CHAPTER 2

CONDITIONAL TESTS FOR THE SHAPE PARAMETER IN WEIBULL DISTRIBUTION

2.1 INTRODUCTION.

Consider the two parameter Weibull distribution having the pdf,

\[ f_X(x; \theta, \beta) = (\beta/\theta) x^{\beta-1} e^{-x^\beta/\theta}, \quad \theta > 0, \beta > 0, x > 0, \]  

(2.1.1)

where \( \beta \) is the shape parameter and \( \theta \) is the scale parameter. This model includes the exponential distribution with constant hazard function for \( \beta = 1 \) and provide an increasing hazard function for \( \beta > 1 \) and decreasing hazard function for \( \beta < 1 \). This model had been derived in an analysis of breaking strengths by Waloddi Weibull in 1939. This distribution had also been earlier derived by Fisher and Tippett in 1928 as the third asymptotic distribution of extreme values. Consequently, in some applications there may be theothermal reasons for choosing the Weibull model based on extreme value theory. As an example, suppose \( X \) represents the strength of a chain of \( n \) links and if \( X_i \) denote the strength of its \( i \)th link, then \( X = \min(X_i) \). Thus the distribution of \( X \) is the distribution of a minimum. For many different types of \( X_i \) variable the limiting distribution of the minimum approaches a Weibull distribution as \( n \to \infty \). The breaking strength of a ceramic would be
similar if the ceramic breaks at its weakest flow. W.A. Thompson (1969, p. 153) discusses another type of application related to traffic flow, where a driver's speed is constrained by the slowest driver. Thus, the Weibull model may sometimes be suggested by theoretical considerations, particularly when related to extreme value characteristics.

Weibull distribution is quite popular as a life-testing model and for many other applications where a skewed distribution is required. Probably the main justification for consideration of the Weibull distribution is that it has been shown experimentally to provide a good fit for many different types of characteristics derived in an analysis of breaking strengths. This model is quite flexible and has the advantage of having a closed form of CDF.

Thoman et.al. (1969) have considered the problem of testing of hypotheses regarding the shape parameter in the Weibull distribution based on MLE. They have also provided the power of the test for different values of the shape parameter under the alternatives. Mann et.al. (1973) have studied a goodness-of-fit test for two-parameter Weibull distribution. Bain and Engelhardt (1986) have proposed a modified version of Thoman et.al. (1969) test statistic whose asymptotic distribution may be approximated to a chi-squared distribution.
In this chapter, we propose a test for testing $H_0: \beta = \beta_0$ against $H_1: \beta > \beta_0 \ (H_2: \beta < \beta_0)$ treating $\theta$ as the nuisance parameter. Section 2.2 discusses the test procedure based on a complete sample. In sections 2.3 and 2.4, we obtain the tests based on a Type-2 censored sample. We also study the properties of the test statistics, simulate the percentiles of the distributions of the test statistics and carry out the power studies in the respective sections.

2.2. A TEST BASED ON A COMPLETE SAMPLE.

In the following subsections we derive the expression for the test statistic and suggest the test procedure. We also study the asymptotic property of the null distribution of the test statistic. Since the expression for the exact distribution is difficult to find, we simulate the percentiles of the exact distribution of the test statistic. We also carry out the Monte-Carlo power study of the test.

2.2.1 Derivation of the test.

Let $X_1, X_2, \ldots, X_n$ be a random sample from (2.1.1). Making the transformation $Y_i = \frac{X_i}{\beta_0}$, the problem of testing can be reduced to $H_0: \nu = 1$ against $H_1: \nu > 1 \ (H_2: \nu < 1)$, where $\nu = \frac{\beta}{\beta_0}$, based on the random observations $Y_1, Y_2, \ldots, Y_n$. Then the joint density of $Y_1, Y_2, \ldots, Y_n$ is

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\begin{align*}
  f_Y(y;\theta,\nu) &= (\nu/\theta)^n \prod_{i=1}^{n} y_i^{\nu-1} \exp\left(-\frac{\nu y_i}{\theta}\right). \tag{2.2.1}
\end{align*}

0 < y_i < \infty, \ i=1,2,\ldots,n; \theta,\nu > 0. \text{ Here } T = \Sigma_{i=1}^{n} Y_i^{\nu}, \nu \text{ known, is the complete sufficient statistic for } \theta. \text{ having the pdf}

\begin{align*}
  f_T(t;\theta) &= (\theta^{n-1})e^{-t/\theta} t^{n-1}, t > 0. \tag{2.2.2}
\end{align*}

Since \( Y_i \) is Weibull(\( \theta,\nu \)) and \( Z = \Sigma_{j \neq i} Y_j^{\nu} \) follows Gamma(\( \theta, n-1 \)), and they are independent, their joint pdf is given by

\begin{align*}
  f_{Y_i,Z}(y_i,z;\theta,\nu) &= \nu y_i^{\nu-1} z^{n-2} e^{-(y_i^{\nu}+z)/\theta} / (\theta^{n-1} y_i > 0, z > 0).
\end{align*}

Making the transformations \( T = Y_i^{\nu} + Z \) and \( Y_i = Y_i^{\nu} \), we get the joint density of \( Y_i \) and \( T \) as

\begin{align*}
  f_{Y_i,T}(y_i,t;\theta,\nu) &= \nu y_i^{\nu-1} (t-y_i^{\nu})^{n-2} e^{-(y_i^{\nu})/\theta} / (\theta^{n-1} (n-1)).
\end{align*}

0 < y_i^{\nu} < t < \infty. \text{ Then the conditional distribution of } Y_i \text{ given } T = t \text{ is}

\begin{align*}
  f_{Y_i/T}(y_i/t;\nu) &= (n-1)\nu (y_i^{\nu-1}/t)(1-y_i^{\nu}/t)^{n-2}, \tag{2.2.3}
\end{align*}

0 < y_i^{\nu} < t, \nu > 0.
It can be seen that the conditional distribution \((2=2=3)\) does not contain the nuisance parameter \(\Theta\). Moreover, it is the uniformly minimum variance unbiased estimate of the density \(f_Y(y,\Theta,\nu)\). Hence we, shall derive the expression for the test statistic \(Q\) defined in (1.1), for testing \(H_0: \mu = 1\) against \(H_1: \mu \neq 1\) considering that the data \(Y = (y_1, y_2, \ldots, y_n)'\) have come from the estimate of \(f_Y(y; \Theta, \nu)\) given in (2.2.3), for known value \(t\) of \(T\) instead from \(f_Y(y; \Theta, \nu)\). For this, we obtain \(\mu_i\) and \(\sigma_{ij}\) as follows:

\[
E(Y_i^r | T = t) = \int_0^{t^r/\nu} (n-1)\nu(y_i^{r+1}/t)(1-y_i^{\nu}/t)^{n-2}dy_i
\]

\[
= (n-1)t^{r/\nu} B(n-1, r/\nu+1), \quad (2.2.4)
\]

which yields

\[
\mu_i = E_{H_0}(Y_i | t) = t/n \quad (2.2.5)
\]

and

\[
\sigma_{ii} = V_{H_0}(Y_i | t) = (n-1)t^2/(n^2(n+1)). \quad (2.2.6)
\]

To find \(\sigma_{ij} = \text{COV}_{H_0}(Y_i, Y_j | t), \ i \neq j\), we need the conditional density of \((Y_i, Y_j)\) given \(T = t\). Letting \(Z = \sum_{k=1, j} Y_k\), the joint pdf of \(Y_i, Y_j\) and \(Z\) is

\[
f_{Y_i, Y_j, Z}(y_i, y_j, z; \Theta, \nu) = \nu^2(y_i y_j)^{\nu-1}z^{-3}\nu^{\nu+\nu z}/(\Theta^n T(n-2)).
\]

Now, making the transformations, \(T = \sum_{i, j} y_i^\nu + z\), \(Y_i = Y_i\) and \(Y_j = Y_j\), we
\[ f_{Y_i, Y_j, \mathcal{T}}(y_i, y_j, t; \theta, \nu) = \nu^2(y_i y_j)^{\nu-1} (t-y_i y_j)^{n-3} e^{-t/\theta} / (\theta^{n+2(n-2)}), \]

for \(0 < y_i, y_j < \infty, y_i y_j < t < \infty\). Hence, the conditional pdf of \((Y_i, Y_j)\) given \(T=t\) is

\[ f_{Y_i, Y_j|T}(y_i, y_j|t; \nu) = (n-1)(n-2)(\nu/t)^2(y_i y_j)^{\nu-1}(1-y_i y_j/t)^{n-3}. \]

(2.2.7)

\[ y_i, y_j > 0, y_i y_j < t. \] Using (2.2.7), we get,

\[ E_{H_0}(Y_i^s Y_j^r | T=t) = (n-1)(n-2)B(r+1, n-2)B(r+1, n+r-1)t^{2r}. \]

(2.2.8)

Thus

\[ E_{H_0}(Y_i Y_j | T=t) = t^2/(n(n+1)), \]

and hence

\[ \sigma_{ij} = \text{COV}_{H_0}(Y_i Y_j | T=t) = -t^2/(n^2(n+1)). \]

It is interesting to note that,

\[ \sigma_{ij} = -\sigma_{ii}/(n-1). \]

(2.2.9)

Using (2.2.6) and (2.2.9), the variance-covariance matrix \( \Sigma_0 \) of \( \sim | T=t \), under \( H_0 \), is given by
\[ \Sigma_0 = \frac{t^2}{n(n+1)} \begin{bmatrix} 1-1/n & -1/n & -1/n & \ldots & -1/n & -1/n \\ -1/n & 1-1/n & -1/n & \ldots & -1/n & -1/n \\ \vdots & & & & & \end{bmatrix} \]

\[ = \frac{t^2}{n(n+1)} [I_n - E_{nn}/n], \quad (2.2.10) \]

where \( I_n \) is the identity matrix of order \( n \) and \( E_{nn} \) is the matrix of order \( nxn \) with all elements unity. Then the g-inverse of \( \Sigma_0 \) is

\[ \Sigma_0^- = n(n+1)[I_n + E_{nn}/n]/t^2. \quad (2.2.11) \]

Hence, the expression for the test statistic is

\[ \Omega_1 = (Y-\mu_0)' \Sigma_0^- (Y-\mu_0) \]

\[ = n(n+1)(Y- t \frac{1}{n} E_n)'(I_n + \frac{1}{n} E_{nn})(Y- t \frac{1}{n} E_n)/t^2 \]

\[ = n(n+1)[ \sum_{i=1}^{n} \frac{Y_i^2}{t^2} - 1/n ]. \]

This statistic is based on the given condition \( T=t \). In practice, hardly we know the value \( t \) of \( T \) in advance. So, we modify this statistic by replacing \( t \) by its corresponding random variable \( T \) and propose the new statistic,

\[ \Omega_1^* = n(n+1)[ \sum_{i=1}^{n} \frac{Y_i^2}{T^2} - 1/n], \quad (2.2.12) \]

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\( T = \sum_{i=1}^{n} Y_i \). The exact mean and variance of \( Q_1^* \) can be obtained as follows:

Let \( V = \sum_{i=1}^{n} \frac{Y_i^2}{T^2} \). Then, under \( H_0 \),

\[
E_{H_0}(V) = \mathbb{E}[V|T=t]
\]

\[
= \mathbb{E}(n \mathbb{E}[Y_i^2|T=t]/T^2)
\]

\[
= \frac{2}{(n+1)}
\]

\[
E_{H_0}(V^2) = \mathbb{E}[V^2|T=t]
\]

\[
= \mathbb{E}((n\mathbb{E}[Y_i^4|t]+n(n-1)\mathbb{E}[Y_i^2Y_j^2|t])/T^4).
\]

Upon using (2.2.4) and (2.2.8), this yields

\[
E_{H_0}(V^2) = \frac{4(n+5)/((n+1)^2(n+2)(n+3))}{(n+1)^2(n+2)(n+3))}.
\]

Therefore,

\[
E_{H_0}(Q_1^*) = n-1
\]

and

\[
V_{H_0}(Q_1^*) = \frac{4n^2(n-1)/((n+2)(n+3))}{(n+1)^2(n+2)(n+3))}.
\]

Hence, we propose the test statistic \( Q_1^* \) in standardized form for
testing $H_0: \beta = \beta_0$ against $H_1: \beta > \beta_0$ ($H_2: \beta < \beta_0$) based on the random sample $(X_1, X_2, \ldots, X_n)$ from (2.1.1) as

$$Q_1^{*} = \left[ \frac{Q_1^{*} - E_{H_0}(Q_1^{*})}{V_{H_0}(Q_1^{*})} \right]^{1/2}$$

$$= \left[ \frac{(n+2)(n+3)/(n-1) \sum_{i=1}^{n} x_i - 2\beta_0}{\sum_{i=1}^{n} x_i - 2(n+1)\beta_0} \right]^{1/2}.$$  (2.2.14)

To determine the test procedure, we compute $E_{H_1}(Q_1^{*}(sd))$ for different values of $(\beta/\beta_0)$, $\Theta$ and $n$. These values are given in Table 2.1.

<table>
<thead>
<tr>
<th>$\beta/\beta_0$</th>
<th>$\Theta=0.5$</th>
<th>$\Theta=1.0$</th>
<th>$\Theta=2.0$</th>
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The table shows that $E_{H_1}(Q_1^{*}(sd)) < 0$ ($\geq 0$) for $\beta > \beta_0$ ($0 < \beta < \beta_0$) irrespective of the value of the nuisance parameter $\Theta$. Hence, the test procedure is to reject $H_0: \beta = \beta_0$ in favour of $H_1: \beta > \beta_0$ ($H_2: \beta \leq \beta_0$) for small (large) values of $Q_1^{*}(sd)$.  

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2.2.2. Asymptotic null distribution of $Q^*_1$.

Since it is very difficult to obtain the exact distribution of $Q^*_1$ (or $Q^*_1$), we shall derive the asymptotic null distribution of the test statistic $Q^*_1$ using the asymptotic theory of $U$-statistics.

Define, $U_1 = \frac{\beta}{n} \sum_{i=1}^{n} X_i^0/n$ and $U_2 = \frac{2\beta}{n} \sum_{i=1}^{n} X_i^0/n$.

Then $Q^*_1$ reduces to

$$Q^*_1 = (n+1)[U_2/U_1^2-1] = g(U_1, U_2),$$

Now, since

$$E[X_i^0] = \theta^{\beta_0/\beta} \Gamma(\beta_0/\beta + 1),$$

we have

$$E(U_1) = \theta^{\beta_0/\beta} \Gamma(\beta_0/\beta + 1),$$

$$E(U_2) = \theta^{2\beta_0/\beta} \Gamma(2\beta_0/\beta + 1),$$

$$V(U_1) = \theta^{2\beta_0/\beta} \frac{[\Gamma(2\beta_0/\beta + 1) - (\Gamma(\beta_0/\beta + 1))^2]}{n},$$

$$V(U_2) = \theta^{4\beta_0/\beta} \frac{[\Gamma(4\beta_0/\beta + 1) - (\Gamma(2\beta_0/\beta + 1))^2]}{n}.$$
\[ \text{COV}(U_1, U_2) = \text{COV}\left[ \sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i \right]/n^2 \]
\[ = \text{COV}[X_1, X_1]/n \]
\[ = \frac{3\beta_0}{\beta} \left[ \Gamma(3\beta_0/\beta + 1) - \Gamma(\beta_0/\beta + 1) \Gamma(2\beta_0/\beta + 1) \right]. \]

If we write \( U = (U_1, U_2)' \), then the mean vector \( \mu_0 \) and the dispersion matrix \( D_0(U) \) of \( U \) under the null hypothesis are given by

\[ \mu_0 = (\theta, 2\theta^2)' \]

and

\[ D_0(U) = \begin{bmatrix} \theta^2/n & 4\theta^3/n \\ 4\theta^3/n & 20\theta^4/n \end{bmatrix} \]

Then by Hoeffding (1948), the asymptotic null distribution of \( U \sim \) is bivariate normal with mean vector \( \mu_0 \) and dispersion matrix \( D_0(U) \).

Following Rao (1974, pp. 387, 6a.2 (ii)), the approximate distribution of \( Q_1^* \) is given by

\[ Q_1^* = g(U_1, U_2) \sim N\left[ g(\mu_1, \mu_2), \sigma_1^2(\partial g/\partial \mu_1)^2 + 2\sigma_2 \frac{\partial^2 g}{\partial \mu_1 \partial \mu_2} + \sigma_2^2(\partial g/\partial \mu_2)^2 \right]. \]

Using (2.2.16), we have

\[ g(\mu_1, \mu_2) = g(U_1, U_2)|U = \mu_0 = (n+1). \]
\[
\frac{\partial g}{\partial \mu_1} = \frac{\partial g(U_1, U_2)}{\partial U_1} \bigg|_{U=\mu} = -4(n+1)/\theta,
\]
\[
\frac{\partial g}{\partial \mu_2} = \frac{\partial g(U_1, U_2)}{\partial U_2} \bigg|_{U=\mu} = (n+1)/\theta^2,
\]
\[
\frac{2}{\partial \mu_1 \partial \mu_2} = \frac{2}{\partial U_1 \partial U_2} \bigg|_{U=\mu} = -2(n+1)/\theta^3.
\]

With the use of these results (2.2.17) reduces to

\[
Q_1^* \sim N[(n+1), 4(n+1)^2/n]
\]
That is,

\[
\sqrt{n} \left[ Q_1^* - (n+1) \right] / (2(n+1)) \xrightarrow{\mathcal{L}} N(0,1).
\]

Therefore,

\[
Q_1^{*}(sd) = \left[ Q_1^* - (n-1) \right] \sqrt{(n+2)(n+3)/(2n\sqrt{n-1})}
\]

\[
= a_n \sqrt{n} \left[ Q_1^* - (n+1) \right] / (2(n+1)) + \sqrt{(n+2)(n+3)/(n\sqrt{n-1})},
\]
where,

\[
a_n = (1+1/n) \sqrt{(n+2)(n+3)/(n(n-1))}.
\]

Clearly as \(n \to \infty\), \(a_n \to 1\) and the second term in the expression of \(Q_1^{*}(sd)\) tends to zero. Hence \(Q_1^{*}(sd) \to N(0,1)\). This suggests that, for large values of \(n\), one may use the percentiles of standard normal distribution for the test procedure.
2.2.3. Simulated percentiles of the distribution of $Q_{1(sd)}^*$

Since it is difficult to get the expression for the exact distribution of $Q_{1(sd)}^*$, we obtain the percentile points of the distribution of $Q_{1(sd)}^*$ by MonteCarlo experiment. For this, we generate 8000 random samples of size $n = 5(1)20(2)30(10)120$, from Weibull populations with $\beta=1$. Since $Q_{1(sd)}^*$ is scale invariant, the percentile points do not depend on $\theta$, we take $\theta=1$. These simulated percentiles are recorded in Table 2.2.

Table 2.2.
The Percentile points of the distribution of $Q_{1(sd)}^*$.

<table>
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<th>n</th>
<th>α</th>
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<th>.10</th>
<th>.90</th>
<th>.95</th>
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<td>3.2962</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>-1.5409</td>
<td>-1.2397</td>
<td>-1.0505</td>
<td>1.2376</td>
<td>1.8528</td>
<td>3.2954</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>-1.5558</td>
<td>-1.2473</td>
<td>-1.0506</td>
<td>1.2365</td>
<td>1.8404</td>
<td>3.2611</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>-1.5685</td>
<td>-1.2477</td>
<td>-1.0582</td>
<td>1.2354</td>
<td>1.8374</td>
<td>3.2375</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>-1.6270</td>
<td>-1.2908</td>
<td>-1.0935</td>
<td>1.2345</td>
<td>1.8213</td>
<td>3.2034</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>-1.6724</td>
<td>-1.3194</td>
<td>-1.1042</td>
<td>1.2342</td>
<td>1.8206</td>
<td>3.1996</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>-1.7107</td>
<td>-1.3459</td>
<td>-1.1176</td>
<td>1.2344</td>
<td>1.8143</td>
<td>3.0685</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>-1.7631</td>
<td>-1.3639</td>
<td>-1.1327</td>
<td>1.2321</td>
<td>1.8109</td>
<td>3.0680</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>-1.7902</td>
<td>-1.3779</td>
<td>-1.1449</td>
<td>1.2321</td>
<td>1.8102</td>
<td>3.0625</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-1.8036</td>
<td>-1.3988</td>
<td>-1.1506</td>
<td>1.2319</td>
<td>1.7923</td>
<td>3.0428</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>-1.8443</td>
<td>-1.4028</td>
<td>-1.1614</td>
<td>1.2303</td>
<td>1.7918</td>
<td>3.0400</td>
<td></td>
</tr>
</tbody>
</table>
The table shows that the rate of convergence of the distribution of $Q^*_1(s_d)$ to normal seems to be very slow.

2.2.4. Power study.

For power study of the test statistic $Q^*_1(s_d)$, we compute the powers for different values of $n$ and $\beta/\beta_0$ through simulation under Weibull alternatives. We present these results in the following tables.

Table 2.3.

<table>
<thead>
<tr>
<th>Powers of $Q^*_1(s_d)$ test for values of $\beta/\beta_0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta/\beta_0$ = 1.2</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>n</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>60</td>
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<tr>
<td>70</td>
</tr>
<tr>
<td>80</td>
</tr>
<tr>
<td>90</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>120</td>
</tr>
</tbody>
</table>
Table 2.4.

Power of $Q^*_1(s_d)$ test for values of $\beta/\beta_o$.

<table>
<thead>
<tr>
<th>$\beta/\beta_o$</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.6633</td>
<td>.8000</td>
<td>.3311</td>
<td>.5070</td>
<td>.1356</td>
<td>.2745</td>
<td>.0411</td>
<td>.1239</td>
</tr>
<tr>
<td>10</td>
<td>.9213</td>
<td>.9784</td>
<td>.5961</td>
<td>.7999</td>
<td>.2444</td>
<td>.4635</td>
<td>.0635</td>
<td>.1824</td>
</tr>
<tr>
<td>20</td>
<td>.9983</td>
<td>1.0000</td>
<td>.8718</td>
<td>.9608</td>
<td>.4300</td>
<td>.6810</td>
<td>.0996</td>
<td>.2680</td>
</tr>
<tr>
<td>30</td>
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<td>1.0000</td>
<td>.9645</td>
<td>.9934</td>
<td>.5965</td>
<td>.8188</td>
<td>.1375</td>
<td>.3411</td>
</tr>
<tr>
<td>40</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9919</td>
<td>.9993</td>
<td>.7259</td>
<td>.9038</td>
<td>.1786</td>
<td>.3995</td>
</tr>
<tr>
<td>50</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9980</td>
<td>.9998</td>
<td>.8161</td>
<td>.9479</td>
<td>.1974</td>
<td>.4578</td>
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<tr>
<td>60</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9999</td>
<td>1.0000</td>
<td>.8705</td>
<td>.9724</td>
<td>.2451</td>
<td>.5104</td>
</tr>
<tr>
<td>70</td>
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<td>1.0000</td>
<td>.9999</td>
<td>1.0000</td>
<td>.9211</td>
<td>.9851</td>
<td>.2935</td>
<td>.5650</td>
</tr>
<tr>
<td>80</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9504</td>
<td>.9924</td>
<td>.3249</td>
<td>.6133</td>
</tr>
<tr>
<td>90</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9733</td>
<td>.9965</td>
<td>.3611</td>
<td>.6553</td>
</tr>
<tr>
<td>100</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9815</td>
<td>.9983</td>
<td>.4003</td>
<td>.6929</td>
</tr>
<tr>
<td>120</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>.9951</td>
<td>.9993</td>
<td>.4616</td>
<td>.7551</td>
</tr>
</tbody>
</table>

From the Tables we see that even for small samples our test performs reasonably well in identifying Weibull alternatives.

In order to compare our test with chi-squared test, we generate 8000 random samples of size 100 from the Weibull population. Then compute $\chi^2$ by considering 10 class intervals of equal probability. The computed values of the powers of $Q^*_1(s_d)$ and chi-squared test are given in Table 2.5.
Table 2.5.

Powers of $Q^*_{1(sd)}$ and Chi-square test for $n = 100$.

<table>
<thead>
<tr>
<th>$\beta=0.6$</th>
<th>$\beta=0.8$</th>
<th>$\beta=1.2$</th>
<th>$\beta=1.4$</th>
<th>$\beta=1.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^*_{1(sd)}$</td>
<td>0.9815</td>
<td>0.9983</td>
<td>0.4003</td>
<td>0.6929</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>0.9356</td>
<td>0.9791</td>
<td>0.1615</td>
<td>0.3445</td>
</tr>
</tbody>
</table>

The table shows that the power of our test is significantly larger than that of chi-square goodness-of-fit-test.

Thoman et al. (1969) have given power curves of their test for 5 and 10 percentage points for different $n$. We reproduce these powers approximated from the curves, along with the powers of our test for 5% level and give in Table 2.6. The powers of our test are at par those of Thoman et al. (1969) for small samples.

Table 2.6.

Powers of Thoman et al. (1969) test for 5% level.

(The values in the braces show the powers of $Q^*_{1(sd)}$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta/\beta_o = 1.2$</th>
<th>$\beta/\beta_o = 1.4$</th>
<th>$\beta/\beta_o = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.08 (0.10)</td>
<td>0.14 (0.15)</td>
<td>0.38 (0.37)</td>
</tr>
<tr>
<td>10</td>
<td>0.14 (0.13)</td>
<td>0.27 (0.26)</td>
<td>0.76 (0.75)</td>
</tr>
<tr>
<td>20</td>
<td>0.23 (0.21)</td>
<td>0.54 (0.47)</td>
<td>1.00 (0.98)</td>
</tr>
<tr>
<td>30</td>
<td>0.34 (0.27)</td>
<td>0.72 (0.64)</td>
<td>1.00 (1.00)</td>
</tr>
<tr>
<td>50</td>
<td>0.52 (0.37)</td>
<td>0.92 (0.75)</td>
<td>1.00 (1.00)</td>
</tr>
<tr>
<td>70</td>
<td>0.66 (0.49)</td>
<td>1.00 (0.93)</td>
<td>1.00 (1.00)</td>
</tr>
<tr>
<td>90</td>
<td>0.78 (0.58)</td>
<td>1.00 (0.97)</td>
<td>1.00 (1.00)</td>
</tr>
<tr>
<td>120</td>
<td>0.90 (0.69)</td>
<td>1.00 (1.00)</td>
<td>1.00 (1.00)</td>
</tr>
</tbody>
</table>

Bain and Engelhardt (1986) have given a size $\alpha$ test for
testing $H_0: \beta < \beta_0$ against $H_1: \beta > \beta_0$ as: Reject $H_0$, if

$$c r(\beta_0/\hat{\beta})^{1+\beta^2} < \chi^2_{p\alpha(c(r-1))},$$

where, $c=2/[(1+p^2)^2\,c_{22}];\ p=r/n, \ c_{22}=\text{asymptotic variance of } (\hat{\beta}/\beta)$ and $\hat{\beta}$ is the MLE of $\beta$. We compute the powers of the test using the percentiles given in Bain and Engelhardt (1986). Table 2.7 shows the computed powers.

Table 2.7.


<table>
<thead>
<tr>
<th>$\beta/\beta_o$</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.0078</td>
<td>.0498</td>
<td>.0100</td>
<td>.0826</td>
</tr>
<tr>
<td>10</td>
<td>.0148</td>
<td>.0944</td>
<td>.0464</td>
<td>.2104</td>
</tr>
<tr>
<td>20</td>
<td>.0670</td>
<td>.2376</td>
<td>.2322</td>
<td>.5512</td>
</tr>
<tr>
<td>30</td>
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</tr>
<tr>
<td>40</td>
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<td>.3818</td>
<td>.5438</td>
<td>.8256</td>
</tr>
</tbody>
</table>

The powers of our test are better for $n<20$ and are comparable for other cases.

2.3. A TEST BASED ON A TYPE-2 CENSORED SAMPLE.

Many times a situation occurs in which the observations of failures are naturally occurring in order. In this case, it is convenient to terminate the experiment after observing the first $r$ failures from $n$ units. The principal advantage of such censoring,
called Type-2 censoring, is that it may take much less time for the first \( r \) failures of \( n \) items to occur, than for all items in a random sample of size \( r \) to fail. In the following sub-sections we derive the test statistics based on a Type-2 censored sample and study its properties.

2.3.1. Derivation of the test.

Let \( X(1) \leq X(2) \leq \ldots \leq X(r) \), \( r \leq n \), be a Type-2 censored sample of a complete sample of size \( n \), from (2.1.1). Then the joint density of \( X(1) \leq X(2) \leq \ldots \leq X(r) \) is

\[
f_X(x; \theta, \beta) = \frac{n!}{(n-r)!} \frac{\beta^r}{\theta} \prod_{i=1}^{r} x_i^{\beta-1} e^{-\left[ \left( \sum_{i=1}^{r} x_i \right) + (n-r) \frac{x_r}{\theta} \right]}, \quad 0 < x_1 < x_2 < \ldots < x_r < \infty, \quad \theta, \beta > 0.
\]

Making the transformation \( Y(i) = \beta x_i, \) \( \beta > 0 \), our problem of testing reduces to \( H_0: \nu = 1 \) against \( H_1: \nu > 1 \) \( (H_2: \nu < 1), \nu = \beta / \beta_0 \), based on \( (Y(1), Y(2), \ldots, Y(r)) \). The joint pdf of \( Y(1) \leq Y(2) \leq \ldots \leq Y(r) \) is then

\[
f_Y(y; \theta, \nu) = \frac{n!}{(n-r)!} \frac{\nu^r}{\theta} \prod_{i=1}^{r} y_i^{\nu-1} e^{-\left[ \left( \sum_{i=1}^{r} y_i \right) + (n-r) y_r \right]}, \quad 0 < y_1 < y_2 < \ldots < y_r < \infty, \quad \theta, \nu > 0.
\]

For given \( \nu \), \( T = \sum_{i=1}^{r} y_i + (n-r) y_r \) is the complete sufficient statistic for \( \theta \), having the pdf
\[ f_T(t; \theta) = t^{r-1}e^{-t/\theta}/(\theta^r r!), \quad t > 0. \tag{2.3.3} \]

To obtain the joint pdf of \( Y(1), Y(2), \ldots, Y(r-1) \) and \( T \), we make the transformation \( Y_1 = Y_1, Y_2 = Y_2, \ldots, Y_{r-1} = Y_{r-1}, \)
\[ T = \sum_{i=1}^{r-1} Y(i)^\nu + (n-r+1)Y(r)^\nu. \]
Then the Jacobian of transformation is
\[ |J| = [(n-r+1)\nu Y(r)]^{-1}, \]
and hence the joint pdf is
\[ f(y(1), y(2), \ldots, y(r-1), t) = \frac{n!(\nu-r+1)!}{(n-r+1)! (\theta)}^{r-1} \prod_{i=1}^{r-1} y(i)^{\nu-1} e^{-t/\theta}, \]
\[ 0 < y(1)^\nu \leq \ldots \leq y(r-1)^\nu \leq (t - \sum_{i=1}^{r-1} y(i)^\nu)/(n-r+1) < \infty. \]
Therefore, the conditional pdf of \( Y_1, Y_2, \ldots, Y_{r-1} \) given \( T = t \) is
\[ f(y(1), y(2), \ldots, y(r-1) | t) = \frac{n!(r-1)!}{(n-r+1)! (\theta)}^{r-1} \prod_{i=1}^{r-1} y(i)^{\nu-1}, \tag{2.3.4} \]
\[ 0 < y(1)^\nu \leq \ldots \leq y(r-1)^\nu \leq (t - \sum_{i=1}^{r-1} y(i)^\nu)/(n-r+1). \]

The conditional distribution (2.3.4) does not contain the nuisance parameter \( \theta \). Hence, we shall derive the expression for the test statistic \( Q \) defined in (1.1), for testing \( H_0 : \nu = 1 \) against \( H_1 : \nu > 1 \) (\( H_2 : \nu < 1 \)). For this, it is sufficient to consider the data \( Y = (Y(1), Y(2), \ldots, Y(r-1))' \) and the given value \( t \) of \( T \). First we
obtain $\mu_i = E_{H_0}(Y(i)|T=t)$ and $\sigma_{ij} = \text{COV}_{H_0}(Y(i), Y(j)|T=t)$ as follows.

We know that $Z_i = (n-i+1)(Y(i) - Y(i-1))$, $i=1,2,...,r$, $Y(0)=0$, $\nu$ known, are iid exponential random variables with mean $\theta$ and $\Sigma Z_i = T$. On the same line as in Section (2.2.1), the pdf of $Z_i$ given $t$ is

$$f_{Z_i|T}(z_i|t) = (r-1)(1-z_i/t)^{r-2}/t, \ 0<z_i<t.$$  \hfill (2.3.5)

Since $Y(i) = \Sigma_{j=1}^{i} Z_j/(n-j+1)$, under $H_0$

$$\mu_i = E_{H_0}[Y(i)|T=t] = \Sigma_{j=1}^{i} E(Z_j|T=t)/(n-j+1)$$
$$= \left(\frac{t}{r}\right)\Sigma_{j=1}^{i} \frac{1}{(n-j+1)} \hfill (2.3.6)$$

$$\sigma_{ii} = V_{H_0}(Y(i)|T=t)$$
$$= V_{H_0}(\Sigma Z_j|T=t)/(n-j+1)$$

$$= \Sigma_{j=1}^{i} V_{H_0}(Z_j|T=t)/(n-j+1)^2 + \Sigma_{j=1}^{i} \Sigma_{k=1}^{i} \text{COV}_{H_0}(Z_j, Z_k|T=t)$$
$$= \Sigma_{j=1}^{i} V_{H_0}(Z_j|T=t)/(n-j+1)^2 - \Sigma_{j=1}^{i} \Sigma_{k=1}^{i} V_{H_0}(Z_j|T=t)/(r-1)(n-j+1)(n-k+1)$$
\[
\begin{align*}
&= \frac{1}{(r-1)} \sum_{j=1}^{i} \frac{V_{H_0}(Z_j | T=t)}{(n-j+1)^2} - \sum_{j=1}^{i} \sum_{k=1}^{r-1} \frac{V_{H_0}(Z_j | T=t)}{(r-1)(n-j+1)(n-k+1)} \\
&= rv\left(a_i - b_{ii}/r\right)/(r-1) \quad (2.3.7)
\end{align*}
\]

and

\[
\begin{align*}
\sigma_{ii} &= \text{COV}_{H_0}[Y_{(i)}, Y_{(1)} | T=t], \ i \neq 1, \ i < 1 \\
&= \sum_{j=1}^{i} \sum_{k=1}^{r-1} \frac{\text{COV}_{H_0}(Z_j, Z_k | T=t)}{(n-j+1)(n-k+1)} \\
&= \sum_{j=1}^{i} \frac{V_{H_0}(Z_j | T=t)}{(n-j+1)^2} - \sum_{j=1}^{i} \sum_{k=1}^{r-1} \frac{V_{H_0}(Z_j | T=t)}{(r-1)(n-j+1)(n-k+1)} \\
&= \frac{rv}{(r-1)} \left[a_i - b_{ii}/r\right], \quad (2.3.8)
\end{align*}
\]

where

\[
v = V_{H_0}(Z_i | t=t) = \frac{(r-1)t^2}{r^2(r+1)}
\]

\[
a_i = \sum_{j=1}^{i} 1/(n-j+1)^2, \ i = 1, 2, \ldots, r-1
\]

and

\[
b_{ii} = \sum_{j=1}^{i} \sum_{k=1}^{r-1} 1/((n-j+1)(n-k+1)), \ i, l = 1, 2, \ldots, r-1.
\]
Using (2.3.7) and (2.3.8), the variance-covariance matrix $\Sigma_0$ of $(Y|T=t)$, under $H_0$ is given by

$$\Sigma_0 = k[A - \frac{1}{r} B], \tag{2.3.9}$$

where

$$k = rv/(r-1),$$

$$A = \begin{bmatrix}
a_1 & a_1 & a_1 & \cdots & a_1 \\
a_1 & a_2 & a_2 & \cdots & a_2 \\
& \cdots & \cdots & \cdots & \cdots \\
a_1 & a_2 & a_3 & \cdots & a_{r-1}
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
b_{11} & b_{12} & b_{13} & \cdots & b_{1(r-1)} \\
b_{12} & b_{22} & b_{23} & \cdots & b_{2(r-1)} \\
& \cdots & \cdots & \cdots & \cdots \\
b_{1(r-1)} & b_{2(r-1)} & b_{3(r-1)} & \cdots & b_{(r-1)(r-1)}
\end{bmatrix}.$$
\[
A^{-1} = \begin{bmatrix}
2 + (n-1)^2 & -(n-1)^2 & 0 & 0 \\
-(n-1)^2 & (n-1)^2 + (n-2)^2 & ... & 0 & 0 \\
& & & & \\
& & & & \\
0 & 0 & ... & (n-r+3)^2 + (n-r+2)^2 & -(n-r+2)^2 \\
0 & 0 & ... & -(n-r+2)^2 & (n-r+2)^2 \\
\end{bmatrix}
\]

and

\[
D = \begin{bmatrix}
1 & 1 & \ldots & \ldots & (n-r+2) \\
1 & 1 & \ldots & \ldots & (n-r+2) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(n-r+2) & (n-r+2) & \ldots & \ldots & (n-r+2)^2 \\
\end{bmatrix}
\]

Therefore, the expression for the test statistic based on the censored sample \(Y(1), Y(2), \ldots, Y(r-1)\) and the value \(t\) of \(T\) is

\[
\Omega_2 = (Y - \mu_0)' \Sigma_0^{-1}(Y - \mu_0)
\]

\[
= \frac{[\Omega_{21} + \Omega_{22}]}{k}. \quad (2.3.10)
\]

Here,

\[
\Omega_{21} = (Y - \mu_0)' A^{-1}(Y - \mu_0)
\]

\[
= \sum_{j=1}^{r-2} \left( (n-j+1)^2 + (n-j)^2 \right) \left( Y(j) - \mu_j \right)^2
\]

\[
- 2 \sum_{j=1}^{r-2} \left( (n-j)^2 (Y(j) - \mu_j)(Y(j+1) - \mu_{j+1}) \right)
\]
\[ +(n-r+2)^2(Y_{r-1} - \mu_{r-1})^2 \]

\[
= \sum_{j=1}^{r-1} (n-j+1)^2 (Y_j - \mu_j)^2 + \sum_{j=1}^{r-2} (n-j)^2 [(Y_j - \mu_j) - \frac{r-2}{r-1}]^2
\]

\[
= n^2(Y_1 - \mu_1)^2 + \sum_{j=1}^{r-2} (n-j)^2 [(Y_j - Y_{j+1}) - \frac{(\mu_j - \mu_{j+1})}{r-1}]^2
\]

\[
= \sum_{j=0}^{r-2} [(n-j)(Y_{j+1} - Y_j) - t/r]^2
\]

\[
= \sum_{i=1}^{r-1} [(n-i+1)(Y_{i+1} - Y_i) - t/r]^2
\]

and

\[
Q_{22} = (Y - \mu_0)' D (Y - \mu_0)
\]

\[
= \sum_{i=1}^{r-2} (Y_i - \mu_i)^2 + (n-r+2)(Y_{r-1} - \mu_{r-1})^2
\]

\[
= [(n-r+1)(Y_r - Y_{r-1}) - t/r]^2.
\]

Hence,

\[
Q_2 = \frac{r(r+1)^2}{t^2} \sum_{i=1}^r [(n-i+1)(Y_i - Y_{i-1}) - t/r]^2. \tag{2.3.11}
\]

This statistic is again based on the given condition \(T=t\). So, we
modify this statistic by replacing \( t \) by its corresponding random variable \( T \) and propose the new statistic for testing \( H_0: \beta = \beta_0 \) against \( H_1: \beta > \beta_0 \) \( (H_2: \beta < \beta_0) \) based on a Type-2 censored sample \( X(1) \leq X(2) \leq \ldots \leq X(r) \) from (2.3.2) as

\[
Q^*_2 = r(r+1)[ \Sigma_{i=1}^r (n-i+1)^2 (X(i) - X(i-1))^2 / T^2 - 1/r ], \quad (2.3.12)
\]

\[
T = \Sigma_{i=1}^r \frac{X(i) + (n-r)X(r)}{\beta_0}.
\]

To find the exact mean and variance of \( Q^*_2 \), under \( H_0 \), we again consider \( Z_i = (n-i+1)(X(i) - X(i-1)) \), \( i=1,2,\ldots,r \), \( X(0)=0 \). Then we have \( Q^*_2 = r(r+1)[ \Sigma_{i=1}^r Z_i^2 / T^2 - 1/r ] \), \( T = \Sigma_{i=1}^r Z_i \). Following as in Section 2.2.1., we have

\[
E_{H_0}(V) = 2/(r+1) \quad \text{and} \quad E_{H_0}(V^2) = 4(r+5)/[(r+1)^2(r+2)(r+3)],
\]

for \( V = \Sigma_{i=1}^r Z_i^2 / T^2 \). This gives,

\[
E_{H_0}(Q^*_2) = r-1
\]

\[
V_{H_0}(Q^*_2) = 4r^2(r-1)/((r+2)(r+3)).
\]

Hence, the standardized form of \( Q^*_2 \) is
\[ Q^*_2(s_d) = \frac{\Omega^*_2 - E_{H_0}(Q^*_2)}{\sqrt{V_{H_0}(Q^*_2)}} \]

\[ = \left[ \frac{(r+2)(r+3)}{(r-1)} \right]^{1/2} \left[ \frac{\sum_{i=1}^r (n-i-1)^2 (X_{(i)} - X_{(i-1)})^2}{2(\sum_i X_{(i)} + (n-r)X_{(r)})^2} - 1 \right]. \]

To see the direction of the test procedure since, we could not find the expression for \( E_{H_1}(Q^*_2(s_d)) \), we simulate its values for \( n=10, 20, 30 \) and for different values of \( \beta \) under 10% censoring, we present these values in form of curves shown in fig.2.1.

It is seen from the curves that \( E_{H_1}(Q^*_2(s_d)) > E_{H_0}(Q^*_2(s_d)) \) for \( \beta < 1 \). Hence, the test procedure is to reject \( H_0 \) for large values of \( Q^*_2(s_d) \) for \( \beta < 1 \). For \( \beta > 1 \), \( E_{H_1}(Q^*_2(s_d)) < E_{H_0}(Q^*_2(s_d)) \) for some value of \( \beta \) in (1.0, 1.2) and \( E_{H_1}(Q^*_2(s_d)) > E_{H_0}(Q^*_2(s_d)) \) for \( \beta \geq 1.2 \). Thus, we reject \( H_0 \) for large values of \( Q^*_2(s_d) \) when \( \beta \geq 1.2 \).
Fig. 2.1
2.3.2. Simulated percentiles of the distribution on $Q^*_2(sd)$.

We obtain the percentile points of the distribution of $Q^*_2(sd)$ by Monte-Carlo simulation. We generate 5000 random samples of size $n$ from Weibull population with $\theta=\beta=1$ and then construct type-2 censored samples with 10% censoring proportion. Using these samples we simulate the percentiles of the distribution of $Q^*_2(sd)$ and present in the Table 2.8.

Table 2.8.
The percentile points of the distribution of $Q^*_2(sd)$ for 10% censoring proportion.

<table>
<thead>
<tr>
<th>$n$</th>
<th>.01</th>
<th>.05</th>
<th>.95</th>
<th>.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-1.3886</td>
<td>-1.1603</td>
<td>0.9526</td>
<td>2.0692</td>
</tr>
<tr>
<td>20</td>
<td>-1.4992</td>
<td>-1.2160</td>
<td>0.4453</td>
<td>1.2293</td>
</tr>
<tr>
<td>30</td>
<td>-1.5523</td>
<td>-1.2189</td>
<td>0.1367</td>
<td>0.8243</td>
</tr>
<tr>
<td>40</td>
<td>-1.5845</td>
<td>-1.2609</td>
<td>-0.0599</td>
<td>0.6732</td>
</tr>
<tr>
<td>50</td>
<td>-1.6526</td>
<td>-1.3074</td>
<td>-0.2425</td>
<td>0.5336</td>
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<tr>
<td>60</td>
<td>-1.6958</td>
<td>-1.3334</td>
<td>-0.4027</td>
<td>0.4372</td>
</tr>
<tr>
<td>70</td>
<td>-1.7209</td>
<td>-1.3401</td>
<td>-0.5133</td>
<td>0.4297</td>
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<td>80</td>
<td>-1.7328</td>
<td>-1.3450</td>
<td>-0.6157</td>
<td>0.4272</td>
</tr>
<tr>
<td>90</td>
<td>-1.7551</td>
<td>-1.3465</td>
<td>-0.7404</td>
<td>0.4168</td>
</tr>
<tr>
<td>100</td>
<td>-1.7599</td>
<td>-1.3592</td>
<td>-0.7946</td>
<td>0.3800</td>
</tr>
<tr>
<td>120</td>
<td>-1.8096</td>
<td>-1.3911</td>
<td>-0.8553</td>
<td>0.3725</td>
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</table>

2.3.3. A Power study.

We compute the power of the test statistic $Q^*_2(sd)$ for Weibull alternatives through simulation for different values of $n$ and $\beta/\beta_0$, when censoring proportion is 10%. These powers are presented in Table 2.9 and Table 2.10.
Table 2.9.

Powers of $Q^*_2$ test for values of $\beta / \beta_0$.

<table>
<thead>
<tr>
<th>$\beta / \beta_0$</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0.0420</td>
<td>0.1310</td>
<td>0.0548</td>
<td>0.1732</td>
<td>0.1254</td>
<td>0.2960</td>
<td>0.1736</td>
<td>0.3762</td>
</tr>
<tr>
<td>1.4</td>
<td>0.1056</td>
<td>0.2602</td>
<td>0.1768</td>
<td>0.3632</td>
<td>0.4058</td>
<td>0.6362</td>
<td>0.5572</td>
<td>0.7600</td>
</tr>
<tr>
<td>1.8</td>
<td>0.2970</td>
<td>0.6000</td>
<td>0.4906</td>
<td>0.7602</td>
<td>0.8938</td>
<td>0.9754</td>
<td>0.9600</td>
<td>0.9940</td>
</tr>
<tr>
<td>2.0</td>
<td>0.3546</td>
<td>0.7456</td>
<td>0.6022</td>
<td>0.8730</td>
<td>0.9562</td>
<td>0.9956</td>
<td>0.9932</td>
<td>0.9996</td>
</tr>
<tr>
<td>10</td>
<td>0.4026</td>
<td>0.8456</td>
<td>0.6794</td>
<td>0.9468</td>
<td>0.9802</td>
<td>0.9994</td>
<td>0.9980</td>
<td>1.0000</td>
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<tr>
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<td>0.4434</td>
<td>0.8662</td>
<td>0.7650</td>
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<td>1.0000</td>
<td>0.9994</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 2.10.

Power of $Q^*_2$ test for values of $\beta < \beta_0$.

<table>
<thead>
<tr>
<th>$\beta / \beta_0$</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.8854</td>
<td>0.9616</td>
<td>0.5028</td>
<td>0.7132</td>
<td>0.2076</td>
<td>0.4092</td>
<td>0.0818</td>
<td>0.2190</td>
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<tr>
<td>0.4</td>
<td>0.9660</td>
<td>0.9994</td>
<td>0.8436</td>
<td>0.9380</td>
<td>0.4490</td>
<td>0.6558</td>
<td>0.1830</td>
<td>0.3632</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.9530</td>
<td>0.9870</td>
<td>0.6318</td>
<td>0.8136</td>
<td>0.2970</td>
<td>0.5182</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9960</td>
<td>0.9996</td>
<td>0.8126</td>
<td>0.9480</td>
<td>0.3964</td>
<td>0.6922</td>
</tr>
<tr>
<td>10</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9994</td>
<td>0.9998</td>
<td>0.8810</td>
<td>0.9838</td>
<td>0.7644</td>
<td>0.8198</td>
</tr>
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<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9284</td>
<td>0.9970</td>
<td>0.5056</td>
<td>0.8926</td>
</tr>
</tbody>
</table>

2.4. A MODIFIED TEST STATISTIC BASED ON CENSORED SAMPLE.

We see that, the test statistic $Q^*_2$ obtained in section 2.3 is not a general form of $Q^*_1$. That is, for $r=n$, $Q^*_2$ does not reduce to $Q^*_1$. We suggest in this section a generalized form $Q^*_3$ of $Q^*_1$, namely

$$Q^*_3 = (n+1) \left[ \frac{n \{ \sum_{i=1}^{r} X(i) + (n-r) X(r) \}^{2 \beta_0}}{\sum_{i=1}^{r} \left( \frac{\beta_0}{\beta_0 + (n-r) X(r)} \right)^{2 \beta_0}} \right] - 1. \quad (2.4.1)$$

42
This can also be written in alternative form as

\[ Q_3^* = (n+1) \left[ \frac{\sum_{i=1}^{r} (n-i+1)(X(i)-X(i-1))}{\sum_{j=1}^{r} (n-r)/(n-j+1)} \right] - 1 \]

Using (2.3.6) and (2.3.7), we have

\[ E_{H_0} [X(i) | T=t] = E_{H_0} [Y_i | T=t] \]

\[ = \sigma_{ii} + \mu_i^2 \]

\[ = \frac{t^2}{r(r+1)} \left[ \sum_{i=1}^{n} 1/(n-j+1)^2 + (\sum_{j=1}^{i} 1/(n-j+1))^2 \right]. \]

Therefore, after algebraic simplifications, we have

\[ E_{H_0} \left[ \sum_{i=1}^{r} (n-i+1)(X(i)-X(i-1)) | T=t \right] = \frac{2t^2}{r(r+1)} \sum_{i=1}^{r} \sum_{j=1}^{i} 1/(n-j+1) \]

\[ = \frac{2t^2}{r(r+1)} \left[ r - \sum_{j=1}^{r} (n-r)/(n-j+1) \right] \]

Hence,

\[ E_{H_0} (Q_3^*) = EE(Q_3^* | T=t) \]

\[ = (n+1) \left[ \frac{2n}{r(r+1)} \sum_{j=1}^{r-j+1} \frac{r-j+1}{n-j+1} - 1 \right]. \]

(2.4.2)

Since the exact distribution of \( Q_3^* \) as well as the expression
for $E_{H_1}(Q_3^*)$ have not been mathematically tractable, we simulate $E_{H_1}(Q_3^*)$ for different values of $\beta/\beta_0$ and $n$, which suggest the direction for the test procedure.

The table shows that $E_{H_1}(Q_3^*) > E_{H_0}(Q_3^*)$ for $0 < \beta < \beta_0$ and $E_{H_1}(Q_3^*) \leq E_{H_0}(Q_3^*)$ for $\beta > \beta_0$. Hence the test procedure is to reject $H_0: \beta = \beta_0$ against $H_1: \beta > \beta_0$ ($H_2: \beta < \beta_0$) for small (large) values of $Q_3^*$.

The simulated percentile points of the distribution of $Q_3^*$ for 10% censoring proportion are given in Table 2.12.

### Table 2.11.
The values of $E_{H_1}(Q_3^*)$ for 10% censoring

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta/\beta_0=.4$</th>
<th>$\beta/\beta_0=.6$</th>
<th>$\beta/\beta_0=1.0$</th>
<th>$\beta/\beta_0=1.2$</th>
<th>$\beta/\beta_0=1.4$</th>
<th>$\beta/\beta_0=1.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>17.93</td>
<td>8.46</td>
<td>6.25</td>
<td>4.83</td>
<td>3.86</td>
<td>3.18</td>
</tr>
<tr>
<td>20</td>
<td>25.86</td>
<td>17.69</td>
<td>12.95</td>
<td>9.98</td>
<td>7.82</td>
<td>6.39</td>
</tr>
</tbody>
</table>

### Table 2.12.
The percentile points of the distribution of $Q_3^*$ for 10% censoring proportion.

<table>
<thead>
<tr>
<th>$n$</th>
<th>.01</th>
<th>.05</th>
<th>.95</th>
<th>.99</th>
</tr>
</thead>
<tbody>
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<td>10</td>
<td>1.6391</td>
<td>2.4914</td>
<td>11.6207</td>
<td>15.3972</td>
</tr>
<tr>
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<td>5.2679</td>
<td>7.0314</td>
<td>20.6826</td>
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<td>30</td>
<td>9.9169</td>
<td>12.1463</td>
<td>29.1607</td>
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<tr>
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<td>14.5478</td>
<td>17.1806</td>
<td>37.4102</td>
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<tr>
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<td>29.6824</td>
<td>33.5736</td>
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<tr>
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<td>35.3202</td>
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<td>61.9655</td>
<td>96.7872</td>
<td>105.9404</td>
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</table>
The powers for different values of $n$ and $\beta/\beta_0$ from Weibull alternatives are simulated and are shown in Table 2.13 and 2.14.

### Table 2.13.

<table>
<thead>
<tr>
<th>$\beta/\beta_0$</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
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### Table 2.14.

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<th>1%</th>
<th>5%</th>
<th>1%</th>
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<td>$\beta/\beta_0 = .6$</td>
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<td>$\beta/\beta_0 = .8$</td>
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<tr>
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<td>1%</td>
<td>5%</td>
<td>1%</td>
<td>5%</td>
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