CHAPTER 7
OTHER CONDITIONAL TESTS BASED ON NEYMAN-PEARSON APPROACH

7.1. INTRODUCTION.

In chapter 2 through chapter 5, we have obtained some conditional tests based on quadratic forms for different distributions such as Weibull, Gamma, mixture of degenerated (degenerated at zero) and Weibull distribution. And in chapter 6, we have generalized the method for any density $f(x; \theta, \beta)$. Lehmann (1976) discusses UMP unbiased tests for multiparameter exponential families using conditional distributions. In this chapter we obtain tests for Weibull distribution, Gamma distribution and for Weibull process based on Neyman-Pearson approach under the conditional set up. It is seen that some tests are uniformly most powerful tests under the unconditional set up also.

7.2. CONDITIONAL TEST FOR THE SHAPE PARAMETER IN THE WEIBULL DISTRIBUTION.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Weibull distribution defined by the pdf

$$ f_X(x; \theta, \beta) = (\beta/\theta)x^{\beta-1}e^{-x^\beta/\theta}, \quad x > 0, \theta, \beta > 0. \quad (7.2.1) $$

Then the joint pdf of $(X_1, X_2, \ldots, X_n)$ is
\[ f_X(x; \theta, \beta) = (\beta/\theta)^n \prod_{i=1}^{n} x_i^{\beta - 1} e^{-\sum_{i=1}^{n} x_i^{\beta}/\theta}. \] (7.2.2)

For known \( \beta \), \( T_\beta = \sum_{i=1}^{n} x_i^{\beta} \) is the sufficient statistic for \( \theta \). Now, given \( (x_1, x_2, \ldots, x_n) \), we can compute \( t_\beta = \sum_{i=1}^{n} x_i^{\beta} \) for \( \beta = \beta_0 \) and \( \beta = \beta_1 \). Then the conditional distribution of \( X = (X_1, X_2, \ldots, X_{n-1}) \) given \( T_\beta = t_\beta \) is

\[ f(x_1, x_2, \ldots, x_{n-1} | t_\beta) = \Gamma n \beta^{n-1} \prod_{i=1}^{n-1} x_i^{\beta - 1}/t_\beta^{n-1}. \] (7.2.3)

\[ \sum_{i=1}^{n} x_i^{\beta} \leq t_\beta, \quad 0 < x_i < \infty, \] which is free from the parameter \( \theta \). Therefore, we shall consider the problem of testing \( H_0: \beta = \beta_0 \) against \( H_1: \beta = \beta_1 > \beta_0 \) \( (H_2: \beta = \beta_1 < \beta_0) \) treating that the observations have come from (7.2.3). In this conditional set up, using Neymann-Pearson approach, we have

\[ \lambda = \frac{f_{H_1}(x_1, x_2, \ldots, x_{n-1} | t_\beta)}{f_{H_0}(x_1, x_2, \ldots, x_{n-1} | t_\beta)} \]

\[ = \left( \frac{\beta_1 t_\beta}{\beta_0 t_\beta} \right)^{n-1} \prod_{i=1}^{n-1} x_i^{\beta_1 - \beta_0}. \] (7.2.4)

We reject \( H_0 \) in favour of \( H_1 \) if \( \lambda \geq c_\alpha \), where \( c_\alpha \) is the \( \alpha \)th percentile of the distribution of \( \lambda \) under \( H_0 \). Now, \( \lambda \geq c_\alpha \) only if
\[ n \sum_{i=1}^{n-1} \ln(x_i) \geq c_\alpha(t_{\beta_0}) \cdot \]

Note that here \((x_i, i=1,2,...,n-1)\) is a realization from (7.2.3) under \(H_0\). \(c_\alpha(t_{\beta_0})\) is to be obtained using the size condition

\[ \frac{n-1}{\sum_{i=1}^{n-1} \ln(x_i)} \geq c_\alpha(t_{\beta_0}) | t_{\beta_0} \alpha, \]

for given value of \(\alpha\). Using the transformation,

\[ W_i = \left(\frac{t_{\beta_0}}{t_{\beta_0}}\right) \cdot i=1,2,...,n-1. \]

we have

\[ \frac{n-1}{\sum_{i=1}^{n-1} \ln(W_i)} \leq c_\alpha^*(t_{\beta_0}) | t_{\beta_0} \alpha, \]

where \(c_\alpha^*(t_{\beta_0}) = \beta_0 c_\alpha(t_{\beta_0}) - \ln(t_{\beta_0})\). The joint pdf of \(W_1,W_2,...,W_{n-1}\) under \(H_0\), is

\[ f(w_1,w_2,...,w_{n-1} | t_{\beta_0}) = (n-1)!, \]

\[ 0 < \sum_{i=1}^{n-1} w_i \leq 1, \quad 0 < w_i \leq 1, \quad i=1,2,...,n-1. \]

It can be seen that the distribution given in (7.2.6) does not depend on \(t_{\beta_0}\). This fact helps us to obtain the \(\alpha\)-th percentiles of the distribution of

\[ -\sum_{i=1}^{n} \ln(W_i) \] in (7.2.5). In practice, it is difficult to get the
exact distribution of \( -\sum_{i=1}^{n-1} \ln(W_i) \), hence we obtain the \( \alpha \)-percentiles through simulation. For this, we derive the conditional distribution of \( W_i \) given \( (w_1, w_2, \ldots, w_{n-1}) \).

The joint pdf of \( W_1, W_2, \ldots, W_i \ i \leq n-1 \), is given by

\[
f(w_1, w_2, \ldots, w_i)
\]

\[
= (n-1)! \int_{0}^{1} \ldots \int_{0}^{1} dw_i \ldots dw_i+1
\]

\[
= (n-1)! \int_{0}^{1} \ldots \int_{0}^{1} (1 - \sum w_j) dw_i \ldots dw_i+1
\]

\[
= (n-1)! \int_{0}^{1} \ldots \int_{0}^{1} (1 - \sum w_j)^2 dw_i \ldots dw_i+1.
\]

Successively evaluating the remaining integrals we get, finally,

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\begin{equation}
  f(w_1, w_2, \ldots, w_i) = \frac{(n-1)!}{(n-1-i)!} \left(1 - \sum_{j=1}^{i} w_j\right)^{n-i-1}, \quad (7.2.7)
\end{equation}

\(0 < \sum_{j=1}^{i} w_j < 1, 0 < w_j < 1, j=1,2,\ldots,i.\) The marginal pdf of \(W_i\) is given by

\begin{equation}
  f(w_i) = (n-1) (1-w_i)^{n-2}, 0<w_i<1. \quad (7.2.8)
\end{equation}

Then the conditional pdf of \(W_i\) given \(w_1, w_2, \ldots, w_i-1\) is

\begin{equation}
  f(w_i|w_1, w_2, \ldots, w_{i-1}) = \frac{(n-i)!}{w_{i-1}^{*}} \left(1 - w_i/w_{i-1}^{*}\right)^{n-i-1}, \quad (7.2.9)
\end{equation}

\(0 < w_i < w_{i-1}^{*}\), where \(w_{i-1}^{*} = 1 - \sum_{j=1}^{i-1} w_j\). Now we generate 5000 values of \(W = (W_1, W_2, \ldots, W_{n-1})\) using (7.2.8) and (7.2.9) and provide the simulated percentiles in Table 7.1.

\begin{table}[h]
\centering
\begin{tabular}{c|cccc}
\hline
\(\alpha\) & .01 & .05 & .95 & .99 \\
\hline
\(n\) &  &  &  &  \\
10 & 21.2774 & 22.0891 & 30.0826 & 33.0041 \\
20 & 60.5778 & 62.2331 & 73.9863 & 77.2279 \\
30 & 106.5365 & 108.2875 & 122.6585 & 126.6168 \\
40 & 155.6249 & 158.0954 & 174.9125 & 178.9849 \\
50 & 208.0253 & 210.9621 & 229.2118 & 234.2231 \\
60 & 262.2556 & 265.4741 & 285.7267 & 290.2925 \\
70 & 318.7497 & 322.0091 & 343.8069 & 349.8369 \\
80 & 376.0086 & 379.7300 & 403.7801 & 409.7236 \\
90 & 435.3227 & 439.6501 & 464.4130 & 470.0324 \\
100 & 495.5522 & 499.9197 & 526.4383 & 532.8998 \\
120 & 619.3477 & 624.4763 & 653.1782 & 660.0628 \\
\hline
\end{tabular}
\caption{Percentile points of \(-\sum_{i=1}^{n-1} \ln(W_i)\).}
\end{table}
Since $c^\alpha$ in (7.2.5) does not depended upon the value $\beta_1$ of $\beta$, except that $\beta_1 > \beta_0$, the test remains the same for $H_0: \beta = \beta_0$ against $H_1: \beta > \beta_0$. Hence, under the conditional set up, the test is uniformly most powerful.

By replacing $t_{\beta_1}$ by $T_{\beta_1} = \frac{\sum X_i}{\sqrt{n}}$ and $t_{\beta_0}$ by $T_{\beta_0} = \frac{\sum X_i}{\sqrt{n}}$ in (7.2.4), we get an alternative test procedure, which rejects $H_0$ when

$$
\lambda^* = \frac{(n-1) (\beta_1 - \beta_0)^{1/(n-1)}}{\sum X_i / \sum X_i} > c_{\alpha}^{**},
$$

where $c_{\alpha}^{**}$ is the $\alpha$th percentile of the null distribution of the test statistic. The percentiles of the null distribution of the test statistic are obtained through simulation. The simulated percentiles for $(\beta_1/\beta_0) = 1.2(2.0)2.0$ and $3.0$ are give in Table 7.2.

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<th>$\beta_1/\beta_0$</th>
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<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>3.0</th>
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<td>.05</td>
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<td>.05</td>
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<td>.450</td>
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<td>.359</td>
<td>.108</td>
<td>.078</td>
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</table>
The Monte Carlo power study for the test based on \( \lambda^* \) has been carried out for different values of \((\beta_1/\beta_0)\) and the results are provided in Table 7.3.

### Table 7.3.
**Power of \( \lambda^* \) test for different value of \((\beta_1/\beta_0)\) and \(\alpha=0.05\)**

<table>
<thead>
<tr>
<th>(\beta_1/\beta_0)</th>
<th>(n=10)</th>
<th>(n=20)</th>
<th>(n=30)</th>
<th>(n=40)</th>
<th>(n=50)</th>
<th>(n=80)</th>
<th>(n=100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.2)</td>
<td>0.1284</td>
<td>0.2138</td>
<td>0.3064</td>
<td>0.3992</td>
<td>0.4722</td>
<td>0.6634</td>
<td>0.7212</td>
</tr>
<tr>
<td>(1.4)</td>
<td>0.2622</td>
<td>0.5116</td>
<td>0.7088</td>
<td>0.8436</td>
<td>0.9498</td>
<td>0.9892</td>
<td>0.9966</td>
</tr>
<tr>
<td>(1.6)</td>
<td>0.4326</td>
<td>0.7700</td>
<td>0.9304</td>
<td>0.9810</td>
<td>0.9984</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(1.8)</td>
<td>0.5874</td>
<td>0.9264</td>
<td>0.9876</td>
<td>0.9992</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(2.0)</td>
<td>0.7100</td>
<td>0.9824</td>
<td>0.9988</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(3.0)</td>
<td>0.9878</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
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</tbody>
</table>

It is seen that the power of the test for \( H_0: \beta=\beta_0 \) against \( H_1: \beta=\beta_1 \) for any value \( \beta^* (>\beta_1) \) of \( \beta \) is approximately equal to the power of the test for \( H_0: \beta=\beta_0 \) against \( H_1: \beta^*=\beta^* \). Therefore the \( \lambda^* \) test can be used for testing \( H_0: \beta=\beta_0 \) against \( H_1: \beta=\beta_1 \). Table 7.4 gives such values of powers of \( \lambda^* \) test for \( H_0: \beta=\beta_0 \) against \( H_1: \beta=1.2\beta_0=\beta_1 \), considering \((\beta/\beta_0)>1.2\)

### Table 7.4.
**Powers of the \( \lambda^* \) test for \( H_0: \beta=\beta_0 \) vs \( H_1: \beta=1.2\beta_0=\beta_1 \), \((\beta/\beta_0)=1.2\)**

<table>
<thead>
<tr>
<th>(\beta/\beta_0=1.4)</th>
<th>(\beta/\beta_0=1.6)</th>
<th>(\beta/\beta_0=1.8)</th>
<th>(\beta/\beta_0=2)</th>
<th>(\beta/\beta_0=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n=10)</td>
<td>0.2588</td>
<td>0.4180</td>
<td>0.5722</td>
<td>0.7092</td>
</tr>
<tr>
<td>(n=20)</td>
<td>0.5048</td>
<td>0.7574</td>
<td>0.9202</td>
<td>0.9760</td>
</tr>
<tr>
<td>(n=30)</td>
<td>0.7030</td>
<td>0.9268</td>
<td>0.9902</td>
<td>0.9996</td>
</tr>
<tr>
<td>(n=40)</td>
<td>0.8334</td>
<td>0.9834</td>
<td>0.9996</td>
<td>1.0000</td>
</tr>
<tr>
<td>(n=60)</td>
<td>0.9352</td>
<td>0.9986</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(n=80)</td>
<td>0.9880</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(n=100)</td>
<td>0.9948</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

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Let $X_1, X_2, \ldots, X_n$ be a random sample from

$$f_X(x; \theta, \beta) = \frac{1}{\theta^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\theta}, \quad x > 0, \theta, \beta > 0. \quad (7.3.1)$$

The joint pdf of $X = (X_1, X_2, \ldots, X_n)$ is

$$f_{X}(x; \theta, \beta) = \frac{1}{\theta^{n\beta} (\Gamma(\beta))^n} \prod_{i=1}^{n} \frac{x_i^{\beta-1}}{e^{x_i/\theta}}. \quad (7.3.2)$$

For known $\beta$, $T = \sum_{i=1}^{n} X_i$ is sufficient statistic for $\theta$, having the pdf

$$f_T(t; \theta, \beta) = \frac{1}{\Gamma(n\beta) \theta^n} e^{-t/\theta} t^{n-1}, \quad t > 0. \quad (7.3.3)$$

Then the conditional distribution of $(X_1, X_2, \ldots, X_{n-1})$ given $T=t$ is

$$f(x_1, x_2, \ldots, x_{n-1} | T=t) = \frac{\Gamma(n\beta)}{\Gamma(\beta) \theta^{n}\Gamma(\beta)^{n-1}} \left( t - \sum_{i=1}^{n-1} x_i \right)^{\beta-1} \prod_{i=1}^{n} x_i^{\beta-1}, \quad (7.3.4)$$

for $n-1 \leq t$, $0 < x_i < \infty$, which is free from the parameter $\theta$.

We shall consider the problem of testing $H_0: \beta = \beta_0$ against $H_1: \beta > \beta_0$ treating that the observations $x_1, x_2, \ldots, x_{n-1}$ have come from (7.3.4). In this set up using Neymann-Pearson approach, we
have

\[ \lambda = \frac{f_{H_1}(x_1, x_2, \ldots, x_{n-1} | t)}{f_{H_0}(x_1, x_2, \ldots, x_{n-1} | t)} \]

\[ = \frac{\Gamma(n\beta_1)}{\Gamma(n\beta_0)} \left( \frac{\Gamma(\beta_0)}{\Gamma(\beta_1)} \right)^n \frac{(t - \Sigma x_i)^{\beta_1 - \beta_0 - 1}}{t^{n(\beta_1 - \beta_0)}} \prod_{i=1}^{n-1} x_i^{\beta_1 - \beta_0}. \]  

(7.3.5)

We reject \( H_0 \) if \( \lambda \geq c_\alpha(\beta_0, \beta_1) \) where \( c_\alpha(\beta_0, \beta_1) \) is the \( \alpha \)th percentile of the distribution of \( \lambda \) under \( H_0 \). Now, \( \lambda \geq c_\alpha \), only if,

\[ (t - \Sigma x_i) \prod_{i=1}^{n-1} x_i^{\beta_1 - \beta_0} \geq c_\alpha(\beta_0, \beta_1) \]

\[ [1 - \Sigma (x_i / t)] \prod_{i=1}^{n-1} (x_i / t) \geq c_\alpha(\beta_0, \beta_1). \]

Here \( c_\alpha(\beta_0, \beta_1) \) is to be obtained using size condition

\[ P_{H_0} \left\{ \prod_{i=1}^{n-1} (x_i / t) \geq c_\alpha(\beta_0, \beta_1) \right\} = \alpha. \]

Making the transformation,

\[ W_i = X_i / t, \quad i = 1, 2, \ldots, n-1, \]

we have

\[ P_{H_0} \left\{ \prod_{i=1}^{n-1} W_i \geq c_\alpha(\beta_0, \beta_1) | t \right\} = \alpha. \]

Viz.
\[ \int_{c_\alpha(\beta_0, \beta_1)}^{\infty} f_U(u, \beta_0) du = \alpha, \]  
\[ (7.3.6) \]

where

\[ U = (1 - \sum_{i=1}^{n-1} w_i) \prod_{i=1}^{n-1} w_i. \]

and the joint pdf of \( W_1, W_2, \ldots, W_{n-1} \), given \( t \) under \( H_0 \)

\[ f(w_1, w_2, \ldots, w_{n-1} | t) = \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} (1 - \sum_{i=1}^{n-1} w_i)^{\beta_0-1} \prod_{i=1}^{n-1} w_i^{\beta_0-1}. \]  
\[ (7.3.7) \]

\[ \sum_{i=1}^{n-1} w_i \leq 1, 0 < w_i \leq 1, i=1,2,\ldots,n-1. \]  
It is noted that the distribution given in (7.3.7) does not depend on \( t \). In practice, it is difficult to get the exact distribution of \( U \), hence we obtain the \( \alpha \)-th percentiles of the distribution of \( U \) through simulation. For this, we obtain the distribution of \( W_i \) given \( (w_1, w_2, \ldots, w_{i-1}) \).

We have from (7.3.7)

\[ f(w_1, w_2, \ldots, w_i) = \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} \prod_{j=i+1}^{n-1} w_j^{\beta_0-1} \int_{0}^{\infty} (1 - \sum_{j=1}^{n-1} w_j) \prod_{i=1}^{n-1} w_i^{\beta_0-1} dw_{n-1} \cdots dw_{i+1} \]
\[
\frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} \frac{2^{\beta_0-1}}{(1-\Sigma w_j)^{\beta_0-1}} \beta_0-1
\]

\[
= \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} B(\beta_0,\beta_0) \int_{0}^{\infty} ... (1-\Sigma w_i)^{\beta_0-1}
\]

\[
= \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} B(\beta_0,\beta_0) \int_{0}^{\infty} ... (1-\Sigma w_i)^{\beta_0-1}
\]

\[
\left[1 - \frac{w_{n-2}}{(1 - \Sigma w_i)}\right]^{2^{\beta_0-1}} dw_{n-2} ... dw_{i+1}
\]

\[
= \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} B(\beta_0,\beta_0) \int_{0}^{\infty} ... (1-\Sigma w_i)^{\beta_0-1}
\]

\[
= \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{\beta_0-1} B(\beta_0,\beta_0) \int_{0}^{\infty} ... (1-\Sigma w_i)^{\beta_0-1}
\]

\[
\prod_{j=1}^{i} w_j^{\beta_0-1} \int_{0}^{\infty} ... (1-\Sigma w_i)^{\beta_0-1}
\]

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Continuing the process of evaluating the integrals successively, we get

\[
f(w_1, w_2, \ldots, w_i) = \frac{\Gamma(n\beta_0)}{(\Gamma\beta_0)^n} \prod_{j=1}^{i} w_j^{(n-i)\beta_0^*} B(\beta_0, \beta_0) B(\beta_0, 2\beta_0) \ldots B(n-i)\beta_0, \beta_0).
\]

(7.3.8)

\[
0 < w_j < 1, \quad j = 1, 2, \ldots, i. \quad \text{The conditional distribution of } W_i \text{ given } (w_1, w_2, \ldots, w_{i-1}) \text{ is, then,}
\]

\[
f(w_i | w_1, w_2, \ldots, w_{i-1}) = \frac{\Gamma((n-i+1)\beta_0)}{\Gamma\beta_0^*((n-i)\beta_0)} \left[1 - \frac{w_i}{w_{i-1}}\right]^{(n-i)\beta_0^*} \left[\frac{w_i}{w_{i-1}}\right]^{\beta_0^*} \frac{1}{w_{i-1}}, \quad (7.3.9)
\]
0 < w_i < w_i-1, where \( w_i^* = 1 - \sum_{j=1}^{i-1} w_j \). The marginal pdf of \( W_i \) is given by

\[
f(w_i) = \frac{\Gamma(n\beta_0)}{\Gamma(\beta_0)^{n}} (1 - w_i)^{(n-1)\beta_0 - 1} \beta_0^{-1} w_i, \quad 0 < w_i < 1.
\]

(7.3.10)

Using (7.3.9) and (7.3.10) one can generate random samples \( (W_1, W_2, \ldots, W_{n-1}) \) and simulate the percentiles of the distribution of \( U \).

From (7.3.6), we see that the constant \( c_a(\beta_0, \beta_1) \) does not depend on \( \beta_1 \). Hence, the test is valid for composite alternative \( H_1: \beta > \beta_0 \).

Since the joint distribution of \( W_1, W_2, \ldots, W_{n-1} \) does not depend on the value \( t \) of \( T \), taking \( W_i \) by \( X_i/T \), we have,

\[
P_{H_0} \left[ \prod_{i=1}^{n-1} \frac{X_i}{\sum_{i=1}^{n} X_i} \geq c_a(\beta_0) \right] = \alpha.
\]

\[
P_{H_0} \left[ \prod_{i=1}^{n} \left( \frac{X_i}{\sum_{i=1}^{n} X_i} \right) \geq c_a(\beta_0) \right] = \alpha.
\]

\[
P_{H_0} \left[ \left( \prod_{i=1}^{n} X_i^{1/n} / \left( \sum_{i=1}^{n} X_i / n \right)^n \right) \geq c_a(\beta_0) \right] = \alpha.
\]

\[
P_{H_0} \left[ (\bar{X}/\bar{X}) \geq c_a(\beta_0) \right] = \alpha.
\]
where \( \bar{x} \) is the geometric mean and \( \bar{x} \) is the arithmetic mean of \( x_1, x_2, \ldots, x_n \) and \( c_\alpha(\beta_0) \) is the \( \alpha \)-th percentile of the distribution of \( \bar{x}/\bar{X} \). The simulated percentiles for different values of \( \beta_0 \) are given in Tables 7.5 to 7.7.

Table 7.5.
Percentile points of the distribution of \( \bar{x}/\bar{X} \) (for \( \beta_0=1 \)).

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<th>( \alpha = .05 )</th>
<th>( \alpha = .95 )</th>
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Table 7.6.
Percentile points of the distribution of \( \bar{x}/\bar{X} \) (for \( \beta_0=1.2 \)).

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Table 7.7.
Percentile points of the distribution of $\bar{X}/\bar{X}$ (for $\beta_0=1.4$).

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7.4. CONDITIONAL LIKELIHOOD TEST FOR THE WEIBULL PROCESS

A Weibull process is defined by the intensity function

$$\nu(x) = (\beta/\theta)x^{\beta-1}$$

and the mean value function

$$m(x) = x^{\beta/\theta}$$

with scale parameter $\theta>0$ and shape parameter $\beta>0$. For $\beta=1$, the process reduces to an homogeneous Poisson process. Let $X_1, X_2, \ldots, X_n$ denote the first $n$ successive times of occurrences of a Weibull process with mean value function given above. If $Z_j = m(X_j)$, $j=1, 2, \ldots, n$, then $Z_1, Z_2, \ldots, Z_n$ are distributed as the first $n$ successive occurrence times of a Poisson process with intensity one (Theorem 9.1.1, page 417, Bair and Engelhardt (1991)). Therefore, by the property of Poisson process, $Z_j-Z_{j-1}$, $j=1, 2, \ldots, n$; $Z_0=0$ are independent exponential random variables.
with mean one. The joint density function of \( Z_1, Z_2, \ldots, Z_n \) is given by

\[
f(z_1, z_2, \ldots, z_n) = e^{-z_1}, \quad 0 < z_1 < z_2 < \ldots < z_n
\]  

(7.4.1)

and hence that of \( X_1, X_2, \ldots, X_n \) is

\[
f(x_1, x_2, \ldots, x_n; \theta, \beta) = (\frac{\beta}{\theta})^n \prod_{i=1}^{n} x_i^{\beta - 1} e^{-x_i^{\beta / \theta}},
\]

(7.4.2)

\( 0 < x_1 < x_2 < \ldots < x_n < \infty ; \theta > 0, \beta > 0 \). It can be seen that \( (X_n, \sum_{i=1}^{n} \ln(X_i)) \) is jointly sufficient statistic for \( (\theta, \beta) \). For known \( \beta \), \( X_n \) is the complete sufficient statistic for \( \theta \), having the pdf

\[
f_{X_n}(x_n; \theta, \beta) = \frac{\beta}{\theta^n(n-1)!} e^{-x_n^{\beta / \theta}} x_n^{(n-1)\beta}, \quad x_n > 0.
\]

(7.4.3)

The conditional distribution of \( (X_1, X_2, \ldots, X_{n-1}) \) given \( X_n = x_n \) is

\[
f(x_1, x_2, \ldots, x_{n-1} | x_n) = (n-1)! (\frac{\beta}{x_n})^{n-1} \prod_{i=1}^{n-1} (x_i/x_n)^{\beta - 1},
\]

(7.4.4)

\( 0 < x_1 < x_2 < \ldots < x_{n-1} < x_n ; \beta > 0 \), which is free from the nuisance parameter \( \theta \). Hence we shall consider the problem of testing \( H_0: \beta = \beta_0 \) against \( H_1: \beta = \beta_1 > \beta_0 \) considering that the data have come from \( (7.4.4) \). Using Neymann-Pearson approach, we have

\[
\lambda = \frac{f_{H_1}(x_1, x_2, \ldots, x_{n-1} | x_n)}{f_{H_0}(x_1, x_2, \ldots, x_{n-1} | x_n)}
\]
We reject $H_0$, if
\[ \lambda \geq C_\alpha(\beta_0, \beta_1) \]

viz
\[
\sum_{i=1}^{n-1} \ln(x_i/x_n) \geq C^*(\beta_0, \beta_1),
\]

where $C^*(\beta_0, \beta_1)$ is to be obtained using the size condition

\[
P_{H_0} \left[ \sum_{i=1}^{n-1} \ln(x_i/x_n) \geq C^*(\beta_0, \beta_1) \mid X_n = x_n \right] = \alpha \quad (7.4.6)
\]

Making the transformation
\[ W_i = (x_i/x_n)^\beta, \quad i = 1, 2, \ldots, n-1 \]

in (7.4.4), we get the joint pdf of $W_1, W_2, \ldots, W_{n-1}$ as
\[
f(w_1, w_2, \ldots, w_{n-1} \mid x_n) = (n-1)!.
\quad (7.4.7)
\]

$0 < w_1 < w_2 < \ldots < w_{n-1} < 1$. It is seen that the density given in (7.4.7) does not depend on $x_n$ and $W_1, W_2, \ldots, W_{n-1}$ are the order statistics of a random sample of size $(n-1)$ from uniform $(0,1)$ distribution. Using the above transformation in (7.4.6) we get
\[ P_{H_0} \left[ \sum_{i=1}^{n-1} \beta_0^{-1} \ln(W_i) \geq C^*_{\alpha (\beta_0, \beta_1)} \right] = \alpha \]

that is

\[ P_{H_0} \left[ -2 \sum_{i=1}^{n-1} \ln(W_i) \leq C_{\alpha} \right] = \alpha. \]

Since the distribution of \(-2 \sum_{i=1}^{n-1} \ln(W_i)\) is \(\chi^2\) with \(2(n-1)\) df, the last result gives

\[ P_{H_0} \left[ \chi^2_{2(n-1)} \leq C_{\alpha} \right] = \alpha. \]

Thus \(C_{\alpha}\) is the \(\alpha^{th}\) percentile of \(\chi^2\) distribution with \(2(n-1)\) df.