CHAPTER 1

INTRODUCTION

General Relativity theory is the most widely accepted theory of gravitation today. Among the four forces of interaction, gravitation is the dominant interaction on macroscopic scale. It extends the concept of space- time as a 4-dimensional continuum of special relativity by assigning to it a Riemannian non-flat geometry describable by a metric:

$$ds^2 = g_{ij} dx^i dx^j \tag{1.1}$$

of signature + - - - [It is conventional to choose signature - + + + also]. The metric coefficients g_{ij} which in general are functions of coordinates, play the role of gravitational potential and are governed by the Einstein's field equations

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -\frac{8\pi G}{c^2} T_{ij} \tag{1.2}$$

where $\Re_{ij}$ is the Ricci tensor, $\Re$ the Ricci scalar of the space-time and T_{ij} denotes the energy-momentum tensor which contains all the information about the physical content of the space-time.

Einstein's field equations comprise of a set of ten second order non-linear partial differential equations connecting the metric and physical variables. Thus the concept of gravitational force in the classical Newtonian theory of gravitation is understood as a geometric phenomenon incorporated in the curvatures of a Riemannian space-time, in General Relativity.

Though the justification for accepting General Relativity comes from experimental observations limited to the solar system only, it has widely ranging applications in Cosmology and Astrophysics, the areas wherein the gravitational interactions play a dominant role .Its applications in Cosmology have lead us to believe that we are living in an expanding universe which was born out of a big bang. Its applications in Astrophysics have thrown light on the final outcome of stellar structures collapsing under self gravitation. Nothing is known with certainty about how the collapse ,that is contraction under self gravity ,proceeds. However General Relativity theory predicts that the final outcome of the collapse is a superdense star of the type — white dwarf,neutron star or a black hole. It is in these respective areas of Cosmology and Astrophysics involving strong gravitational interactions that the relativity theory finds its applications as the most useful theory of gravitation. The observations in Cosmology and Astrophysics pertaining to above situations enhance the acceptability of General Relativity.

One of the most important problems in General Relativity is to obtain exact solutions of Einstein's field equations. Exact solutions play significant role in the development of understanding of many areas of gravitational significance, such as the study of effects of gravitation in solar system, gravitational radiation theory, black holes and gravitational collapse, Cosmology and dynamics of early universe. Another reason for its importance is that obtaining exact solutions is difficult due to the non-linear nature of Einstein's field equations. This may be the reason why Schutz (1985) remarked :*'finding exact solutions of Einstein's field equations is an art which requires the successful combination of useful coordinates, simple geometry, good intuition and most case luck'*.

General Relativity theory has developed over a period of nine decades into a highly matured discipline. A large number of exact solutions of Einstein's field equations —of physical significance and also devoid of any physical significance — are known (Kramer, Stephani, MacCallum and Herlt (1980)). Accordingly the discovery of new solutions which have interesting physical applications or unusual geometrical properties or which illustrate some point of important mathematical relevance will only provide deeper insights into the structure of General Relativity which is clothed in a set of highly non-linear partial differential equations. In this thesis we have reported some such new solutions of Einstein's field equations and discussed their geometrical and physical significance.

The physical 3-space associated with the Schwarzschild interior metric representing the gravitational field within a spherically symmetric distribution of homogeneous perfect fluid in equilibrium or Einstein's metric representing the static model of the universe, de Sitters and Robertson-Walker metrics representing the model for expanding universe, have the geometry of a 3-sphere — a closed 3-surface. Following this observation Vaidya and Tikekar (1982) investigated the physical significance of the space-time with the physical 3-space associated with them having the geometry of a 3-spheroid with cartesian equation

$$\frac{w^2}{b^2}+\frac{x^2+y^2+z^2}{R^2}=1 \tag{1.3}$$

representing a closed 3-surface immersed in a 4-dimensional Euclidean space with coordinates x, y, z and w. They showed that the space-time in the interior of highly compact relativistic objects such as superdense stars may have this kind of geometry, the departure from spherical geometry providing the law for variation of density in such objects.

Vaidya and Tikekar's investigation has promted us to undertake the study of space-times with the associated physical 3-space having the geometry of a pseudo 3- spheroid, a 3-surface with cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1 \quad , \tag{1.4}$$

immersed in a 4-dimentional Euclidean space with metric:

$$d\sigma^2 = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \quad , \tag{1.5}$$

where

$$K = 1 + \frac{b^2}{R^2}. \tag{1.6}$$

We call this 3-surface a 3-pseudo spheroid because the sections w = constant with $w^2 < b^2$ represent spheres of imaginary radius.

This thesis comprises of various aspects of physical and mathematical significance of relativistic space-times endowed with a pseudo spheroidal geometry for their accompanying physical spaces. A chapterwise summary of the results of our investigations is given below.

CHAPTER-2: Fluid Distributions on Pseudo spheroidal Space-times -I

Relativistic models of spherical stars in equilibrium are obtained by treating them as filled with spherical distributions of matter wherein the equilibrium is maintained by counterbalancing the inward force of self gravity by the outward force of fluid pressure.Gravitational significance of equilibrium models of this type of distributions with physical content in the form of perfect fluid have been investigated by many researchers,considering space-times with definite geometry or otherwise.We have investigated in this chapter the gravitational significance of space-times whose t=constant sections have the geometry of 3-pseudo spheroid and matter content in the form of perfect fluid. The form of the metric of pseudo spheroidal space-times is derived and geometrical features of the space-time are discussed. A new exact solution of Einstein's field equations is obtained. The space-time metric of this solution is joined smoothly with Schwarzschild exterior metric, which uniquely describes the space-time outside a massive sphere, across the boundary of the distribution. The suitability of this solution to describe the model of a spherical superdense star in equilibrium is explored (Tikekar and Thomas-1993,1998).

By assigning the value 2×10^{14} gm cm^{-3} for the matter density at the boundary, numerical estimates of mass and size are obtained for different values of the density variation parameter λ which is the ratio of the matter density at the boundary to density at the centre by adopting the scheme given by Vaidya and Tikekar (1982).

CHAPTER-3: Fluid Distributions on Pseudo spheroidal Space-times-II

The usual procedure of assigning an equation of state for the matter content of a fluid sphere in equilibrium was replaced in Chapter-2 by adopting the requirement of pseudo spheroidal geometry for its space-time while obtaining the solution of Einstein's field equations. Subsequently it is highly necessary to carefully examine the physical viability of the class of models obtained on the basis of this solution. We have shown in this chapter that the models based on this solution comply with the physical requirements such as positivity of matter density and pressure, energy conditions, fulfilment of causality conditions etc. Following the method of Bardeen,Thorne and Meltzer (1966) as used by Knutsen (1988) to investigate the stability of Vaidya-Tikekar models, we have examined the stability of these models using Chandrasekhar's pulsation equation and found that the models with $0.3 \leq \lambda \leq 0.7$ are stable. This analysis indicates that the space-time with pseudo spheroidal geometry for its spatial sections t =constant are suitable for describing interiors of superdense fluid stars in equilibrium (Tikekar and Thomas-1998).

CHAPTER - 4: Charged Dust Spheres on Pseudo spheroidal Space-times.

A spherically symmetric distribution of matter in the form of perfact fluid keeps its equilibrium by counterbalancing the self gravitation by the repulsive radial fluid pressure. If the matter content of the distribution is in the form of dust, there is no force to counter the collapse. In such cases the collapse may be countered by the electrostatic repulsive force if the matter is accompanied by some charge. Bonnor (1960, 65), De and Raychoudhary (1968), Cooperstock and de la Cruz (1978), Bonnor and Wikramsuriya (1975), Tikekar (1984) have examined this possibility. We have shown in this Chapter that the pseudo spheroidal space-time can be used as a background space-time for describing a distribution of charged dust in equilibrium. Tikekar (1984) has shown that spheroidal space-times are useful in describing spherical distribution of charged dust in equilibrium, and their boundary radius is restricted by $a < 0.894\ R$, where R is the curvature parameter. It is shown that for charged dust distributions in equilibrium on pseudo spheroidal space-time, the boundary radius does not have such a restriction.

CHAPTER-5: Charged Fluid Spheres on Pseudo spheroidal Space-times.

The study of gravitational significance of distributions of charged perfect fluids in equilibrium has received considerable attention in the recent past. Krori and Barua (1975), GJG Juvenicus (1976), Pant and Shah (1979), Chang (1983), Tikekar (1984, 1985), Patel and Pandya (1983), Patel and Kopper (1987), Patel and Mehta (1995) have reported solutions of Einstein's field equations representing space-times of spherical static charged fluid distributions.This problem is relevant for two reasons. The Coulombian force of repulsion contributes additonal force to the fluid pressure against the inward force of self gravitation and electromagnetic field energy also acts as a source producing curvature in space-time.

In this chapter we have studied coupled Einstein-Maxwell equations on the background of pseudo spheroidal space-time representing space-time inside a charged perfect fluid in equilibrium and obtained a solution which is a charged analogue of the solution obtained in Chapter-2. When the curvature parameter K=2, the charge disappears and the solution reduces to the solution (2.29) obtained in Chapter-2. Numerical estimates of mass, size and charge are obtained for different values of the density variation parameter λ. Consequences of various physical reqirements are examined using analytical as well as numerical methods.

CHAPTER-6: Einstein Clusters on Pseudo spheroidal Space-times.

Einstein showed that spherical material systems comprising of dust particles in motion along non-intersecting circular trajectories around a common centre also constitute non-collapsing spherical distributions of matter in equilibrium. Such systems are called as Einstein clusters. Gilbert (1954) and Hogan (1973) derived the expressions for energy-momentum tensor in curvature co-ordinates for such systems. Florides (1974) gave a general scheme for constructing solutions of Einsteins field equations for Einstein clusters. Following Florides scheme Patel (1984) has shown that static spheroidal space-times of Vaidya - Tikekar type are appropriate to describe the space-times of Einstein clusters. In this chapter we have studied Einstein Clusters on the background of pseudo spheroidal space-times and have shown that space-times with $K>3$ are suitable for representing Einstein clusters of finite extent (Tikekar and Thomas-1993).

CHAPTER-7: Anisotropic Fluid Distributions on Pseudo spheroidal Space-times.

Spherical distributions of matter in the form of perfect fluid in equilibrium is an idealization rather than a reality when we consider situations wherein high densities of matter are involved. Theoretical studies of Ruderman (1972) and Canuto (1974) have indicated that matter content of certain stellar objects may have anisotropy in pressure. It is expected that at least in certain density ranges matter may be anisotropic (Cosenza, Herrera, Esculpi and Written –1981). Several researchers have investigated solutions of Einstein's field equations for anisotropic matter distributions.

In this Chapter we discuss fluid distributions with anisotropic pressure on pseudo spheroidal space-time. A new exact solution for anisotropic fluid distribution is obtained by prescribing a suitable expression governing the anisotropy . A particular model with curvature parameter $K=2$, is studied in detail. For this model the density and radial pressure are found to be positive throughout the distribution. The positivity of tangential pressure imposes restriction on the boundary radius $a \geq \sqrt{2}R$. This in turn imposes restriction on the density variation parameter $\lambda \leq 0.093$ which means a high degree of density variation as one proceeds from centre to boundary. The solution with boundary radius $a = \sqrt{2}R$ is found to have the noteworthy feature that both radial and tangential pressures vanish at the boundary.

CHAPTER-8: Core-envelope Models on Pseudo spheroidal Space-times.

A spherical distribution of matter in equilibrium consisting of two regions– core and envelope – containing different fluid distributions have been extensively studied in the context of General Relativity (Hartle (1978), Iyer and Vishveshwara (1985), Durgapal and Gehlot (1969), Kopper and Patel (1987)).The models based on such solutions are called core-envelope models. A common feature of the core-envelope models reported in literature is that their core and envelope regions both contain distributions of perfect fluids. Core-envelope models incorporating anisotropic fluid distributions are seldom discussed. In this Chapter we have investigated this possibility and discussed a core-envelope model with core consisting of an anisotropic distribution of matter described by the metric of solution of Chapter-7, surrounded by an envelope of perfect fluid described by the metric of solution of Chapter-2. The physical plausibility of these models is examined. The core-envelope models for which density variation parameter $\lambda < 0.093$ are found to be physically plausible.

CHAPTER-9: Static Solutions of Einstein's Field Equations in Higher Dimensions on Pseudo spheroidal Space-times

Developments in superstring theories indicate the requirement of dimensions greater than four for the space-time manifold. Solutions of Einstein's field equations in higher dimensions are believed to be of physical relevance possibly at the extremely early times of the evolution of universe. Implications of the theories in higher dimensional space-times, in spite of the lack of observational evidence supporting their existence, have been investigated by several authors (Myer and Perry (1986), Xu Dianyan (1988), Iyer and Visveshwara (1989)).

In this chapter we have extended the formalism of 4-dimensional pseudo spheroidal space-time of Chapter-2, to higher dimensions and obtained a new class of solutions for Einstein's field equations for perfect fluids (Tikekar and Thomas-1995).

CHAPTER-10: Non-adiabatic Gravitational Collapse on Pseudo spheroidal Space-times.

A star heavier than a few solar masses passes in the state of gravitational collapse when it has exhausted all its nuclear fuel.The first theoretical model for studying gravitational collapse of spherical dust distribution was proposed by Oppenheimer and Volkoff (1939). Since then several studies aimed to understand various aspects of the gravitational collapse in the framework of General Relativity have been reported.(Oppenheimer and Snyder (1939), Misner and Sharp (1964), Berkenstein (1971), Santos (1985), de Oliveira, Santos and Kolassis (1985)).

We have investigated in this Chapter the gravitational collapse of a spherical distribution of perfect fluid accompanied by heat flux in the radial direction, on the background of a pseudo spheroidal space-time following the approach of Tikekar and Patel (1992). Various aspects of the collapse have been extensively studied using both analytical as well as numerical methods.

Thus we found that the space-times whose physical space having the geometry of a 3-pseudo spheroid are suitable for describing different aspects of strong gravitational fields of compact matter distributions.