CHAPTER 2

FLUID DISTRIBUTIONS ON PSEUDO SPHEROIDAL SPACE-TIMES - I

1. INTRODUCTION

General Relativity establishes relationship between matter energy and the geometry of the physical 3-space. The physical 3-space associated with the Schwarzschild interior metric representing the gravitational field within a spherically symmetric distribution of homogeneous perfect fluid in equilibrium, or Einstein's metric representing static model of the universe, or de Sitter's and Robertson-Walker metrics representing the models for expanding universe, have the geometry of 3-sphere. Following this observation Vaidya and Tikekar (1982) considered the space-times whose associated 3-spaces have the geometry of 3-spheroid and showed that such space-times are appropriate to describe relativistic models of superdense spherical distribution of matter in equilibrium such as neutron stars. However there are many other exact solutions (Kramer, Stephani, MacCallum and Herlt-1980) which satisfy the required physical conditions yet obscure its explicit 3-space geometry. So it is highly pertinent to undertake study of space-times whose 3-spaces have a definite geometry, without sacrificing the relevant physical conditions. Keeping this in view we have investigated the gravitational significance of space-times whose t=constant sections have the geometry of a 3-pseudo spheroid.

The form of the space-time metric and some of its general features are discussed in Section-2. Einstein's field equations, assuming physical content of the space-time to be in the form of perfect fluid are written in Section-3. An out line of the scheme for numerical estimation of different physical parameters is described in Section-4 and a new exact closed form solution is obtained in Section-5.

2. PSEUDO SPHEROIDAL SPACE-TIME METRIC.

A 3- pseudo spheroid immersed in the four- dimensional Euclidean space with metric,

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2, \tag{2.1}$$

will have the Cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1, \tag{2.2}$$

where b and R are constants.

The sections x=constant, y=constant and z=constant of (2.2) represent respectively hyperboloids of two sheets while the sections w=constant are spheres of real radius when $w^2 > b^2$ and of imaginary radius when $w^2 < b^2$ and therefore we call it 3-pseudo spheroid.

On introducing the parametrization

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \\ w &= b\sqrt{1+\frac{r^2}{R^2}}, \end{aligned} \tag{2.3}$$

the metric (2.1) on pseudo spheroid assumes the form

$$d\sigma^2 = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{2.4}$$

where

$$K = 1 + \frac{b^2}{R^2}. \tag{2.5}$$

The form of the metric (2.4) shows that the pseudo spheroidal 3-space is spherically symmetric. It is further regular and has $K > 1$. Its geometry is governed by two curvature parameters R and K. It is flat when $K = 1$ and degenerates into open hyperboloid when $K = 0$.

The metric (2.4) has the following interesting behaviour.

As $r \to \infty$, it degenerates into

$$d\sigma^2 = K dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{2.6}$$

the metric of 3-space associated with the minimally curved space-times studied by Dadhich (1997).

We will consider the space-time with metric

$$ds^2 = e^{\nu(r)} dt^2 - d\sigma^2 \quad ,$$

$$\text{i.e.} \quad ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \tag{2.7a}$$

with

$$e^{\lambda} = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}}. \tag{2.7b}$$

The physical space obtained as t = constant section of (2.7) has the geometry of a 3-pseudo spheroid immersed in a 4-dimensional Euclidean space.

The space-time metric (2.7) is the pseudo spheroidal counterpart of the spheroidal space-time metric obtained by Vaidya and Tikekar (1982) for which

$$e^{\lambda} = \frac{1 - K\dfrac{r^2}{R^2}}{1 - \dfrac{r^2}{R^2}}.$$

with prescription $K < 1$. The spheroidal space-time of Vaidya and Tikekar is singular at $r = R$ and is limited to $r < R$ The pseudo-spheroidal space-time of (2.7) however is regular every where for all r. We shall study the nature of matter distribution in the form of perfect fluid on the background of the space-time of metric (2.7).

3. MATTER DISTRIBUTION

If the physical content of the space-time of the metric (2.7) is a spherical distribution of perfect fluid, it should satisfy Einstein's field equations

$$\Re_i^j - \frac{1}{2}\Re\delta_i^j = -\frac{8\pi G}{c^2}T_i^j \tag{2.8}$$

where $\Re_i^j$ denotes the Ricci tensor and $\Re$ the Ricci scalar of the space-time, expressions for which are given in an Appendix. Equations (2.8) form a system of ten non-linear partial differential equations of second order, connecting the metric variables with the physical parameters, such as matter density and fluid pressure. T_{ij} represents the energy momentum tensor describing the physical content of the space-time.

We take the spherical distribution of matter in the form of perfect fluid and investigate the gravitational significance with background metric in the form given by equation (2.7).

The energy-momentum tensor for a perfect fluid is

$$T_i^j = \left(\rho + \frac{P}{c^2}\right)u_i u^j - \frac{P}{c^2}\delta_i^j . \qquad (2.9)$$

Here ρ, P and u^i respectively denote the matter density, fluid pressure and the unit time like 4-velocity field of the fluid. For fluids in equilibrium

$$u^i = \left(e^{-\frac{\nu}{2}}, 0, 0, 0\right). \qquad (2.10)$$

The Einstein's field equations (2.8) for the space-time metric (2.7) subsequently lead to the following set of equations (Appendix)

$$-\frac{8\pi G}{c^2}T_0^0 = -\frac{8\pi G}{c^2}\rho. = \frac{-3(K-1)}{R^2}.\frac{1+\frac{K}{3}\frac{r^2}{R^2}}{\left(1+K\frac{r^2}{R^2}\right)^2}, \qquad (2.11)$$

$$-\frac{8\pi G}{c^2}T_1^1 = \frac{8\pi G}{c^2}.\frac{P}{c^2} = \frac{\left(1+\frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{(K-1)}{R^2}}{1+K\frac{r^2}{R^2}}, \qquad (2.12)$$

$$-\frac{8\pi G}{c^2}T_2^2 = \frac{8\pi G}{c^2}\frac{P}{c^2} = \frac{1+\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r}\right) - \frac{r(K-1)\left(\frac{\nu'}{2}+\frac{1}{r}\right)}{R^2\left(1+K\frac{r^2}{R^2}\right)^2}, \qquad (2.13)$$

$$-\frac{8\pi G}{c^2}T_3^3 = -\frac{8\pi G}{c^2}T_2^2.$$

Here and in what follows an overhead prime indicates a differentiation with respect to r.

4. LAW OF VARIATION OF DENSITY

Equations (2.11) through (2.13) form a system of three equations in three unknowns ρ, P, ν. The equation (2.11) determines the matter density ρ in terms of the curvature parameters R and K and hence constitutes the law governing its variation throughout the configuration. Differentiating (2.11) with respect to r, we get

$$\frac{8\pi G}{c^2}\frac{d\rho}{dr} = -\frac{2K(K-1)r}{R^2}\cdot\frac{\left(5+K\frac{r^2}{R^2}\right)}{R^2\left(1+K\frac{r^2}{R^2}\right)^3}. \tag{2.14}$$

Since $K > 1$, the density gradient is negative and hence the matter density of the distribution is a decreasing function of r. It has its maximum value

$$\frac{8\pi G}{c^2}\rho(o) = \frac{3(K-1)}{R^2} \tag{2.15}$$

at the centre and at the boundary $r=a$, attains the value

$$\frac{8\pi G}{c^2}\rho(a) = \frac{3(K-1)\left(1+\frac{K}{3}\frac{a^2}{R^2}\right)}{R^2\left(1+K\frac{a^2}{R^2}\right)^2}. \tag{2.16}$$

We introduce the parameter

$$m(r) = \frac{4\pi G}{c^2}\int_0^r \xi^2 \rho(\xi)d\xi \tag{2.17}$$

associated with the mass content of the matter within the spherical region of radius r. Using expression (2.11) in (2.17) we find

$$m(r) = \frac{(K-1)r^3}{2R^2\left(1+K\frac{r^2}{R^2}\right)}. \tag{2.18}$$

Vaidya and Tikekar (1982) have developed a scheme to determine the total mass and size of the matter content of a fluid sphere on spheroidal space-time. The scheme can be extended to cover the case of pseudo spheroidal space-times case as well. Accordingly we introduce the density variation parameter

$$\lambda = \frac{\rho(a)}{\rho(o)} = \frac{1 + \frac{K}{3}\frac{a^2}{R^2}}{\left(1 + K\frac{a^2}{R^2}\right)^2}, \tag{2.19}$$

representing the ratio of the density of matter at the surface $r=a$ to that at the centre $r=0$. Since $\rho(r)$ is a decreasing function of r, $\lambda < 1$.

Using (2.19) in (2.15) we get

$$R^2 = \frac{3(K-1)c^2\lambda}{8\pi G\rho(a)} \quad . \tag{2.20}$$

Equation (2.19) determines $\frac{a^2}{R^2}$ in terms of K and λ as

$$\frac{a^2}{R^2} = \frac{1 - 6\lambda + \sqrt{24\lambda + 1}}{6\lambda K}. \tag{2.21}$$

The algebraic root assigning negative values to $\frac{a^2}{R^2}$ is rejected to ensure that a is real.

Equation (2.20) determines R in terms of λ , $\rho(a)$ and K. Then equation (2.21) determines the boundary radius a of the distribution. Subsequently the mass of the distribution enclosed within a radius $r = a$ can be determined using

$$m(a) = \frac{(K-1)a^3}{2R^2\left(1 + \frac{Ka^2}{R^2}\right)^2} \tag{2.22}$$

which is obtained by putting $r = a$ in equation (2.18).

It is thus seen that the knowledge of the curvature of the physical -space is sufficient to form estimates about the size and the mass of the star model. The verification of the physical conditions like $P \geq 0$, $\rho - 3\frac{P}{c^2} \geq 0$ throughout the distribution is possible only when the expression for pressure is obtained by solving (2.12) and (2.13). This is done in the next section.

5. SOLUTION OF FIELD EQUATIONS.

The consistency condition $T_1^1 = T_2^2$ leads to

$$\left(1+\frac{r^2}{R^2}\right)\left(1+\frac{Kr^2}{R^2}\right)\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right)-\frac{K-1}{R^2}-\frac{(K-1)r\nu'}{2R^2}+\frac{K-1}{R^2}\left(1+\frac{Kr^2}{R^2}\right)=0. \tag{2.23}$$

This is a non-linear equation in ν. We choose new independent and dependent variables z and F defined by

$$z^2 = 1 + \frac{r^2}{R^2}, \tag{2.24a}$$

$$e^{\nu/2} = F, \tag{2.24b}$$

so that the equation (2.23) transforms into the linear differential equation

$$\left(1 - K + Kz^2\right)\frac{d^2F}{dz^2} - Kz\frac{dF}{dz} + K(K-1)F = 0. \tag{2.25}$$

This equation has the same form as the equation obtained by Tikekar (1990) in a spheroidal space-time for which $K < 1$. Since $K > 1$ for pseudo spheroidal space-times, the equation (2.25) is different. Further if we choose a new independent variable u defined by

$$u^2 = \frac{K}{K-1} z^2 \tag{2.26}$$

equation (2.25) assumes the simple form

$$\left(1-u^2\right)\frac{d^2F}{du^2} + u\frac{dF}{du} - (K-1)F = 0. \tag{2.27}$$

The points $u = \pm 1$ are regular singular points and all other points are regular for this linear equation. We seek a series solution in the form $F = \sum_{n=0}^{\infty} C_n u^n$. Substitution then yields the recurrence relation in the form

$$(n+2)(n+1)C_{n+2} = \left(n^2 - 2n + K - 1\right)C_n.$$

for coefficients C_n.

If $n^2 - 2n + K - 1 = 0$ *i.e.* $n = 1 \pm \sqrt{2-K}$ admits integral values of n as solutions either of the two sets of coefficients $\left(C_0, C_2, \ldots\right)$, $\left(C_1, C_3, \ldots\right)$ contains finite number of elements and corresponding term in the solution constitutes a finite polynomial solution of (2.27). We find that n is a positive integer only when K=2. Subsequently F=$F_1 = C_1 u$ is a finite polynomial solution of (2.23). The linear second order differential equation (2.27) is expected to have two linearly independent solutions. The other linearly independent solution will be obtained by using method of variation of parameters. Assuming $F_2 = V(u)u$ as the form of the other linearly independent solution. Substitution in (2.27) leads to the differential equation

$$u\left(1-u^2\right)\frac{d^2V}{du^2} + \left(2-u^2\right)\frac{dV}{du} = 0$$

for V.

This ordinary differential equation admits the solution

$$V = \left[\ln\left(u + \sqrt{u^2 - 1}\right) - \frac{\sqrt{u^2 - 1}}{u} \right].$$

The general solution of the equation (2.27) accordingly is realized in the form

$$F = c_1 u + c_2 \left[u \ln\left(u + \sqrt{u^2 - 1}\right) - \sqrt{u^2 - 1} \right] \qquad (2.28)$$

The space-time metric for this solution reads

$$ds^2 = \left\{ A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}} \ln\left(\sqrt{2}\sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + \frac{2r^2}{R^2}} \right) - \frac{1}{\sqrt{2}}\sqrt{1 + \frac{2r^2}{R^2}} \right] \right\}^2 dt^2$$

$$- \frac{1 + \frac{2r^2}{R^2}}{1 + \frac{r^2}{R^2}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 . \qquad (2.29)$$

The matter density and fluid pressure for the distribution with space-time metric (2.29) have expressions

$$\frac{8\pi G}{c^2} \rho = \frac{3}{R^2} \frac{\left(1 + \frac{2}{3}\frac{r^2}{R^2}\right)}{\left(1 + 2\frac{r^2}{R^2}\right)^2} , \qquad (2.30)$$

$$\frac{8\pi G}{c^4} P = \frac{A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}} \ln\left(\sqrt{2}\sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + \frac{2r^2}{R^2}} \right) + \frac{1}{\sqrt{2}}\sqrt{1 + \frac{2r^2}{R^2}} \right]}{R^2\left(1 + \frac{2r^2}{R^2}\right)\left\{ A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}} \ln\left(\sqrt{2}\sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + \frac{2r^2}{R^2}} \right) - \frac{1}{\sqrt{2}}\sqrt{1 + \frac{2r^2}{R^2}} \right] \right\}}$$

(2.31)

6. MATCHING WITH SCHWARZSCHILD EXTERIOR METRIC

According to Birkoff's theorem Schwarzschild solution is the only spherically symmetric asymptotically flat solution of Einstein's vacuum field equations. This implies that even a radially pulsating or collapsing star will have a static extetior metric. Therefore if the spherical distribution of matter is to be of finite extent of radius a, the metric (2.29) should continuously match with Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2.32}$$

on the boundary $r = a$ where the pressure is expected to vanish. The continuity of the metric coefficients across $r = a$ requires

$$e^{-\lambda(a)} = 1 - \frac{2m}{a} \tag{2.33}$$

and

$$e^{\frac{\nu(a)}{2}} = \sqrt{1 - \frac{2m}{a}} = A\sqrt{1 + \frac{a^2}{R^2}} + B\left[\sqrt{1 + \frac{a^2}{R^2}} \ln\left(\sqrt{2}\sqrt{1 + \frac{a^2}{R^a}} + \sqrt{1 + \frac{2a^2}{R^2}}\right) - \frac{1}{\sqrt{2}}\sqrt{1 + \frac{2a^2}{R^2}}\right] \tag{2.34}$$

The continuity of pressure across $r = a$ determines the boundary radius a as per the equation

$$A\sqrt{1 + \frac{a^2}{R^2}} + B\left[\sqrt{1 + \frac{a^2}{R^2}} \ln\left(\sqrt{2}\sqrt{1 + \frac{a^2}{R^2}} + \sqrt{1 + \frac{2a^2}{R^2}}\right) + \frac{1}{\sqrt{2}}\sqrt{1 + \frac{2a^2}{R^2}}\right] = 0. \tag{2.35}$$

The arbitrary constants A and B are determined by (2.34) and (2.35) as

$$A = \frac{\sqrt{1+\frac{a^2}{R^2}}\ \ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}}+\sqrt{1+\frac{2a^2}{R^2}}\right)+\frac{1}{\sqrt{2}}\sqrt{1+\frac{2a^2}{R^2}}}{\sqrt{2}\left(1+\frac{2a^2}{R^2}\right)} \tag{2.36}$$

$$B = \frac{-\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\left(1+\frac{2a^2}{R^2}\right)} \tag{2.37}$$

Equation (2.33) determines the mass of the distribution enclosed within a radius $r=a$ as

$$m = \frac{a^3}{2R^2\left(1+\frac{2a^2}{R^2}\right)} \tag{2.38}$$

which is the same as that obtained from (2.18) by putting $r = a$ and $K = 2$.

7. SUPERDENSE STAR

One of the important problems of Relativistic Astrophysics is the study of various stages — such as birth, growth, equilibrium and collapse—in the evolution of a star. In general a star exists in a state of hydrostatic equilibrium in which the attractive force of gravitation is counterbalanced by repulsive hydrostatic pressure. When the equilibrium is lost a star begins to contract under its own gravitational field. There are various stages in which equilibrium could be attained by a collapsing star resulting in formation of superdense stars. The models that we have obtained here are appropriate to describe the interior space-time of such superdense stars in equilibrium with densities in the range of 10^{14}-10^{16} gm/cm^3.

In accordance with the method used by Vaidya andTikekar (1982) we take the density at the boundary of the star as $\rho(a) = 2 \times 10^{14} gm / cm^3$, the value given by Rees,Ruffini and Wheeler (1975) and used by Vaidya and Tikekar (1982). Choosing different values for density variation parameter λ we determine corresponding values of curvature parameter R using (2.20) and the boundary radii a of the stars using (2.21). Then the mass of the distribution is determined by (2.38). The values of the arbitrary constants A and B follow from (2.36) and (2.37). The mass m obtained is in units of kms.

The numerical estimates listed in the Table-2.1 provide the following observations :

(i) A typical neutron star will have an approximate radius of 10 kms. The models with $\lambda \geq 0.7$ in the table have their equilibrium radii much smaller than that of a neutron star.

(ii) The models presented here take lesser values for radii a and total mass m, than the corresponding values given by Vaidya and Tikekar(1982) for the curvature parameter $K = -2$ using speroidal space-time.

Since the equation of state is not specified in the above analysis we have to examine the physical requirements carefully. This is done in the next chapter.

TABLE-2.1

Masses and equilibrium radii of superdense star models corresponding to K=2 and

$\rho(a) = 2 \times 10^{14} gm / cm^3$.

λ	*a/R*	*m/a*	*R(km)*	*a(km)*	*m(km)*	$\mathfrak{m}/M_o$	*A*	*B*
0.90	0.181	0.010	26.928	04.875	0.075	0.050	1.094	-0.674
0.80	0.269	0.031	25.388	06.828	0.215	0.146	1.062	-0.639
0.70	0.348	0.048	23.748	08.271	0.403	0.273	1.027	-0.602
0.60	0.428	0.067	21.987	09.429	0.634	0.429	0.987	-0.562
0.50	0.517	0.087	20.071	10.382	0.904	0.613	0.943	-0.518
0.40	0.621	0.109	17.952	11.162	1.216	0.825	0.890	-0.469
0.30	0.757	0.133	15.547	11.771	1.571	1.065	0.826	-0.413
0.20	0.959	0.162	12.694	12.176	1.972	1.337	0.742	-0.345
0.10	1.367	0.197	08.976	12.274	2.421	1.641	0.613	-0.252
0.05	1.907	0.219	06.347	12.107	2.66	1.804	0.501	-0.184

1 M_o =1.475 km.