CHAPTER 1

INTRODUCTION

In statistical inference there are two main problems. Estimation and Testing of hypotheses. In the problem of testing statistical hypotheses a family of probability distributions

\[ \mathcal{F}_\theta = \{ f_X(x; \theta); \theta = (\theta_1, \theta_2, \ldots, \theta_k) \in \Omega \subseteq \mathbb{R}^k \} \]

is assumed for the random phenomenon \( X \) under study. The most common statistical hypotheses are assertions about the unknown parameters in the model \( \mathcal{F}_\theta \). Let \( \Omega_0, \Omega_1 \) be two disjoint subsets of \( \Omega \). Then two hypotheses are framed regarding the parameter \( \theta \). First one is the null hypothesis, \( H_0: \theta \in \Omega_0 \), which is doubted for its suitability. Other, which may be accepted against the rejection of the null hypothesis is termed alternative hypothesis, \( H_1: \theta \in \Omega_1 \). A statistical hypothesis is called simple if it completely specifies the parameters and is called composite if it does not specify completely at least one of the parameters. Usually a statistical test is a partition of sample space \( S \) of random sample \( X=(X_1, X_2, \ldots, X_n) \) into two parts viz. rejection region (critical region), \( C \) and acceptance region \( S-C \). If \( x=x \in C \) then we reject the null hypothesis \( H_0 \) with probability one, otherwise it is accepted. In this process, one may commit two types of errors. That is, reject \( H_0 \) when it is true (Type I error) and accept \( H_0 \) when it
is not true (Type II error). Suppose we consider simple null hypothesis $H_0: \theta = \theta_0$ against the simple alternative $H_1: \theta = \theta_1$, then the probabilities of type I and type II errors are denoted by

$$P(\text{Type I error}) = P(\text{reject } H_0 | \theta = \theta_0)$$

$$= P_{\theta_0} (x \in C)$$

$$= \alpha$$

and

$$P(\text{Type II error}) = P(\text{accept } H_0 | \theta = \theta_1)$$

$$= 1 - P_{\theta_1} (x \in C)$$

$$= \beta.$$ 

It is desirable to choose $C$ such that $\alpha$ and $\beta$ are small. Here $\alpha$ is called the size of the test and $1 - \beta$ is called the power of the test. Actually, it is a common practice to set up the experiment so that the most costly or most dangerous error is associated with the type I error. Since $\alpha$ and $\beta$ cannot be minimized simultaneously, we fix a desirable small value of $\alpha$ and then consider different critical regions of size $\alpha$ and choose the one which gives the smallest value of $\beta$. In case the upper limit on $P(\text{Type I error})$ is fixed then it is called the level of significance.

If we consider composite hypothesis $H_0: \theta \in \Theta_0$, then
P(\text{Type I error}) = P_{\Theta_0}(x \notin C),

where $\Theta_0$ is the true value of $\Theta$ contained in $\Omega_0$. To have the probability of Type I error is less than the prescribed level $\alpha$, it is necessary to find a critical region such that

$$P_{\Theta}(x \in C) \leq \alpha \text{ for all } \Theta \in \Omega_0.$$ 

Thus, for a composite hypothesis,

$$\sup_{\Theta \in \Omega_0} P_{\Theta}(x \in C) \leq \alpha.$$

Here the lefthand side of the inequality is called the size of the test. As in earlier case, we choose that critical region, which has pre-determined level $\alpha$ and smallest $\beta$ or largest power.

A test based on $C^*$ for testing $H_0: \Theta = \Theta_0$ against $H_1: \Theta = \Theta_1$ is said to be a most powerful (MP) test of level $\alpha$ if

(i) $P_{\Theta_0}(x \in C^*) = \alpha$

(ii) $P_{\Theta_0}(x \in C^*) \leq P_{\Theta_1}(x \in C)$
for any other critical region \( C \) of size \( \leq \alpha \). The critical region \( C^* \) is called the best critical region (BCR). It is possible to find the CR, which yields the MP test, using Neymann Pearson Lemma. For this, consider a likelihood function \( f(x_1, x_2, \ldots, x_n; \theta) \) and the ratio

\[
\lambda(x; \theta_0, \theta_1) = \frac{f(x_1, x_2, \ldots, x_n; \theta_1)}{f(x_1, x_2, \ldots, x_n; \theta_0)}.
\]

Suppose the critical region \( C^* \) is the set

\[
C^* = \{x|\lambda(x; \theta_0, \theta_1) \leq k^*\}
\]

where \( k^* \) is a constant such that

\[
P_{\theta_0} [\lambda(x; \theta_0, \theta_1) \leq k^*] = P_{\theta_0} [x \in C^*] = \alpha.
\]

Then \( C^* \) is a BCR and the test based on \( C^* \) is a MP level \( \alpha \) test. It is necessary to know the distribution of \( \lambda \), to find the value of \( k^* \).

If we consider a test for a composite null hypothesis \( H_0: \theta \in \Omega_0 \) against \( H_1: \theta \in (\Omega - \Omega_0) \), then the test based on a critical region \( C \) is to be obtained which has size \( \leq \alpha \) for all \( \theta \in \Omega_0 \). Such a test may be the most powerful for some \( \theta \) in the alternative \( (\Omega - \Omega_0) \) and may not
be most powerful for other values. If the test is most powerful for all \( \theta \in (\Omega - \Omega_0) \), then the test is said to be uniformly most powerful (UMP) test. Thus a test for \( H_0: \theta = \theta_0 \) against \( H_1: \theta \in (\Omega - \Omega_0) \) is said to be UMP test of level \( \alpha \) if

\[
\text{(i) } \sup_{\theta \in \Omega_0} P_\theta (x \in C^*) = \alpha
\]

\[
\text{(ii) } P_\theta (x \in C^*) \geq P_\theta (x \in C)
\]

for all \( \theta \in (\Omega - \Omega_0) \) and for all critical region \( C \) of size \( \leq \alpha \). If \( \mathcal{F}_\theta \) possesses Monotone Likelihood Ratio (MLR) property, then it is possible to get tests that are uniformly most powerful for some hypotheses. \( f(x, \theta) \) is said to have monotone likelihood ratio if there exists a function \( T(x) \) such that for any \( \theta < \theta' \), \( f(x, \theta') / f(x, \theta) \) is a nondecreasing function of \( T(x) \). It is often possible to determine uniformly most powerful tests of one-sided alternatives of the type \( H_0: \theta \leq \theta_0 \) against \( H_1: \theta > \theta_0 \), but it may not be possible to obtain a uniformly most powerful test for a two-sided alternative, \( H_0: \theta = \theta_0 \) against \( H_1: \theta \neq \theta_0 \). In this case, the critical region chosen to reject \( H_0 \) in favour of \( H_1: \theta > \theta_0 \), may not cause the rejection in favour of \( H_1': \theta < \theta_0 \). Hence an ideal critical region is to be selected for the rejection of \( H_0 \) in favour of both the hypotheses \( H_1 \) and \( H_1' \) which gives reasonably good power. It may sometimes be possible to restrict the class of tests considered using a restriction somewhat analogous to unbiasedness in
estimation and obtain a UMP test within the restricted class, called uniformly most powerful unbiased (UMPU) tests.

A test for \( H_0: \theta \in \Omega_0 \) against \( H_1: \theta \in \Omega - \Omega_0 \) is said to be unbiased if the probability of rejecting \( H_0 \) when it is false is at least as large as the probability of rejecting \( H_0 \) when it is true. That is,

\[
P_\theta(x \in C) \leq \alpha, \quad \text{when } \theta \in \Omega_0
\]

\[
\geq \alpha, \quad \text{when } \theta \in (\Omega - \Omega_0).
\]

If within the class of unbiased tests a UMP test exists, then this test is called UMPU test.

Consider a joint density function \( f(x; \theta), \theta \in \Omega \), and the hypothesis \( H_0: \theta \in \Omega_0 \) versus \( H_1: \theta \in (\Omega - \Omega_0) \). The likelihood ratio test principle is to reject \( H_0 \) if \( \lambda(x) \leq k \), where \( \lambda(x) \) is the generalized likelihood ratio.

\[
\lambda(x) = \frac{\sup_{\theta \in \Omega_0} f(x_1, x_2, \ldots, x_n; \theta)}{\sup_{\theta \in \Omega} f(x_1, x_2, \ldots, x_n; \theta)}
\]

\[
= \frac{f(x_1, x_2, \ldots, x_n; \theta_0)}{f(x_1, x_2, \ldots, x_n; \theta)}
\]
where \( \hat{\theta} \) denotes the MLE of \( \theta \) and \( \hat{\theta}_0 \) denotes maximum likelihood estimator under the restriction that \( \theta \in \Theta_0 \). The constant \( k \) would be chosen such that \( P[X(x) \leq k] = \alpha \).

If the above procedures are not suitable, then it may be possible to choose some other statistics on which tests may be based. If a sufficient statistic exists, then it would be best to consider a test statistic which is a function of the sufficient statistic.

Much of the work on alternative test procedures is done by using three approaches: one approach is to derive methods based on Maximum Likelihood Estimators (MLE's). In this case if MLE's are not in closed form then the corresponding test statistic for the test is also not in closed form. As a result the problem of deriving the distribution of the test statistic is not mathematically tractable. A second approach is to consider methods based on best linear estimators. This approach also has difficulties somewhat similar to those for MLE's. A third approach is to find simple alternative tests which can be verified to have good properties. In this thesis, we derive tests for some lifetime distributions based on the last approach.

In \( \mathcal{F}_\Theta \), if \( k > 1 \), then we may not be interested in all \( k \) parameters. Let the hypotheses specify the values of \( \ell \) parameters.
of interest, $\theta = (\theta_1, \theta_2, \ldots, \theta_i), \ell \leq k$. Then remaining parameters
$\theta_n = (\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_m}), \ell + m = k$, are called the nuisance parameters.

Suppose there exists a sufficient statistic $T(X)$ for $\theta_n$ for the
known values of $\theta_i$. Then the conditional distribution of $X$ given
$T=t$ does not depend on $\theta_n$. Therefore, we may consider the data $x$
have come from the conditional distribution $f_{\theta_n}(x|T=t)$ instead of 
$f_{\theta}$ and derive tests, called conditional tests. In order to
construct a conditional test, we consider the quadratic form

$$Q = (X-\mu_0)' \Sigma_0^{-1} (X-\mu_0), \quad (1.1)$$

where $\mu_0=(\mu_1, \mu_2, \ldots, \mu_n)'$ is the mean of $X|t$ and $\Sigma_0=((\sigma_{ij}))$ is the
variance-covariance matrix of $X|t$, computed under the null hypothesis.

In this thesis we obtain tests based on $Q$ for the parameters of Weibull, Gamma and Mixed Failure Time families $f_{\theta}$ of
distributions. We also obtain the tests for Poisson process against Weibull process. The tests are generalized for any
two-parameter family of distributions. The simulated percentiles
and powers of the proposed tests are given in the respective
chapters. The whole study done in the thesis is incorporated in
seven chapters.
In Chapter 2 we consider the lifetime model as Weibull with the pdf

\[ f_X(x; \theta, \beta) = (\beta/\theta)x^{\beta-1}e^{-x^\beta/\theta}, \quad \beta > 0, \theta > 0, x > 0. \]

This model is quite flexible and has the advantage of having a closed form of cdf. Thoman et al. (1969) have considered the problem of testing of hypotheses regarding the parameter \( \beta \) based on MLE. Mann et al. (1973) have studied a goodness-of-fit test for the Weibull distribution. Bain and Engelhardt (1986) have proposed a modified version of Thoman et al. (1969) test statistic whose asymptotic distribution is approximated to a chi-squared distribution. In this chapter we propose conditional tests for \( H_0: \beta = \beta_0 \) against \( H_1: \beta > \beta_0 (H_2: \beta < \beta_0) \) treating \( \theta \) as the nuisance parameter. We obtain and study the tests

\[
Q^*_1(sd) = [(n+2)(n+3)/(n-1)]^{1/2}\left\{ \frac{n}{(n+1)} \sum_{i=1}^{n} \frac{2\beta_0}{2(\sum X_i^\beta)_{i=1}^{n}} - 1 \right\}
\]

based on a complete sample and

\[
Q^*_2(sd) = \left[ \frac{(r+2)(r+3)}{(r-1)} \right]^{1/2}\left\{ \frac{r}{(r+1)} \sum_{i=1}^{r} \frac{(n-i+1)\beta_0}{2\beta_0} \left( X_{(i)} - X_{(i-1)} \right)^2 - 1 \right\}
\]

based on a type 2 censored sample. We also propose a modified form
Chapter 3 deals with the Gamma distribution having pdf

\[ f_X(x; \theta, \beta) = x^{\beta-1} e^{-x/\theta} / (\theta^\beta \Gamma(\beta)), \theta > 0, \beta > 0, x > 0. \]

Using \( W = \bar{X} / \tilde{X} \) or \( S = \ln(\bar{X} / \tilde{X}) \), where \( \bar{X} \) is the sample mean and \( \tilde{X} \) is the geometric mean, several authors such as Bain and Engelhardt (1975), Engelhardt and Bain (1977), Lawless (1982) etc. have suggested tests for hypotheses about the shape parameter \( \beta \). Recently Keating et al. (1990) have simplified the expression for the density of \( W \) and provided the tables for the critical values useful for testing the hypotheses about \( \beta \). Here, we obtain the conditional tests

\[
Q_3^* = (n+1) \left[ \frac{\sum_{i=1}^{r} (n-i+1)(X_{(i)} - X_{(i-1)})}{\sum_{i=1}^{r} (n-i+1)(X_{(i)})^2} \right]^{2/3} - 1
\]

using \( X_i^* \), \( i=1,2,\ldots,n \) and

\[
Q_5^* = \left( \frac{6}{\pi^2} \right) \left[ \frac{\sum_{i=1}^{n} (\ln(X_i / \bar{X}) + S_{n-1})^2}{n \sum_{i=1}^{n} (\ln(X_i / \bar{X}) + S_{n-1})^2} \right] - nb_n \left( \sum_{i=1}^{n} (\ln(X_i / \bar{X}) + S_{n-1})^2 \right).
\]

where

\[
S_{n-1} = \sum_{j=1}^{n-1} \frac{1}{j}, \quad b_n = \frac{(S_{n-1} - n^2 / 6)}{n(S_{n-1} - n^2 / 6 + \pi^2 / 6)}; \quad S_{n-1}^* = \frac{\sum_{i=1}^{n} 1}{j^2}
\]
using $\ln(X_i), i=1,2,\ldots,n$ for testing $H_0: \beta=1$ against $H_1: \beta>1$ ($H_2: \beta<1$). Computing $y=-\ln(F_\theta(x;\beta_0))$ for given $x$ and $\theta=1$ and using $Y_i$'s instead of $X_i$'s in $Q_4^{*}(sd)$, one can test $H_0: \beta=\beta_0$ against $H_1: \beta>\beta_0 (H_2: \beta<\beta_0)$.

A Weibull process is defined by the intensity function
\[
\nu(x) = (\beta/\theta)x^{\beta-1}, \beta>0, \theta>0
\]
and the mean value function
\[
m(x) = x^\beta/\theta, \beta>0, \theta>0.
\]
For $\beta=1$, the process reduces to an homogeneous Poisson process. Cox(1955), Bartholomew(1956), Ascher and Feingold(1978), Crow(1974,1982), Bain and Engelhardt(1980) and Bain et.al.(1985) have discussed tests for $H_0: \nu(x)$ is constant versus $H_1: \nu(x)$ is increasing(decreasing) based on time truncated data. Bain and Engelhardt(1991) have given a test for $H_0: \beta\leq\beta_0$ versus $H_1: \beta>\beta_0$ using the MLE of $\beta$ based on a failure truncated data from a Weibull process. In Chapter 4, we propose a test
\[
Q_6^{*} = 2n(n+1)\sum_{i=1}^{n-1} W_i(W_i - W_{i-1}) + n^2 - 1,
\]
where $W_i = X_i/X_n$, for testing $H_0: \nu(x)$ is constant against $H_1: \nu(x)$ is increasing ($H_2: \nu(x)$ is decreasing) based on the first $n$ successive times of occurrences from the Weibull process. Another test
\[
Q_7^{*} = \sum_{j=1}^{n-1} [1-(n-j)(\ln X_{n-j+1} - \ln X_{n-j})]^2
\]

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based on logarithmic successive times of occurrences is also proposed. A modified form

\[ Q^*_n = \sum_{i=1}^{n-1} (U_i - \bar{U})^2, \]

where \( U_i = (n-i)(V_i - V_{i-1}); \ V_i = \ln Z_i - \ln Z_{i-1}; \ Z_i = m(X_i), i = 1, 2, \ldots, n \) and \( \bar{U} = \frac{\sum_{i=1}^{n} U_i}{(n-1)} \) of the test statistic \( Q^*_n \) is studied. The last section deals with the power study for a broad range of intensities such as logarithmic and exponential intensities.

In practice, many times it happens that when we put units in a life testing experiment, then some of the units fail instantaneously and thereafter the life time of units follow a distribution such as a Weibull. Similarly, in clinical trials it happens that initially a drug have no response with probability \( q \) (0 ≤ q < 1), but once there is a response, length of response follow an Exponential, Weibull or Uniform distribution. Gaver and Lewis (1980), Adke (1988), Jayade and Prasad (1990), Adke and Balkrishna (1992), Jayade (1993) have considered the distribution

\[ F(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-x^\beta/\theta\right) & x \geq 0, 0 < p \leq 1, \theta > 0, \beta > 0 \end{cases} \]

for \( \beta = 1 \) in connections with first order exponential autoregressive process. In Chapter 5, we propose the conditional tests

\[ Q^*_q = (n-1)(r+1) \left[ n \sum_{i=1}^{\beta_0} X_i^2 / \left( \sum_{i=1}^{\beta_0} X_i^2 - 1 \right) \right] / (2n-r-1) \]
and

\[ Q_{10}^* = (n^2-1)p \left[ \frac{\sum_{i=1}^n X_i^2}{(\sum_{i=1}^n X_i^2 - n)} \right] / a^*, \]

for shape parameter \( \beta \), respectively when \( p \) is unknown and \( p \) is known. Here \( P=P(R \geq 1), a^* = 2n + (n-1)(2n+1)p - 2np \). We also obtain the test

\[ Q_{11}^* = \frac{[n(n-1) \sum_{i=1}^n (X_i - r/\theta^2/n)^2 + (n-r) \sum_{i=1}^n (X_i - r/\theta^2/n)^2]}{(r(2n-r-1)\theta^2)} \]

for the scale parameter \( \theta \) for known \( \beta \).

In Chapter 6, we generalize the procedure used in early chapters for the case of two parameter family of distributions. The expression for the test statistic is obtained when \( T=\sum_{i=1}^n d(X_i) \) is sufficient for the nuisance parameter, as

\[ Q_g^* = \frac{1}{n^p(T)} \left[ \sum_{i=1}^n d^2(X_i) - \frac{\sum_{i=1}^n d(X_i)}{n} \right], \]

where \( d \) is one-to-one function of \( X_i \). The expressions for \( Q_g^* \) are obtained for other distributions such as Normal, Poisson etc.

The conditional distribution of the observations given the sufficient statistic does not depend upon the nuisance scale parameter. This has motivated us to construct a test based on Neymann-Pearson approach under the conditional setup. Chapter 7 consists of such conditional tests for the shape parameters in

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Weibull and Gamma distributions and in the initial distribution of the Weibull process. Though the tests are obtained for simple hypotheses it is observed that the test can also be used for composite alternatives.

Some of the results from the thesis are either published or in the process of publication after revision.

We provide, in Appendix, the computer programmes in BASIC for the simulations of percentiles and powers of the tests studied in Chapter 2 through Chapter 6.

At the end we give an extensive Bibliography.