CHAPTER 5
CHARGED FLUID SPHERES ON PSEUDO SPHEROIDAL SPACE-TIMES

1. INTRODUCTION

The study of gravitational significance of distributions of charged perfect fluids in equilibrium is important for two reasons. The Coulombian force of repulsion contributes additional force to the fluid pressure against the inward force of gravitation. Electromagnetic field contains energy and hence it will act as a source for curvature in space-time. Accordingly the problem of solving coupled Einstein-Maxwell equations describing charged fluid distributions has received considerable attention. Krori and Barua (1975) obtained a singularity free solution for static charged fluid spheres. This solution has been analysed in detail by GJG Juvenicus (1976). The solution obtained by Pant and Sah (1979) for static spherically symmetric relativistic charged fluid sphere has Tolman Solution IV as a particular case in the absence of charge. Whitman and Burch (1981) gave a method for solving the coupled Einstein-Maxwell equations for spherically symmetric static systems containing charge, obtained a number of analytic solutions and examined their stability. They showed that Pant and Sah's solution is not stable. Conformally flat interior solutions were obtained by Chang (1983) for charged perfect fluid as well as charged dust distributions.

Tikekar (1984) studied some general aspects of spherically symmetric static distributions of charged fluids and solved coupled Einstein-Maxwell equations for a specific choice of density and pressure. This solution admits Pant and Sah solution as particular case.

Patel and Pandya (1984) presented the charged analogue of Vaidya-Tikekar (1982) solution. Patel and Kopper (1987) solved the Einstein-Maxwell equations on spheroidal space-times by assuming a specific form for the electric field intensity E. Patel and Mehta (1995) obtained solutions for charged spherically symmetric static distributions of matter on spheroidal space-times with a different assumption for E^2. Krishna Rao and Trivedi (1997) developed a formalism of generating new solutions of coupled Einstein-Maxwell equations.

We have investigated the gravitational significance of charged perfect fluid spheres with their associated 3-space, t = constant, as 3-pseudo spheroidal space.

2. SOLUTION OF FIELD EQUATIONS

We begin by assuming that the space-time of a charged perfect fluid in equilibrium is described by

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{5.1}$$

with

$$e^{\lambda} = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}}, \tag{5.2}$$

the metric of the pseudo spheroidal space-time.

It was shown in Chapter-4 ,that the Coupled Einstein-Maxwell equations for charged fluid distributions in equilibrium with (5.1) and (5.2) as the metric of the space-time is a system of three equations:

$$8\pi\rho + E^2 = \frac{3(K-1)\left(1+\frac{Kr^2}{3R^2}\right)}{R^2\left(1+K\frac{r^2}{R^2}\right)^2}, \tag{5.3}$$

$$8\pi P - E^2 = \frac{\left(1+\frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}}{1+K\frac{r^2}{R^2}}, \tag{5.4}$$

$$8\pi P + E^2 = \frac{\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{2r}\right)\left(1+\frac{r^2}{R^2}\right)}{1+K\frac{r^2}{R^2}} - \frac{\frac{(K-1)r}{R^2}\left(\frac{\nu'}{r}+\frac{1}{r}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}. \tag{5.5}$$

The variables ρ , P and E^2 have their usual meaning.

To have specific solutions of these three equations connecting the four variables ρ, P, ν and E, it is necessary to have one more relation.However specific solutions can as well be obtained by prescribing any ad-hoc relation for ρ, P, ν and E. The physical plausibility of the subsequently obtained solution will have to be carefully examined. We will solve the above system of equations by stipulating

$$E^2 = \frac{\alpha^2 r^2}{R^4}\left(1+K\frac{r^2}{R^2}\right)^{-2}, \tag{5.6}$$

where α is a constant.Equations (5.4) ,(5.5) and (5.6) lead to the differential equation

$$\left(1 - K + Kz^2\right)\frac{d^2F}{dz^2} - Kz\frac{dF}{dz} + \left[K(K-1) - 2\alpha^2\right]F = 0, \tag{5.7}$$

where

$$z^2 = 1 + \frac{r^2}{R^2} \text{ and } F = e^{\nu/2}, \tag{5.8}$$

for determination of $\nu(r)$.

If we choose

$$\alpha^2 = \frac{K(K-2)}{2}, \tag{5.9}$$

equation (5.7) admits a closed form solution

$$F = A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}} \ln\left(\sqrt{\frac{K}{K-1}}\sqrt{1 + \frac{r^2}{R^2}} + \frac{1}{\sqrt{K-1}}\sqrt{1 + K\frac{r^2}{R^2}}\right) - \frac{1}{\sqrt{K}}\sqrt{1 + K\frac{r^2}{R^2}}\right] \tag{5.10}$$

where A and B are arbitrary constants of integration.

The space-time metric for this solution can be explicitly written as

$$ds^2 = \left\{ A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}} \ln\left(\sqrt{\frac{K}{K-1}}\sqrt{1+\frac{r^2}{R^2}} + \frac{1}{\sqrt{K-1}}\sqrt{1+K\frac{r^2}{R^2}} \right) - \frac{1}{\sqrt{K}}\sqrt{1+K\frac{r^2}{R^2}} \right] \right\}^2 dt^2$$

$$\frac{1+K\dfrac{r^2}{R^2}}{1+\dfrac{r^2}{R^2}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{5.11}$$

The matter density and fluid pressure are found to be

$$8\pi\rho = \left(\frac{3(K-1)}{R^2} + \frac{K^2}{2R^2}\frac{r^2}{R^2} \right)\left(1+K\frac{r^2}{R^2} \right)^{-2}, \tag{5.12}$$

$$8\pi P = \frac{A\sqrt{1+\dfrac{r^2}{R^2}}\,g(r) + B\left\{ g(r)f(r) + \dfrac{1}{\sqrt{K}R^2}\left[(K-1)\left(1+K\dfrac{r^2}{R^2}\right) - \dfrac{K(K-2)}{2}\dfrac{r^2}{R^2} \right]\sqrt{1+K\dfrac{r^2}{R^2}} \right\}}{\left(1+K\dfrac{r^2}{R^2}\right)^2 \left\{ A\sqrt{1+\dfrac{r^2}{R^2}} + B\left[f(r) - \dfrac{1}{\sqrt{K}}\sqrt{1+K\dfrac{r^2}{R^2}} \right] \right\}}, \tag{5.13}$$

where

$$f(r) = \sqrt{1+\frac{r^2}{R^2}} \ln\left(\sqrt{\frac{K}{K-1}}\sqrt{1+\frac{r^2}{R^2}} + \frac{1}{\sqrt{K-1}}\sqrt{1+K\frac{r^2}{R^2}} \right), \tag{5.14}$$

and

$$g(r) = \left(\frac{3-K}{R^2} \right)\left(1+K\frac{r^2}{R^2} \right) + \frac{K(K-2)}{2R^2}\frac{r^2}{R^2}. \tag{5.15}$$

The electric field intensity and charge density are found from equation (5.6) and (4.13) as

$$E^2 = \frac{K(K-2)}{2R^2}\frac{r^2}{R^2}\left(1+K\frac{r^2}{R^r}\right)^{-2}, \tag{5.16}$$

$$4\pi\sigma = \frac{1}{R^2}\sqrt{\frac{K(K-2)}{2}}\left(3+K\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-2}. \tag{5.17}$$

We observe that E^2 and σ are non-negative since $K \geq 2$.

If the charged fluid distribution is to be of finite extent, the metric (5.11) should continuously match with the external Reissner-Nordstrom metric

$$ds^2 = \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2 - \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{5.18}$$

across the boundary $r = a$ where $P(a)=0$. Here m and $q=q(a)$ respectively denote the total mass and charge of the sphere.

It follows from (4.11) that

$$[q(a)]^2 = \frac{K(K-2)}{2}\frac{a^6}{R^4}\left(1+K\frac{a^2}{R^2}\right)^{-2}. \tag{5.19}$$

The appropriate boundary conditions ensuring continuity of metric coefficients of (5.11) with those of (5.18) read

$$e^{\nu(a)} = \frac{1+\frac{a^2}{R^2}}{1+K\frac{a^2}{R^2}} = 1-\frac{2m}{a}+\frac{q^2}{a^2}. \tag{5.20}$$

These conditions determine the constants m, A and B as

$$m = \frac{\left(\frac{(K-1)}{2} + \frac{K(3K-4)}{4} \cdot \frac{a^2}{R^2}\right)\frac{a^3}{R^2}}{\left(1 + K\frac{a^2}{R^2}\right)^2}, \tag{5.21}$$

$$A = \frac{\frac{\sqrt{K}}{2}\left[(3-K) + \frac{K(K-2)}{2}\left(1 + K\frac{a^2}{R^2}\right)^{-1}\right]f(a)}{1 + K\frac{a^2}{R^2}} + \frac{\frac{(K-1)}{2} - \frac{K(K-2)}{4}\left(1 + K\frac{a^2}{R^2}\right)\frac{a^2}{R^2}}{\sqrt{1 + K\frac{a^2}{R^2}}} \tag{5.22}$$

$$B = -\frac{\frac{\sqrt{K}}{2}\left[(3-K)\left(1 + K\frac{a^2}{R^2}\right) + \frac{K(K-2)}{2}\frac{a^2}{R^2}\right]}{\sqrt{1 + \frac{a^2}{R^2}}\left(1 + K\frac{a^2}{R^2}\right)^2}. \tag{5.23}$$

When K=2, the electromagnetic field disappears and the space-time metric (5.11) corresponds to the solution obtained in Chapter-2 for uncharged fluid distribution.

3. DISCUSSION

It is evident from equation (5.12) that matter density is positive. The gradients of density and pressure are given by

$$8\pi\frac{d\rho}{dr} = \frac{\frac{Kr}{R^4}\left(12 - 11K - K^2\frac{r^2}{R^2}\right)}{\left(1 + K\frac{r^2}{R^2}\right)^3}, \tag{5.24}$$

$$8\pi\frac{dP}{dr}=-8\pi(\rho+P)\left[\frac{r}{2}\cdot\frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\left(8\pi P-E^2+\frac{1}{r^2}\right)-\frac{1}{2r}\right]+2EE'+\frac{4E^2}{r},$$

(5.25)

where ρ, P and E are given by equations (5.12), (5.13) and (5.16).

The equation (5.24) indicates that the matter density of the distribution decreases radially outward. The expression (5.13) for pressure is highly complex and hence it is not possible to examine in general the positivity of pressure throughout.Similarly the expression (5.25) for pressure gradient is also complex. Hence we adopt numerical methods.

We shall discuss a method for obtaining the estimates of total mass m,charge q and boundary radius a of the charged fluid spheres in equilibrium similar to the one discussed in Chapter-2 for fluid spheres on pseudo spheroidal space-time.

The matter density has its maximum value

$$8\pi\rho(0)=\frac{3(K-1)}{R^2}, \tag{5.26}$$

at the centre.It attains the value

$$8\pi\rho(a)=\frac{\left[\frac{3(K-1)}{R^2}+\frac{K^2}{2R^2}\frac{a^2}{R^2}\right]}{\left(1+K\frac{a^2}{R^2}\right)^2}. \tag{5.27}$$

on the boundary.

We introduce the density variation parameter

$$\lambda = \frac{\rho(a)}{\rho(0)} = \frac{1 + \frac{K^2}{6(K-1)}\frac{a^2}{R^2}}{\left(1 + K\frac{a^2}{R^2}\right)^2}, \tag{5.28}$$

as the ratio of the density at the surface $r=a$ to the density at the centre $r=0$. Since $\rho(r)$ is decreasing $\lambda < 1$.

Equation (5.28) determines $\frac{a^2}{R^2}$ in terms of K and λ as

$$\frac{a^2}{R^2} = \frac{K - 12\lambda(K-1) + \sqrt{24\lambda(K-1)(5K-6) + K^2}}{12\lambda K(K-1)}. \tag{5.29}$$

The algebraic root assigning negative values to $\frac{a^2}{R^2}$ is rejected to ensure that a is real.

Now using (5.26) and (5.28) we get

$$R^2 = \frac{3(K-1)\lambda}{8\pi\rho(a)}. \tag{5.30}$$

This equation determines R for given values of λ and $\rho(a)$. Equation (5.29) determines the boundary radius a and consequently (5.19) and (5.21) respectively give the total charge and mass of the configuration . A and B follow from (5.22) and (5.23).

Taking matter density at the boundary of the star as 2×10^{14} gm/cm^3, the estimates of mass, size and other relevent quantities are obtained for different values of λ. These estimates are displayed in Table-5.1.

In general the charged fluid distribution also must confirm with the physical requirements stipulated in Section-1 of Chapter-3 . It is difficult to verify all the requirements analytically.However we have obtained the numerical estimates of $8\pi\rho$, $8\pi P$,

$8\pi(\rho - 3P)$, $\frac{dP}{d\rho}$, $\frac{P}{\rho}$ and γ for a model with λ=0.4 of Table-5.1 . These estimates are shown in Table-5.2. It is found from Table-5.2 that the pressure is decreasing as one proceeds from centre to boundary. The strong energy condition and the causality requirement are fulfilled throughout the distribution. Moreover $\frac{P}{\rho}$ is found to be decreasing radially outward and the adiabatic index $\gamma > 1$ throughout indicating that the temparature is decreasing from centre to boundary.

This analysis indicates model with $\lambda = 0.4$ is a physically viable model of a static superdense charged matter distributions. Thus pseudo spheroidal space-times are useful to describe physically viable superdense distributions of charged matter in equilibrium.

TABLE-5.1

Masses and equilibrium radii of superdense star models corresponding to *K=2.1* and $\rho(a) = 2 \times 10^{14} gm / cm^3$.

λ	a / R	m / a	*R(km)*	*a(km)*	*m(km)*	$m / M_{\odot}$	A	B	*q*
0.9	0.176	0.016	28.243	4.966	0.080	0.054	1.074	-0.624	0.016
0.8	0.261	0.033	26.628	6.954	0.230	0.156	1.043	-0.594	0.134
0.7	0.338	0.051	24.908	8.421	0.431	0.292	1.008	-0.561	0.252
0.6	0.416	0.071	23.060	9.596	0.678	0.460	0.970	-0.526	0.395
0.5	0.501	0.092	21.051	10.561	0.971	0.659	0.926	-0.487	0.563
0.4	0.603	0.116	18.829	11.346	1.311	0.889	0.874	-0.442	0.757
0.3	0.733	0.142	16.306	11.953	1.699	1.152	0.811	-0.391	0.978
0.2	0.927	0.174	13.314	12.346	2.142	1.452	0.728	-0.328	1.226
0.1	1.318	0.213	9.414	12.409	2.462	1.791	0.602	-0.602	1.503

TABLE -5.2

The values of $\bar{\rho} = 8\pi\rho \times 10^4$, $\bar{P} = 8\pi P \times 10^4$, $\bar{\rho} - 3\bar{P}$, $\frac{dp}{d\rho}$, $\frac{P}{\rho}$, γ for different

values of $x = \frac{r^2}{R^2}$ for the model with λ=0.4 of Table-5.1.

x	$\bar{\rho}$	$\bar{P}$	$\bar{\rho}-3\bar{P}$	$\frac{dp}{d\rho}$	$\frac{P}{\rho}$	γ
0.000	93.08	3.99	81.11	0.0646	0.043	1.572
0.025	85.43	3.49	74.96	0.0662	0.041	1.689
0.050	78.78	3.04	69.65	0. 0678	0.039	1.823
0.75	72.95	2.64	65.02	0. 0692	0.036	`1.977
0.100	67.82	2.29	60.97	0.0704	0.034	2.159
0.125	63.28	1.96	57.39	0.0715	0.031	2.378
0.150	59.22	1.67	54.21	0.0726	0.028	2.645
0.175	55.59	1.41	51.38	0.0735	0.025	2.979
0.200	52.33	1.16	48.84	0.0743	0.023	3.4133
0.225	49.38	0.94	46.55	0.0751	0.019	3.998
0.250	46.71	0.74	44.48	0.0757	0.016	4.834
0.275	44.28	0.56	42.60	0.0763	0.013	6.126
0.300	42.06	0.39	40.89	0.0768	0.009	8.391
0.325	40.02	0.23	39.33	0.0771	0.005	13.411
0.350	38.15	0.09	37.89	0.0775	0.002	34.049
0.364	37.17	0.00	37.17	0.0777	0.000	264.637