CHAPTER 3

FLUID DISTRIBUTIONS ON PSEUDO SPHEROIDAL SPACE-TIMES-II

1. PHYSICAL PLAUSIBILITY

In this chapter we discuss the physical plausibility and stability of the solution obtained for perfect fluid distribution, on the background of the pseudo spheroidal space-time, using analytical and numerical methods. Since the usual procedure of assuming an equation of state of matter was replaced by the requirement of pseudo spheroidal geometry for the space-time of the fluid distribution it is necessary to examine carefully the physical plausibility of the solution.The requirements a physically acceptable solution is expected to fulfil were elaborately stated by Knutsen (1987,1988).

(i) The matter density ρ and pressure P should be non negative throughout the distribution. i. e. $\rho \geq 0$ and $P \geq 0$. They should also be decreasing functions of r as one proceeds from centre to boundary. *i.e.* $\frac{d\rho}{dr} < 0$ and $\frac{dP}{dr} < 0$.

(ii) In general the fluid distribution should confirm with the requirements of strong energy conditions throughout the distribution. which imply $\rho - 3P > 0$. For superdense matter these conditions may be relaxed to the requirement $\rho - P > 0$ throughout, an implication of weak energy conditions.

(iii) According to the causality requirement no signal can propagate with speed greater than the speed of light. In isentropic fluids the speed of sound is given by $\sqrt{\frac{dP}{d\rho}}$.

Since perfect fluid is isentropic one expects that $\sqrt{\frac{dP}{d\rho}} < c$.

(iv) For a negative temperature gradient, the ratio $\frac{P}{\rho}$ should be decreasing outwards.

(v) A necessary condition for the stability of the model is that the adiabatic index $\gamma = \frac{\rho + P}{P} \frac{dP}{d\rho}$ should be larger than $4/3$. When $\gamma > 1$, the temperature decreases radially outward. Also γ should be increasing outward from centre.

We choose a new variable $x = 2\frac{r^2}{R^2}$ and express the physical variables ρ and P in a more convenient form as (using geometric units $c = 1$, $G = 1$)

$$8\pi\rho = \frac{1}{R^2}\frac{x+3}{(x+1)^2}, \tag{3.1}$$

$$8\pi P = \frac{1}{R^2}\left[\frac{4(x+2)}{x+1}\frac{1}{F}\frac{dF}{dx} - \frac{1}{x+1}\right], \tag{3.2}$$

where

$$F = C_1\sqrt{x+2} + C_2\left[\sqrt{x+2}\,\ln\left(\sqrt{x+1}+\sqrt{x+2}\right) - \sqrt{x+1}\right], \tag{3.3}$$

and $C_1 = \frac{A}{\sqrt{2}}$ and $C_2 = \frac{B}{\sqrt{2}}$ are arbitrary constants.

The pressure isotropy equation (2.23) reduces to

$$4(x+1)(x+2)\frac{d^2F}{dx^2} - 2\frac{dF}{dx} + F = 0. \tag{3.4}$$

Using equations (3.1), (3.2) and (3.4) we express the gradients of matter density and pressure as

$$\frac{d\rho}{dx} = -\frac{1}{8\pi R^2} \cdot \frac{x+5}{(x+1)^3}, \qquad (3.5)$$

$$\frac{dP}{dx} = -(\rho + P)\frac{1}{F}\frac{dF}{dx}, \qquad (3.6)$$

and

$$\frac{dP}{d\rho} = \frac{8\pi(\rho + P)R^2(x+1)^3}{x+5}\frac{1}{F}\frac{dF}{dx}. \qquad (3.7)$$

From equation (3.3) we obtain

$$\frac{1}{F}\frac{dF}{dx} = \frac{C_1 + C_2 \ln\left(\sqrt{x+1} + \sqrt{x+2}\right)}{C_1(x+2) + C_2\left[(x+2)\ln\left(\sqrt{x+1} + \sqrt{x+2}\right) - \sqrt{x+1}\sqrt{x+2}\right]}. \qquad (3.8)$$

Substituting the expression for $\frac{1}{F}\frac{dF}{dx}$ in (3.2), we get

$$8\pi P = \frac{C_1\sqrt{x+2} + C_2\left[\sqrt{x+2}\ln\left(\sqrt{x+1} + \sqrt{x+2}\right) + \sqrt{x+1}\right]}{R^2(x+1)\left\{C_1\sqrt{x+2} + C_2\left[\sqrt{x+2}\ln\left(\sqrt{x+1} + \sqrt{x+2}\right) - \sqrt{x+1}\right]\right\}}. \qquad (3.9)$$

At the boundary $r=a$, let $x(a) = b = \frac{2a^2}{R^2}$. The continuity of pressure across the boundary implies that $P(b) = 0$. *i.e.*,

$$C_1\sqrt{b+2} + C_2\left[\sqrt{b+2}\ln\left(\sqrt{b+1} + \sqrt{b+2}\right) + \sqrt{b+1}\right] = 0 . \qquad (3.10)$$

The continuity of the interior space-time metric with the Schawarzschild exterior metric at the boundary implies

$$C_1\sqrt{b+2} + C_2\left[\sqrt{b+2}\ln\left(\sqrt{b+1} + \sqrt{b+2}\right) - \sqrt{b+1}\right] = \sqrt{\frac{b+2}{2(b+1)}} . \qquad (3.11)$$

The arbitrary constants C_1 and C_2 obtained from (3.10) and (3.11) read

$$C_1 = \frac{\sqrt{b+1} + \sqrt{b+2}\,\ln\left(\sqrt{b+1} + \sqrt{b+2}\right)}{b+1}, \tag{3.12}$$

$$C_2 = -\frac{\sqrt{b+2}}{b+1}. \tag{3.13}$$

Using these values of C_1 and C_2 the expressions for $\frac{1}{F}\frac{dF}{dx}$, P, $\rho - P$, and $\rho - 3P$

become

$$\frac{1}{F}\frac{dF}{dx} = \frac{r(x)[l(b) - l(x)] + s(x)}{2(x+2)\{r(x)[l(b) - l(x)] + s(x) + q(x)\}}, \tag{3.14}$$

$$8\pi P = \frac{r(x)[l(b) - l(x)] + s(x) - q(x)}{R^2(x+1)\{r(x)[l(b) - l(x)] + s(x) + q(x)\}}, \tag{3.15}$$

$$8\pi(\rho - P) = \frac{2}{R^2(x+1)^2}\left\{\frac{r(x)[l(b) - l(x)] + s(x) + (x+2)q(x)}{r(x)[l(b) - l(x)] + s(x) + q(x)}\right\}, \tag{3.16}$$

$$8\pi(\rho - 3P) = \frac{2}{R^2(x+1)^2}\left\{\frac{(2x+3)q(x) - xr(x)[l(b) - l(x)] - xs(x)}{r(x)[l(b) - l(x)] + s(x) + q(x)}\right\}, \tag{3.17}$$

where

$$\begin{aligned} q(x) &= \sqrt{x+1}\sqrt{b+2}, \\ r(x) &= \sqrt{x+2}\sqrt{b+2}, \\ s(x) &= \sqrt{x+2}\sqrt{b+1}, \\ l(x) &= \ln\left(\sqrt{x+1} + \sqrt{x+2}\right). \end{aligned} \tag{3.18}$$

We note that $r(x) > s(x) \geq q(x)$.

Equations (3.1), (3.15) and (3.5) show that matter density and fluid pressure are positive and gradient of matter density is negative throughout the distribution. Equation (3.14) shows that $\frac{1}{F}\frac{dF}{dx}$ is positive. Hence from (3.6) we observe that the gradient of pressure is negative. From (3.16) it follows that the condition $\rho - P > 0$ is fulfilled throughout the distribution. The condition $\rho - 3P \geq 0$ is satisfied both at the centre and at the boundary. Using numerical procedure it is found that the specific models reported in Table-2.1 of Chapter-2 comply with the requirement $\rho - 3P \geq 0$, throughout their region of validity.

Substituting the values of $\rho, P,$ and $\frac{1}{F}\frac{dF}{dx}$ in (3.7) one gets

$$\frac{dP}{d\rho} = \left\{\frac{x+1}{x+5}\right\}\left\{\frac{r(x)[l(b)-l(x)]+s(x)}{r(x)[l(b)-l(x)]+s(x)+q(x)}\right\}$$
$$\times\left\{\frac{r(x)[l(b)-l(x)](x+2)+s(x)(x+2)+q(x)}{r(x)[l(b)-l(x)](x+2)+s(x)(x+2)+q(x)(x+2)}\right\}. \tag{3.19}$$

It is apparent from this equation that $\frac{dP}{d\rho} < 1$, and subsequently the fluid distribution satisfies causality requirements.

The expression for $\frac{P}{\rho}$ is given by

$$\frac{P}{\rho} = \left\{\frac{x+1}{x+3}\right\}\left\{\frac{r(x)[l(b)-l(x)]+s(x)-q(x)}{r(x)[l(b)-l(x)]+s(x)+q(x)}\right\}. \tag{3.20}$$

The expression (3.20) shows that the requirement $\frac{P}{\rho} < 1$ is fulfilled throughout. For the specific models of Table- 2.1 it has been verified by numerical procedures that $\frac{P}{\rho}$ decreases radially outward.

The adiabatic index γ has expression

$$\gamma = \left\{\frac{2}{(x+2)(x+5)}\right\}$$

$$\times\left[\frac{\{r(x)[l(b)-l(x)](x+2)+s(x)(x+2)+q(x)\}^2\{r(x)[l(b)-l(x)]+s(x)\}}{\left(\{r(x)[l(b)-l(x)]+s(x)\}^2-\{q(x)\}^2\right)\left(r(x)[l(b)-l(x)]+s(x)+q(x)\right)}\right] \tag{3.21}$$

Owing to the complexity of the expression (3.21), it is difficult to show analytically that $\gamma>1$. However, for the models of the Table-2.1, it is found, using numerical methods, that $\gamma>1$ throughout.

2. DYNAMIC STABILITY

A static solution of Einstein's field equations representing the space-time of a spherical fluid distribution in equilibrium will be appropriate for describing interior space-time of a superdense star in equilibrium if it is stable for radial modes of pulsation. Chandrashekhar (1964) gave a method for studying dynamic stability of a spherically symmetric distribution of matter under small radial adiabatic perturbations in the context of General Relativity and determined a sufficient condition for stability of a static spherical fluid distribution. A normal mode of radial oscillation for an equilibrium configuration $\delta r = \xi(r)\exp(i\omega t)$ is stable when the frequency of oscillation ω is real and unstable when ω is imaginary. Chandrashekhar's (1964) pulsation equation for equilibrium configurations of fluids described by metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\alpha(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{3.22}$$

is

$$\omega^2 \int_{centre}^{boundary} \exp\left(\frac{3\alpha + \nu}{2}\right)(\rho + P)\frac{u^2}{r^2}dr = \int_{centre}^{boundary} \exp\left(\frac{3\nu + \alpha}{2}\right)\left(\frac{\rho + P}{r^2}\right)$$

$$\times\left\{\left[-\frac{2}{r^2}\frac{d\nu}{dr} - \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2 + 8\pi\exp(\alpha)\right]u^2 + \frac{dP}{d\rho}\left(\frac{du}{dr}\right)^2\right\}dr \ , \qquad (3.23)$$

where

$$u = \xi r^2 e^{-\nu/2} . \qquad (3.24a)$$

For pseudo spheroidal space-time of (3.22)

$$e^{\alpha} = \frac{1 + 2\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}} . \qquad (3.24b)$$

The condition to be satisfied at the boundary $r = a$ of the distribution is that the Lagrangian change in pressure ΔP should vanish.

That is

$$\Delta P = \left[-e^{\nu/2}\frac{\gamma P}{r^2}\frac{du}{dr}\right]_{r=a} = 0 , \qquad (3.25)$$

where γ denotes the adiabatic index of matter. Therefore we must have for stability

$$\frac{du}{dr} = 0 , \qquad (3.26)$$

at $r = a$.

We follow the method of Bardeen, Thorne and Meltzer (1966) as used by Knutsen (1988) to investigate the stability of Vaidya-Tikekar models. We choose

$$u = R^3 x^{3/2}\left(1 + a_1 x + b_1 x^2 + \ldots\ldots\right) \qquad (3.27)$$

as trial function where $x = 2\frac{r^2}{R^2}$.

The condition (3.26) implies

$$3 + 5a_1 b + 7b_1 b^2 + \ldots \qquad = 0 \tag{3.28a}$$

where

$$b = 2\frac{a^2}{R^2}. \tag{3.28b}$$

We write the pulsation equation (3.23) for the metric (3.22) in the form

$$\omega^2 \int_{centre}^{boundary} \exp\left(\frac{3\alpha + \nu}{2}\right)(\rho + P)\frac{u^2}{r^2}dr = \int_0^{b} R_1 R_2 \left[\left(R_3 + R_4 + R_5\right)R_6 + R_7 R_8\right]dx, \tag{3.29}$$

where

$$R_1 = e^{\frac{3\nu+\alpha}{2}} = \left\{C_1\sqrt{x+2} + C_2\left[\sqrt{x+2}\ln\left(\sqrt{x+1} + \sqrt{x+2}\right) - \sqrt{x+1}\right]\right\}^3 \sqrt{\frac{2(x+1)}{x+2}},$$

$$R_2 = \rho + P = \frac{1}{4\pi(x+1)^2}\left\{\frac{r(x)[l(b) - l(x)](x+2) + s(x)(x+2) + q(x)}{r(x)[l(b) - l(x)] + s(x) + q(x)}\right\},$$

$$R_3 = -\frac{2}{r}\frac{d\nu}{dr} = -\frac{2}{r}\frac{d\nu}{dx}\frac{dx}{dr} = -\frac{16}{R^2}\frac{1}{F}\frac{dF}{dx}.$$

$$= -\frac{8}{R^2}\left\{\frac{r(x)[l(b) - l(x)] + s(x)}{(x+2)\{r(x)[l(b) - l(x)] + s(x) + q(x)\}}\right\}$$

$$R_4 = -\frac{1}{4}\left(\frac{dv}{dr}\right)^2 = -\frac{1}{4}\left(\frac{dv}{dx}\frac{dx}{dr}\right)^2 = -\frac{1}{4}\left[2\frac{dF}{dx}.\frac{1}{F}\right]^2\left(\frac{dx}{dr}\right)^2$$

$$= -\frac{2x}{R^2(x+2)^2}\left\{\frac{r(x)[l(b)-l(x)]+s(x)}{r(x)[l(b)-l(x)]+s(x)+q(x)}\right\}^2 ,$$

$$R_5 = 8\pi Pe^{\alpha} = \frac{2(x+1)}{R^2(x+2)(x+1)}\left\{\frac{r(x)[l(b)-l(x)]+s(x)-q(x)}{r(x)[l(b)-l(x)]+s(x)+q(x)}\right\},$$

$$R_6 = \frac{u^2}{r^2} = 2R^4x^2\left(1+a_1x+b_1x^2+\ldots\right)^2,$$

$$R_7 = \frac{dP}{d\rho} = \left\{\frac{x+1}{x+5}\right\}\left\{\frac{r(x)[l(b)-l(x)]+s(x)}{r(x)[l(b)-l(x)]+s(x)+q(x)}\right\}$$

$$\times\left\{\frac{r(x)[l(b)-l(x)](x+2)+s(x)(x+2)+q(x)}{r(x)[l(b)-l(x)](x+2)+s(x)(x+2)+q(x)(x+2)}\right\} ,$$

The equation (3.27) gives

$$\frac{du}{dr} = \frac{du}{dx}.\frac{dx}{dr} = \sqrt{2}xR^2\left(3+5a_1x+7b_1x^2+\ldots\right),$$

so that

$$R_8 = \frac{1}{r^2}\left(\frac{du}{dr}\right)^2 = 4R^2x\left(3+ba_1x+7b_1x^2+\ldots\right)^2.$$

The integral I, on the right hand side of equation (3.29) can be numerically evaluated for different specific values of b for different choices small, large, positive or negative values of the constants a_1 and b_1. Our computations obtain positive values for I for $0.243 \leq b \leq 1.146$ i.e. $.3 \leq \lambda \leq .7$ for different choices small, large, positive or negative values of the constants a_1 and b_1.

We have reported in Table - 3.1 the values of I, evaluated numerically for certain specific choices of a_1 and b_1 for the model with $\lambda = 0.4$ of Table-2.1.

This analysis indicates that these models with $.3 \leq \lambda \leq .7$ will be stable for radial modes of pulsation. The static space-time (2.7) with pseudo spheroidal geometry for its spatial sections t = constant thus admits the possibilities of describing space-time of superdense stars in equilibrium.

TABLE 3.1

The values of the integral, on the right hand side of equation (3.29) for a model with λ = 0.4 of Table-2.1, for specific choices of constants a_1 and b_1.

a_1	b_1	Integral
0.000	-0.717	1.476
-0.776	0.000	0.536
-0.858	1.000	0.084
1.000	-1.641	3.447
5.000×10^2	-4.627×10^2	1.076×10^5
-5.410×10^2	5.000×10^2	1.246×10^5
1.000×10^5	-9.240×10^4	4.272×10^9
-1.000×10^5	9.240×10^4	4.272×10^9