CHAPTER 4

CHARGED DUST SPHERES ON PSEUDO SPHEROIDAL SPACE-TIMES

1. INTRODUCTION

The equilibrium in a spherically symmetric distribution of perfect fluid is maintained by the repulsive pressure responsible for counterbalancing the gravitational attraction. For the matter distributions in the form of dust , there exists no such force to counter the collapse. In such situations the collapse of the distribution to a singularity can be averted if the matter is accompanied by the presence of some electrical charge. A systematic study of electromagnetic fields in the context of general relativity was due to Rainich (1925).The equilibrium of charged dust spheres within the frame work of general relativity was examined critically by Papapetrou and Majumdar (1947). Bonnor (1960) showed that if the ratio of charge density to matter density is constant throughout a charged dust distribution in equilibrium, the equilibrium is stable. De and Raychaudhari (1968) have shown that this result is a consequence of Einstein-Maxwell equations which are singularity free. Cooperstock and de la Cruz (1978) have studied relativistic spherical distributions of charged perfect fluids in equilibrium and obtained an explicit solution of coupled Einstein-Maxwell equations in the interior of a sphere containing uniformly charged dust in equilibrium , assuming a constant non gravitational energy density. Bonnor and Wickramsuriya (1975) have obtained a static interior dust metric with matter density increasing outward.

Interior solutions of Einstein-Maxwell equations for charged dust spheres assuming that the physical space has a 3-spheroidal geometry have been obtained by Tikekar (1984). We have investigated in this chapter the nature of charged dust distributions on the background of the pseudo-spheroidal space-time. It is found that coupled Einstein-Maxwell equations admit physically viable closed form regular solutions in this set up.

2. EINSTEIN-MAXWELL EQUATIONS

We begin by assuming that the space-time of the charged perfect fluid distribution in equilibrium is described by the metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \qquad (4.1a)$$

with

$$e^{\lambda} = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}}, \qquad (4.1b)$$

of a pseudo spheroidal space-time.

If the physical content of the space-time of metric (4.1) is a spherical distribution of charged perfect fluid ,it should satisfy the Einstein-Maxwell equations

$$\Re_i^j - \frac{1}{2}\Re\delta_i^j = -8\pi T_i^j, \qquad (4.2a)$$

with $c = G = 1$.

The energy-momentum tensor T_i^j for a distribution of matter in the form of charged perfect fluid has the form (Cooperstock and Cruz-1978)

$$T_j^i = (\rho + P)u_i u^j - P\delta_i^j + E_i^j, \qquad (4.2b)$$

where

$$E_i^j = \frac{1}{4\pi}\left(-F_{ik}F^{kj} + \tfrac{1}{4}F_{mn}F^{mn}\delta_i^j\right) , \tag{4.2c}$$

is the energy-momentum tensor in the electromagnetic field associated with the charge distribution.

Here ρ, P and u^i respectively denote the matter density, fluid pressure and the unit time-like four-velocity of the fluid. F_{ij} are components of electromagnetic field tensor satisfying Maxwell's equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \quad , \tag{4.3}$$

$$\frac{\partial}{\partial x^k}\left(F^{ik}\sqrt{-g}\right) = 4\pi\sqrt{-g}J^i \ . \tag{4.4}$$

The four current J^i is defined as

$$J^i = \sigma u^i \tag{4.5}$$

where σ denotes the charge density of the distribution.

For static distributions

$$u^i = \left(e^{-\nu/2}, 0, 0, 0\right) \ . \tag{4.6}$$

The spherical symmetry implies that electromagnetic field tensor has

$$F_{10} = -F_{01} \tag{4.7}$$

as its only non-vanishing component.

Maxwell's equations (4.3) and (4.4) admit

$$F^{01} = \frac{e^{-\left(\frac{\lambda+\nu}{2}\right)}}{r^2}\int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr \ , \tag{4.8}$$

and

$$F_{01} = g_{00}g_{11}F^{01} = -\frac{e^{\left(\frac{\lambda+\nu}{2}\right)}}{r^2}\int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr \tag{4.9}$$

as their solution.

Let

$$E^2 = -F_{01}F^{01} \tag{4.10}$$

denote the electric filed intensity.

Equations (4.8) and (4.9) determine E^2 as

$$E^2 = \frac{1}{r^4}\left[\int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr\right]^2 = \frac{[q(r)]^2}{r^4}. \tag{4.11}$$

where $q(r)$ is the total charge contained within a sphere of radius r. It follows that

$$q(r) = 4\pi\int_0^r r^2 \sigma e^{\lambda/2} dr \tag{4.12}$$

From (4.11) and (4.12), we obtain the charge density of the distribution in the form

$$4\pi\sigma = \frac{\sqrt{1+\frac{r^2}{R^2}}}{\sqrt{1+K\frac{r^2}{R^2}}}\frac{1}{r^2}\frac{d\left(r^2 E\right)}{dr}. \tag{4.13}$$

The coupled Einstein-Maxwell equations reduce to

$$8\pi P - E^2 = \frac{\left(1+\frac{r^2}{R^2}\right)\frac{\nu^1}{r} - \frac{K-1}{R^2}}{1+K\frac{r^2}{R^2}}, \tag{4.14}$$

$$8\pi P + E^2 = \frac{\left(\frac{\nu''}{2}+\frac{\nu'}{4}+\frac{\nu'}{2r}\right)\left(1+\frac{r^2}{R^2}\right)}{1+K\frac{r^2}{R^2}} - \frac{\frac{r(K-1)}{R^2}\left(\frac{\nu'}{r}+\frac{1}{r}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}, \qquad (4.15)$$

$$8\pi\rho + E^2 = \frac{\frac{3(K-1)}{R^2}\left(1+\frac{K}{3}\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}. \qquad (4.16)$$

Equation (4.14) through (4.16) constitute a system of three equations connecting the metric coefficient ν, the curvature parameters R and K with the physical variables ρ, P and E^2 Equation (4.16) determines the non-gravitational energy-density at every point for specified choice of R and K.

3. SOLUTION OF FIELD EQUATIONS

We shall consider the situation in which the matter content of the charged fluid sphere is in the form of dust with pressure , $P = 0$. Charged dust spheres in equilibrium belong to the interior Papapetrou-Majumdar (1947) class and their metrics can be expressed in the form

$$ds^2 = U^{-2}dt^2 - U^2(dx^2 + dy^2 + dz^2) \quad , \qquad (4.17)$$

where $U=U(x,y,z)$. The metric (4.1a) reduces to the form (4.17) if and only if

$$1+\frac{r\nu'}{2} = e^{\nu/2} = \left(1+K\frac{r^2}{R^2}\right)^{1/2}\left(1+\frac{r^2}{R^2}\right)^{-1/2}. \qquad (4.18a)$$

This equation admits

$$\nu = 2\int \frac{1}{r}\left(1+K\frac{r^2}{R^2}\right)^{1/2}\left(1+\frac{r^2}{R^2}\right)^{-1/2} dr - 2\int \frac{1}{r}dr + \log A^2, \tag{4.18b}$$

as its general solution, where A^2 is the constant of integration.

We write

$$e^{\nu} = \frac{A^2 e^I}{r^2}, \tag{4.19}$$

where

$$I = \frac{1}{2}\int \frac{1}{r}\left(1+K\frac{r^2}{R^2}\right)^{1/2}\left(1+\frac{r^2}{R^2}\right)^{-1/2} dr.$$

On evaluating the integral we get

$$I = 2\sqrt{K}\ln\left\{\sqrt{1+K\frac{r^2}{R^2}}+\sqrt{K}\sqrt{1+\frac{r^2}{R^2}}\right\}$$
$$-\ln\left\{\frac{\sqrt{K^2\left(\frac{r^2}{R^2}\right)^2+K(K+1)\left(\frac{r^2}{R^2}\right)+K}+\sqrt{K}+\frac{K(K+1)}{2\sqrt{K}}\frac{r^2}{R^2}}{K\frac{r^2}{R^2}}\right\}.$$

Substituting this value of I in (4.19), we obtain the expression for e^{ν} in the form

$$e^{\nu} = A^2\frac{\left[\left(\sqrt{1+K\frac{r^2}{R^2}}+\sqrt{K}\sqrt{1+\frac{r^2}{R^2}}\right)^{\sqrt{K}}\right]^2}{\left[\sqrt{1+K\frac{r^2}{R^2}}+\sqrt{1+\frac{r^2}{R^2}}\right]^2}. \tag{4.20}$$

The expression for e^{ν} is regular and positive for all values of r. Hence the space-time metric with e^{ν} as given by (4.20) has the explicit form

$$ds^2 = A^2\left\{\frac{\left(\sqrt{+1+K\frac{r^2}{R^2}}+\sqrt{K}\sqrt{1+\frac{r^2}{R^2}}\right)^{\sqrt{K}}}{\sqrt{1+K\frac{r^2}{R^2}}+\sqrt{1+\frac{r^2}{R^2}}}\right\}^2 dt^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2$$

$$-r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{4.21}$$

The equations (4.14) , (4.16) and (4.13) respectively lead to the expressions for electric field intensity , matter density and charge density as

$$E^2 = \frac{1}{r^2}\left\{1-\sqrt{\frac{1+\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}}\right\}^2 , \tag{4.22}$$

$$8\pi\rho = \frac{2}{r^2}\left[\sqrt{\frac{1+\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}} - \frac{1+\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}\right] + \frac{2(K-1)}{R^2\left(1+K\frac{r^2}{R^2}\right)^2} , \tag{4.23}$$

$$8\pi\sigma = \pm\left\{\frac{2}{r^2}\left[\sqrt{\frac{1+\frac{r^2}{R^2}}{1\div K\frac{r^2}{R^2}}} - \frac{1+\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}\right] + \frac{2(K-1)}{R^2\left(1+K\frac{r^2}{R^2}\right)^2}\right\} . \tag{4.24}$$

Subsequently from equations (4.23) and (4.24) , we get

$$\sigma = \pm\rho \quad , \tag{4.25}$$

which is in agreement with the De-Raychaudari (1968) requirement.

4. PHYSICAL ASPECTS

The matter density and electric field intensity at the centre of the distribution are obtained by letting r to tend to zero in expressions (4.23) and (4.22). When r tends to zero we get

$$8\pi\rho(0) = \frac{3(K-1)}{R^2} , \tag{4.26}$$

$$E(0) = 0 . \tag{4.27}$$

Hence the distribution is regular at the centre. It follows from (4.23) and (4.22) that ρ and E^2 are positive throughout the distribution.

If the spherical charged dust distribution extends to a finite radius a, then the interior metric (4.21) should continuously match with the exterior Reissner-Nordstrom (1916,1918) metric

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{4.28}$$

across the surface $r=a$. Here m denotes the total mass and q denotes the total charge of the distribution.

The appropriate boundary conditions satisfied at the boundary $r=a$ are

$$e^{\nu(a)} = \frac{1 + \frac{a^2}{R^2}}{1 + K\frac{a^2}{R^2}} = 1 - \frac{2m}{a} + \frac{q^2}{a^2} \tag{4.29}$$

They determine the constant A^2 and m as

$$A^2 = \left[\frac{1+\frac{a^2}{R^2}}{1+K\frac{a^2}{R^2}}\right]\left[\frac{\sqrt{1+K\frac{a^2}{R^2}}+\sqrt{1+\frac{a^2}{R^2}}}{\left(\sqrt{1+K\frac{a^2}{R^2}}+\sqrt{K}\sqrt{1+\frac{a^2}{R^2}}\right)^{\sqrt{K}}}\right]^{2}, \tag{4.30}$$

$$m = \frac{(K-1)a^3}{2R^2\left(1+K\frac{a^2}{R^2}\right)}+\frac{q^2}{2a}, \tag{4.31}$$

where the total charge q of the distribution within the radius a is given by

$$q(a) = \left[1-\sqrt{\frac{1+\frac{a^2}{R^2}}{1+K\frac{a^2}{R^2}}}\right]a. \tag{4.32}$$

By giving specific values for K in (4.26) , the value of R can be determined in terms of matter density $\rho(0)$ at the centre. Subsequently equation (4.31) determines the mass m contained within a sphere of radius a. The expression (4.32) determines the total charge required to be present in the distribution which can maintain its equilibrium. Different solutions can be generated by assigning different values to $K>1$.

In the solution given by Tikekar (1984) on the background of spheroidal space-time, the boundary radius of the distribution is restricted by the condition $\frac{a}{R} \leq 0.894$ to comply with the physical requirement $\rho > 0$. A noteworthy feature of the solution obtained on the pseudo spheroidal space-time is that the boundary radius of the charged dust sphere is not restricted by the curvature parameter R.

Thus the space-times whose physical space having pseudo spheroidal geometry are useful to describe physically viable models of charged dust spheres in equilibrium.