals and series. The image contains the following text:

\[ I_{n-2} = \int_0^1 (1-y)^{n-2} \ln(y) \, dy. \]

The integral \( I_{n-2} \) be evaluated as

\[ I_{n-2} = y(1-y)^{n-2} \ln(y) \bigg|_0^1 - \int_0^1 y \, d[(1-y)^{n-2} \ln(y)] \]

\[ = -\int_0^1 y[(1-y)^{n-2}/y - (n-2)(1-y)^{n-3} \ln(y)] \, dy \]

\[ = -\int_0^1 (1-y)^{n-2} \, dy + (n-2) \int_0^1 y(1-y)^{n-3} \ln(y) \, dy \]

\[ = -1/(n-1) - (n-2) \int_0^1 (1-y)^{n-2} \ln(y) \, dy \]

\[ + (n-2) \int_0^1 (1-y)^{n-3} \ln(y) \, dy \]

This gives,

\[ (n-1)I_{n-2} = -1/(n-1) + (n-2)I_{n-3}. \]

With the repeated use of this recurrence relation, we get

\[ I_{n-2} = -S_{n-1}/(n-1), \quad (3.3.3) \]

where

\[ S_{n-1} = \sum_{j=1}^{n-1} 1/j, \quad (3.3.4) \]

Using (3.3.3) in (3.3.2) we have,

\[ \mu_i = \ln(t) - S_{n-1}. \quad (3.3.5) \]

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Note that,
\[ \mu_i = \ln(t) - \ln(n) - (S_{n-1} - \ln(n)) \]
\[ \cong \ln(t/n) - \gamma, \]
where,
\[ \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{j} - \ln(n) \right), \]
the Euler's constant.

Now,
\[ E_{H_0} (Z_i^2 | T=t) = (n-1) \int_{-\infty}^{1} z_i^2 \frac{\exp(z_i/t)}{(1-e^{z_i/t})^{n-2}} dz_i \]
\[ = (n-1) \int_{0}^{1} (1-y)^{n-2} \ln^2(yt) dy \]
\[ = (n-1) \left[ \int_{0}^{1} (1-y)^{n-2} \ln^2(y) dy + \int_{0}^{1} (1-y)^{n-2} \ln^2(t) dy \right] \]
\[ + 2 \int_{0}^{1} (1-y)^{n-2} \ln(y) \ln(t) dy \]
\[ = (n-1) \left[ J_{n-2} + \ln^2(t)/(n-1) + 2\ln(t) I_{n-2} \right], \quad (3.3.6) \]
where,
\[ J_{n-2} = \int_{0}^{1} (1-y)^{n-2} \ln^2(y) dy \]
and \( I_{n-2} \) is same as defined in (3.3.3). The integral \( J_{n-2} \) be evaluated as follows.
\[ J_{n-2} = y(1-y)^{n-2} \ln^2(y) \bigg|_0^1 - \int_{0}^{1} y d[(1-y)^{n-2} \ln^2(y)] \]
\[ \int_0^1 (1-y)^{n-2} \ln(y) \, dy + (n-2) \int_0^1 y(1-y)^{n-3} \ln^2(y) \, dy \]
\[ = -2I_{n-2} - (n-2)[\int_0^1 (1-y)^{n-3} \ln^2(y) \, dy \]
\[ + \int_0^1 (1-y)^{n-2} \ln^2(y) \, dy] \]
\[ = -2I_{n-2} - (n-2)[J_{n-2} - J_{n-3}] \]
This, 
\[ (n-1)J_{n-2} = -2I_{n-2} + (n-2)J_{n-3} \]
\[ = 2S_{n-1}/(n-1) + (n-2)J_{n-3}. \]
The repeated use of the last result gives 
\[ J_{n-2} = 2/(n-1) \sum_{i=1}^{n-1} S_i/n. \]
(3.3.7)
Using (3.3.7) in (3.3.6), we get 
\[ E_{H_0}(Z_i^2|T=t) = 2 \sum_{i=1}^{n-1} S_i/i + \ln^2(t) - 2S_{n-1} \ln(t). \]
Therefore,
\[ \sigma_{ii} = \nu_{H_0}(Z_i|T=t) = 2 \sum_{i=1}^{n-1} S_i/i - S_{n-1} \]
\[ = 2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} 1/(ij) - S_{n-1}^2 \]
\[
\begin{align*}
&= \left( \sum_{j=1}^{n-1} 1/j \right)^2 + \sum_{j=1}^{n-1} 1/j^2 - S_{n-1}^2 \\
&= \sum_{j=1}^{n-1} 1/j^2 \\
&= S_{n-1}^*.
\end{align*}
\] (3.3.8)

where
\[
S_{n-1}^* = \sum_{j=1}^{n-1} 1/j^2.
\]

Using (3.2.7), the conditional distribution of \((Z_1, Z_j)\) given \(T=t\) is

\[
f_{Z_1, Z_j | T}(z_1, z_j | t) = \frac{\beta(z_1 + z_j)}{(\Gamma(\beta))^2 T((n-2)\beta) t^{2\beta}} \frac{e^{\beta(z_1 + z_j)}}{(1 - e^{z_1/t - z_j/t})^{(n-2)\beta-1}},
\] (3.3.9)

\(0 \leq z_1 + z_j < t, -\infty \leq z_1, z_j \leq \infty.\) Then under \(H_0,\)

\[
E_{H_0}(Z_1 Z_j | T=t) = \frac{1}{t^2} \int_0^{\infty} \int_0^{\infty} z_1 z_j e^{z_1 + z_j} (1 - e^{z_1/t - z_j/t})^{n-2} dz_1 dz_j
\]

\[
= \int_0^{\infty} \int_0^{\infty} (1-y_i - y_j)^{n-3} \ln(ty_i) \ln(ty_j) dy_i dy_j
\]

\[
= \int_0^{\infty} \int_0^{\infty} (1-y_i/(1-y_j))^{n-3} \ln(ty_j) dy_i dy_j
\]

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Using (3.3.2) and (3.3.3) for first integral and integrating by parts the second integral we get,

\[
E_{H_0}(Z_i Z_j | T=t) = (\ln(t) - S_{n-1})(\ln(t) - S_{n-2} - 1/(n-1)) \\
+ \int_0^1 (1-y_j)^{n-1} [\ln(1-y_j)]/y_j \, dy_j
\]

\[
= (\ln(t) - S_{n-1})^2 - \int_0^\infty u e^{-nu}/(1-e^{-u}) \, du
\]

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\[
E_{H_0}(Z_i Z_j | T=t) = (\ln(t) - S_{n-1})^2 + \sum_{j=1}^{n-1} \frac{1}{j^2} - \frac{n^2}{6}.
\]

Thus

\[
\sigma_{ij} = \text{COV}_{H_0}(Z_i, Z_j | T=t)
= S_{n-1}^* - \frac{n^2}{6}.
\] (3.3.10)

Using (3.3.8) and (3.3.10), the variance-covariance matrix \( \Sigma_0 \) of \( (Z_i | T=t) \), under \( H_0 \), is

\[
\Sigma_0 = \begin{bmatrix}
S_{n-1}^* & S_{n-1}^* - \frac{n^2}{6} & \cdots & S_{n-1}^* - \frac{n^2}{6} \\
S_{n-1}^* - \frac{n^2}{6} & S_{n-1}^* & \cdots & S_{n-1}^* - \frac{n^2}{6} \\
S_{n-1}^* - \frac{n^2}{6} & S_{n-1}^* - \frac{n^2}{6} & \cdots & S_{n-1}^*
\end{bmatrix}
= (S_{n-1}^* - \frac{n^2}{6}) E_{nn} + \frac{n^2}{6} I_n.
\]

Using the fact that \( \sum_{j=0}^{\infty} \frac{1}{(n+j)^2} = \frac{n^2}{6} - \frac{\pi^2}{6} \), we have finally,

\[
E_{H_0}(Z_i Z_j | T=t) = (\ln(t) - S_{n-1})^2 + \sum_{j=1}^{n-1} \frac{1}{j^2} - \frac{n^2}{6}.
\]

Considering \( a = S_{n-1}^* \) and \( b = S_{n-1}^* - \frac{n^2}{6} \) and using the result (1.1.)
given on page 67 in Rao (1974), we get
\[
\Sigma_0^{-1} = \frac{1}{(a-b)} I_n - \frac{b}{[a+(n-1)b](a-b)} E_{nn}
\]

\[
= 6[I_n - bE_{nn}]/\Pi^2.
\]

where,

\[
b_n = \frac{S_{n-1}^* - \Pi^2/6}{n(S_{n-1}^* - \Pi^2/6) + \Pi^2/6}.
\]

Hence, the expression for the test statistic is

\[
Q_5 = (Z-\mu_0)'\Sigma_0^{-1}(Z-\mu_0)
\]

\[
= (6/\Pi^2)[\sum_{i=1}^{n} (Z_i-\mu_1)^2 - nb_n(\sum_{i=1}^{n} (Z_i-\mu_1))^2],
\]

(3.3.12)

where \(\mu_1\) is same as given in (3.3.5). Replacing constant \(t\) by the corresponding random variable \(T\), we get the test statistic as

\[
Q_5^* = (6/\Pi^2)\left\{\sum_{i=1}^{n} \left[\ln(X_i/\sum X_i) + S_{n-1}\right]^2 - nb_n(\sum_{i=1}^{n} \left[\ln(X_i/\sum X_i) + S_{n-1}\right])^2\right\}
\]

(3.3.13)

Since we could not find the expression for \(E_{H_1}(Q_5^*)\), we simulate its values for \(n=10,20,30\) and for different values of \(\beta\). These values are presented in the form of curves as shown in fig.3.1.
Fig. 3.1.
It is seen from the curves that $E_{H_1}(Q^*_5) > E_{H_0}(Q^*_5)$ for $\beta < 1$.

Hence the test procedure is to reject $H_0$ for large values of $Q^*_5$ for $\beta < 1$. For $\beta > 1$, $E_{H_1}(Q^*_5) < E_{H_0}(Q^*_5)$ for some values of $\beta$ in $(1.0, 1.3)$ and $E_{H_1}(Q^*_5) > E_{H_0}(Q^*_5)$ for $\beta \geq 1.3$. Thus we reject $H_0$ for large values of $Q^*_5$ when $\beta \geq 1.3$.

3.3.2. Simulation study for the percentiles and the powers.

The percentiles of the distribution of $Q^*_5$ is computed using Monte-Carlo method by generating 5000 random samples of size $n = 10(10)120$. The values are given in Table 3.7.

| Table 3.7. Percentile points for $Q^*_5$. |
|---|---|---|---|---|
| $n$ | .01 | .05 | .95 | .99 |
| 10 | 6.3933 | 7.4157 | 16.2734 | 38.9599 |
| 20 | 14.4432 | 16.2613 | 32.4687 | 71.5915 |
| 30 | 22.7854 | 25.3559 | 51.0867 | 112.1822 |
| 40 | 31.1898 | 34.4877 | 67.7390 | 152.0391 |
| 50 | 40.0422 | 44.1712 | 85.2819 | 184.2534 |
| 60 | 49.3785 | 53.6784 | 104.8490 | 232.8567 |
| 70 | 57.8462 | 63.6455 | 124.3701 | 277.4249 |
| 80 | 67.2796 | 73.1359 | 146.5859 | 343.6443 |
| 90 | 77.1920 | 82.9297 | 167.1981 | 394.3276 |
| 100 | 86.6124 | 93.3655 | 185.4655 | 428.2257 |
| 120 | 105.1137 | 113.0643 | 235.9415 | 564.8487 |

Using the above percentiles, the powers are computed for different values of $n$ and $\beta$. They are given in Table 3.8 and 3.9.
Table 3.8.
Power of $Q_5^*$ test for values of $\beta<1$.

<table>
<thead>
<tr>
<th>n</th>
<th>$\beta=0.2$</th>
<th>$\beta=0.4$</th>
<th>$\beta=0.6$</th>
<th>$\beta=0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>10</td>
<td>0.9694</td>
<td>0.9880</td>
<td>0.6180</td>
<td>0.7692</td>
</tr>
<tr>
<td>20</td>
<td>0.9988</td>
<td>1.0000</td>
<td>0.8934</td>
<td>0.9538</td>
</tr>
<tr>
<td>30</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.9630</td>
<td>0.9886</td>
</tr>
<tr>
<td>40</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9910</td>
<td>0.9962</td>
</tr>
<tr>
<td>50</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9990</td>
<td>0.9998</td>
</tr>
<tr>
<td>60</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9994</td>
<td>1.0000</td>
</tr>
<tr>
<td>70</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>80</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>90</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>100</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>120</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3.9.
Power of $Q_5^*$ test for values of $\beta>1$.

<table>
<thead>
<tr>
<th>n</th>
<th>$\beta=1.6$</th>
<th>$\beta=1.8$</th>
<th>$\beta=2$</th>
<th>$\beta=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>20</td>
<td>0.0019</td>
<td>0.0738</td>
<td>0.0038</td>
<td>0.1374</td>
</tr>
<tr>
<td>30</td>
<td>0.0030</td>
<td>0.2000</td>
<td>0.0044</td>
<td>0.3548</td>
</tr>
<tr>
<td>40</td>
<td>0.0188</td>
<td>0.3658</td>
<td>0.0596</td>
<td>0.5796</td>
</tr>
<tr>
<td>50</td>
<td>0.0964</td>
<td>0.5622</td>
<td>0.2420</td>
<td>0.7668</td>
</tr>
<tr>
<td>60</td>
<td>0.2140</td>
<td>0.6578</td>
<td>0.4374</td>
<td>0.8698</td>
</tr>
<tr>
<td>70</td>
<td>0.3204</td>
<td>0.7674</td>
<td>0.6016</td>
<td>0.9390</td>
</tr>
<tr>
<td>80</td>
<td>0.4370</td>
<td>0.8186</td>
<td>0.7298</td>
<td>0.9525</td>
</tr>
<tr>
<td>90</td>
<td>0.5310</td>
<td>0.8650</td>
<td>0.8320</td>
<td>0.9750</td>
</tr>
<tr>
<td>100</td>
<td>0.6890</td>
<td>0.9420</td>
<td>0.9190</td>
<td>0.9920</td>
</tr>
<tr>
<td>120</td>
<td>0.7920</td>
<td>0.9650</td>
<td>0.9800</td>
<td>0.9990</td>
</tr>
</tbody>
</table>

From the table it is seen that for $\beta<1$, $Q_5^*$ is better than $Q_4^*$ and other tests described in section 3.2, in identifying the Gamma alternatives. As $\beta$ tends to zero, the test performs well even for small samples. However, the test performs badly for $\beta>1$, especially for small samples, when $\beta$ is not far from 1.