CHAPTER 10

NON -ADIABATIC GRAVITATIONAL COLLAPSE ON PSEUDO SPHEROIDAL SPACE-TIME

1.INTRODUCTION:

We have discussed in the preceeding chapters spherical distributions of matter in equilibrium. When the thermonuclear sources of energy in a star are exhausted it begins to collapse due to the absence of any outward force to counterbalance the inward gravitational force. The final fate of such a massive collapsing star is a white dwarf, a neutron star or a black hole, depending upon the total mass of the configuration.

The first theoretical study of gravitational collapse in relativistic set up is due to Oppenheimer and Volkoff (1939). They showed that Einstein's field equations do not possess any static solutions for spherical distributions of cold neutrons if its total mass is greater than 0.7 solar mass. Oppenheimer and Snyder (1939) investigated the adiabatic gravitational collapse assuming the energy-momentum tensor as that of incoherent dust. They have shown that for this idealized case the total time of collapse for an observer comoving with the stellar matter is finite. Misner and Sharp (1964) have given relativistic equations for adiabatic , spherically symmetric gravitational collapse of matter distributions in the form of perfect fluid. Berkenstein (1971) has derived the equations of gravitational collapse of charged fluid distributions.

A realistic analysis of radiating spherical bodies associated with radial heat flux has been carried out by Santos (1985), de Oliveira, Santos and Kolassis (1985). Further details of these solutions have been discussed by de Oliveira, Pacheco and Santos (1986). Studies of dynamical instability of non-adiabatic collapse has been studied by Herrera, Denmat and

Santos (1989), Chan, Kichenassamy, Denmat and Santos (1989). de Oliveira and Sntos (1987) have obtained the equations governing non-adiabatic gravitational collapse of charged matter with radial heat flow. Tikekar and Patel (1992) have obtained the appropriate generalization of dynamical equations pertaining to collapse of charged anisotropic matter for spherical collapse obtained by de Oliveira and Santos. Gravitational collapse solutions with and without the presence of shear have been extensively studied by Glass (1981,1989).

Tikekar and Patel (1988) have studied the non-adiabatic gravitational collapse of spherical distributions of matter in the form of perfect fluid associated with radial heat flux on the background of spheroidal space-times and have also obtained exact radiating collapse solutions on a conformally flat space-time background (1991). Kramer (1991) has given a model describing a collapsing sphere with radial heat flow. Recently Bhui (1996) generalized the work of Santos (1985) on non-adiabatic collapse with heat flow to higher dimensional space-time.

In this chapter we have considered the non-adiabatic gravitational collapse of spherical distributions of matter accompanied by heat flux in the radial direction on a pseudo spheroidal space-time and discussed various aspects of the collapse.

2. THE INTERIOR SPACE-TIME.

Following Tikekar and Patel (1992) we write the metric of the interior space-time $M_{(i)}$ of the non-adiabatically collapsing fluid sphere as

$$ds^2_{(i)} = e^{\nu(r,t)}dt^2 - e^{\mu(t)}\left[e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] . \qquad (10.1)$$

When $\mu(t) = o$,the coordinates are the usual Schwarzschild coordinates.

The physical content of $M_{(i)}$ is assumed to be in the form of perfect fluid accompanied by heat flux along the radial direction with the energy-momentum tensor (Weinberg-1972)

$$T_i^j = (\rho + P)u_i u^j - P\delta_i^j + q^j u_i + q_i u^j \quad , \; u^i u_i = 1 \; , \tag{10.2}$$

where ρ is the matter density of the fluid , P is the isotropic pressure, u_i is the 4-velocity and q_i is the space-like radial heat flux vector .Adopting comoving coordinates we write

$$u^i = \left(e^{-\nu/2}, 0, 0, 0\right) \; , \tag{10.3}$$

and

$$q^i = (0, q, 0, 0). \tag{10.4}$$

The heat flux vector q^i is orthogonal to u^i with magnitude $q = q\ (r,t)$.

The Einstein's field equations

$$\Re_i^j - \frac{1}{2}\Re\delta_i^j = -8\pi T_i^j \; , \tag{10.5}$$

(with $G = c = 1$) subsequently provide the following system of four equations:

$$8\pi T_1^1 = -8\pi P = -e^{-(\mu+\lambda)}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{e^{-\mu}}{r^2} + e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right) \; , \tag{10.6}$$

$$8\pi T_2^2 = -8\pi P = -e^{-(\mu+\lambda)}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} - \frac{\nu'\lambda'}{4} - \frac{\lambda'}{2r}\right) + e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right) \; , \tag{10.7}$$

$$8\pi T_3^3 = 8\pi T_2^2 \; ,$$

$$8\pi T_0^0 = 8\pi\rho = e^{-(\mu+\lambda)}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{-\mu}}{r^2} + \frac{3}{4}e^{-\nu}\dot{\mu}^2 \; , \tag{10.8}$$

$$8\pi T_0^1 = 8\pi q e^{\nu/2} = -\frac{1}{2} e^{-(\mu+\lambda)} \dot{\mu} \nu' . \tag{10.9}$$

Here and in what follows an overhead dot denotes a differentiation with respect to t.

3. THE EXTERIOR SPACE-TIME.

The exterior space-time $M_{(e)}$ of the collapsing fluid sphere accompanied by heat flux is appropriately described by Vaidya (1951) metric

$$ds_{(e)}^2 = \left[1 - \frac{2m(v)}{y}\right] dv^2 + 2dydv - y^2 d\theta^2 - y^2 \sin^2\theta d\phi^2, \tag{10.10}$$

where $m(v)$ denotes the total mass enclosed within the spherical region of radius y.

The energy-momentum tensor in this region has expression

$$T_i^j = \varepsilon u_i u^j \;, \quad u_i u^i = 0 \;, \tag{10.11}$$

where ε denotes energy -density of radiation.

A time-like 3-hyper surface $\Sigma_{(b)}$ separates the interior from the exterior and this dividing hypersurface $\Sigma_{(b)}$ distinguishes the two space-time manifolds $M_{(i)}$ and $M_{(e)}$, both of which contain $\Sigma_{(b)}$ as a part of their boundaries.

The intrinsic metric on $\Sigma_{(b)}$ will be

$$ds_{(b)}^2 = d\tau^2 - R^2(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{10.12}$$

4. BOUNDARY CONDITIONS

We follow the method of Isrel (1966) in matching the interior with the exterior at the boundary $\Sigma_{(b)}$.

In order to have a unique intrinsic geometry at the boundary hypersurface $\Sigma_{(b)}$ both $g_{ij\,(i)}$ and $g_{ij\,(e)}$ must include the same intrinsic metric on $\Sigma_{(b)}$.

Therefore

$$ds^2_{(i)} = ds^2_{(e)} = ds^2_{(b)}. \tag{10.13}$$

This condition guarantees the continuity of metric coefficients across the boundary surface $\Sigma_{(b)}$.

Thus the second continuity condition impossed on $\Sigma_{(b)}$ is given by

$$\left[K_{ij}\right] = K_{ij\,(e)} - K_{ij\,(i)} = 0, \tag{10.14}$$

where K_{ij} are components of the extrensic curvature. This condition guarantees the continuity of the first derivatives of the metric coefficients $g_{ij\,(i)}$ and $g_{ij\,(e)}$ across $\Sigma_{(b)}$.

The components of the extrinsic curvature K_{ij} are given by Eisenhart(1947)

$$K_{ij} = -\eta_\alpha \frac{\partial^2 x^\alpha}{\partial \xi^i \partial \xi^j} - \eta_\alpha \Gamma^\alpha_{ab} \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j}, \tag{10.15}$$

where ξ^i are coordinates θ, ϕ, τ on $\Sigma_{(b)}$, x^α are the coordinates appropriate to $M_{(i)}$ and $M_{(e)}$ and $\eta_{\alpha(i)}$ and $\eta_{\alpha(e)}$ are unit normals to $\Sigma_{(b)}$ in $M_{(i)}$ and $M_{(e)}$ respectively.

In terms of the interior coordinates boundary surface $\Sigma_{(b)}$,which is the boundary of the interior material distribution ,will have the equation

$$f(r,t) = r - r_{(b)} \;, \tag{10.16}$$

where $r_{(b)}$ is a constant.

In terms of the exterior coordinates boundary surface $\Sigma_{(b)}$, is given by

$$f(y,\upsilon) = y - r_{(b)}(\upsilon)\,. \tag{10.17}$$

The unit space-like normals $\eta_{\alpha(i)}$ and $\eta_{\alpha(e)}$ to $\Sigma_{(b)}$ respectively in $M_{(i)}$ and $M_{(e)}$ are given by

$$\eta_{\alpha(i)} = \left(0, e^{\frac{\mu+\lambda}{2}}, 0, 0\right) \quad , \tag{10.18a}$$

$$\eta_{\alpha(e)} = \left(-\frac{\partial y}{\partial \tau}, \frac{\partial v}{\partial \tau}, 0, 0\right) . \tag{10.18b}$$

The components of $K_{ij(i)}$ and $K_{ij(e)}$ have the following expressions in the present case

$$K_{\tau\tau(i)} = \left[-e^{-\left(\frac{\mu+\lambda}{2}\right)} \frac{\nu'}{2}\right]_{(b)} \quad ,$$

$$K_{\theta\theta(i)} = \left[r e^{\frac{\mu-\lambda}{2}}\right]_{(b)} \quad , \tag{10.19}$$

$$K_{\phi\phi(i)} = \sin^2\theta \, K_{\theta\theta(i)} \quad ,$$

$$K_{ij(i)} = 0 \quad \text{for all} \quad i \neq j \; .$$

$$K_{\tau\tau(e)} = \left[\frac{\ddot{v}}{\dot{v}} - \frac{m}{y^2}\dot{v}\right]_{(b)} \quad ,$$

$$K_{\theta\theta(e)} = \left[y\left(1 - \frac{2m}{y}\right)\dot{v} + y\,\dot{v}\right]_{(b)} \quad , \tag{10.20}$$

$$K_{\phi\phi(e)} = \sin^2\theta \, K_{\theta\theta(e)} \quad ,$$

$$K_{ij(e)} = 0 \quad \text{for all} \quad i \neq j \quad .$$

The conditions (10.13) imply

$$r_{(b)} e^{\mu/2} = R(\tau) \quad ,$$

$$y_{(b)} = R(\tau) \quad , \tag{10.21}$$

$$\frac{dt}{d\tau} = \dot{t} = e^{-\nu/2} \quad ,$$

$$\left[\frac{1}{\dot{v}^2}\right]_{(b)} = \left[1 - \frac{2m}{y} + 2\frac{dy}{dv}\right]_{(b)}, \tag{10.22}$$

and the condition (10.15) in the light of (10.19) and (10.20) leads to

$$\left[-e^{-\left(\frac{\mu+\lambda}{2}\right)}\frac{\nu'}{2}\right]_{(b)} = \left[\frac{\ddot{v}}{\dot{v}} - \frac{m}{y^2}\dot{v}\right]_{(b)}, \tag{10.23}$$

$$\left[re^{\frac{\mu-\lambda}{2}}\right]_{(b)} = \left[y\dot{y} + y\left(1 - \frac{2m}{y}\right)\dot{v}\right]_{(b)}. \tag{10.24}$$

The total mass $m(v)$ of the collapsing fluid sphere follows from (10.22) and (10.24) as

$$m(v) = \frac{r_{(b)}e^{\mu/2}}{2}\left[1 - e^{-\lambda} + \frac{r^2 e^{\mu-\nu}}{4}\dot{\mu}^2\right]_{(b)}. \tag{10.25}$$

The field equation (10.6) assumes the simple form

$$8\pi P_{(b)} = \left[8\pi q e^{\frac{\mu+\lambda}{2}}\right]_{(b)}, \tag{10.26}$$

on using (10.22) and (10.23) in it.

This relation shows that the pressure at the boundary is directly related to the heat flux q at the boundary. The pressure at the boundary becomes zero only when there is no heat flux along the radial direction across the boundary.

The energy-density of radiation measured by an observer on $\Sigma_{(b)}$ with four velocity u^α is given by

$$e = u^\alpha u^\beta T_{\alpha\beta}, \tag{10.27}$$

where

$$u^\alpha = \left(\dot{v}, \dot{y}, 0, 0\right). \tag{10.28}$$

The Einstein's field equations in $M_{(e)}$ determine

$$8\pi T_{00} = -\frac{2}{y^2}\frac{dm}{dv}, \tag{10.29}$$

as the only surviving component of Einstein tensor.

Hence from equation (10.27) we obtain the energy-density due to radiation as

$$8\pi e = \left[-\frac{2}{y^2}\frac{dm}{dv}\dot{v}^2\right]_{(b)}. \tag{10.30}$$

The total luminosity for an observer at rest at infinity (Lindquist, Schwartz and Misner - 1965):

$$L_\infty = \lim_{y\to\infty,\frac{dy}{d\tau}=0} 4\pi y^2 e = -\frac{dm}{dv}, \tag{10.31}$$

where

$$L = 4\pi y^2 e, \tag{10.32}$$

denotes the luminosity observed on $\Sigma_{(b)}$.

The boundary red shift is given by

$$\frac{dv}{d\tau} = 1 + z_{(b)}. \tag{10.33}$$

The equation (10.30) can be rewritten in the form

$$-\frac{dm}{dv} = 4\pi y^2 e\frac{1}{\dot{v}^2}, \tag{10.34}$$

which on using (10.32) and (10.33) in (10.34), leads to

$$L_\infty = \frac{L}{\left[1 + z_{(b)}\right]^2}. \tag{10.35}$$

Equation (10.24) can be written in the form

$$\dot{v} = \left[\frac{re^{\frac{\mu-\lambda}{2}} - y\dot{y}}{y - 2m}\right]_{(b)}, \tag{10.36}$$

which in the light of (10.21) takes the form

$$\dot{\upsilon} = \left[\frac{re^{\frac{\mu-\lambda}{2}} - r^2 e^{\mu}\frac{\dot{\mu}}{2}}{re^{\mu/2} - 2m}\right]_{(b)} . \tag{10.37}$$

It is observed from equation (10.34) that $L_\infty \to 0$ when $\dot{\upsilon} \to \infty$. That is when $re^{\mu/2} \to 2m$. Thus when the collapsing star becomes a black hole i.e when $re^{\mu/2} = 2m$, the boundary redshift becomes infinity.

5. SOLUTION OF FIELD EQUATIONS

When $\mu(t) = 0$ and $\nu = \nu(r)$ in the space-time metric (10.1), we get the usual spherically symmetric static metric in Schwarzschild coordinates. If the matter content of the space-time is in the form of perfect fluid, then the field equations are

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} , \tag{10.38}$$

$$8\pi P_0 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} , \tag{10.39}$$

$$8\pi P_0 = e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} - \frac{\nu'\lambda'}{4} - \frac{\lambda'}{2r}\right) , \tag{10.40}$$

where ρ_0 and P_0 denote the proper density of fluid and isotropic pressure respectively. Using the above set of equations (10.38) through (10.40), we write the field equations (10.6) through (10.9) as

$$8\pi\rho = 8\pi\rho_0 e^{-\mu} + \frac{3}{4} e^{-\nu} \dot{\mu}^2 , \tag{10.41}$$

$$8\pi P = 8\pi P_0 e^{-\mu} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\upsilon}}{2}\right), \tag{10.42}$$

$$8\pi q = -\frac{1}{2} e^{-(\mu+\lambda)} e^{-\nu/2} \dot{\mu} \nu' . \tag{10.43}$$

The pressure isotropy equation $T_1^1 = T_2^2$ obtained from (10.6) and (10.7) is

$$\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} - \frac{\nu'}{2r} - \frac{\lambda'}{2r} - \frac{1}{r^2} + \frac{e^{\lambda}}{r^2} = 0 , \tag{10.44}$$

which is identical with the pressure isotropy equation in the static case.

The boundary condition (10.26) when used in (10.42) we get

$$\left[\frac{1}{2} e^{\frac{\nu-\lambda}{2}} \dot{\mu} \nu'\right]_{(b)} = \left[e^{\mu/2}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{1}{2}\dot{\mu}\dot{\nu}\right)\right]_{(b)} . \tag{10.45}$$

Putting $e^{\nu/2} = S(t)$ and assuming $\nu(r,t) = \nu(r)$, a function of r only, the above equation reduces to

$$2S\ddot{S} + \dot{S}^2 - 2\alpha\dot{S} = 0 , \tag{10.46}$$

where

$$\alpha = \left[e^{\frac{\nu-\lambda}{2}} \frac{\nu'}{2}\right]_{(b)} . \tag{10.47}$$

The equation (10.46) possesses a first integral

$$\dot{S} = -\frac{2\alpha}{b}\left(\frac{1 - b\sqrt{S}}{\sqrt{S}}\right) , \tag{10.48}$$

and admits the solution

$$t - t_0 = \frac{S}{2\alpha} + \frac{\sqrt{S}}{\alpha b} + \frac{1}{\alpha b^2} \ln\left|b\sqrt{S} - 1\right| , \tag{10.49}$$

where b and t_0 are arbitrary constants of integration. We choose $b=1$ and reparametrize t, to get

$$\dot{S} = -\frac{2\alpha}{\sqrt{S}}\left(1 - \sqrt{S}\right) , \tag{10.50}$$

$$t = \frac{S}{2\alpha} + \frac{\sqrt{S}}{\alpha} + \frac{1}{\alpha}\ln\left(1 - \sqrt{S}\right) , \tag{10.51}$$

We note that as t gradually increases from $-\infty$, *i.e.*, $S=1$, the fluid gradually starts to collapse. Thus the solutions $g_{00} = \nu(r)$, $g_{11} = e^{\mu(t)+\lambda(r)} = S^2(t)e^{\lambda(r)}$ and t given by (10.51) represents a static fluid at $t = -\infty$ (*i.e* $S=1$) which is collapsing non-adiabatically.

Equation (10.25) gives the total mass of the collapsing fluid sphere as

$$m(v) = \left[2\alpha^2 r^3 e^{-\nu}\left(1-\sqrt{S}\right)^2 + m_0 S\right]_{(b)} , \tag{10.52}$$

where

$$m_0 = \left[\frac{r}{2}\left(1-e^{-\lambda}\right)\right]_{(b)} , \tag{10.53}$$

is the mass inside $\Sigma_{(b)}$ when $t = -\infty$ (*i.e.* $S = 1$).

The expression for luminosity (10.35) in this case reads

$$L_\infty = \left\{\frac{\left[\alpha r^2 e^{-\frac{\lambda+\nu}{2}} \nu\left(1-\sqrt{S}\right)\right]\left[re^{\mu/2} - 2m\right]}{rS\left[e^{-\lambda/2} - r\dot{S}\right]}\right\}_{(b)} . \tag{10.54}$$

It follows from (10.54) that when the collapsing body becomes a black hole *i.e.* when $re^{\mu/2} = 2m$, $L_\infty = 0$.

6.PHYSICAL REQUIREMENTS AND THERMODYNAMIC RELATIONS

A physically valid non-adiabatically collapsing configuration must comply with the following requirements:

(i) The matter density ρ and pressure P must be finite and positive throughout the configuration.

(ii) ρ and P must be decreasing radially outward.

(iii) $\rho - 3P > 0$, indicating the fulfilment of strong energy condition

(iv) $\frac{dp}{d\rho} < 1$, which implies the fulfilment of causality condition.

We further impose the following thermodynamic relations:

(v) Equations of conservation of matter

$$\left(\varepsilon u^m\right)_{;m} = 0 \tag{10.55}$$

where ε is the rest mass density.

(vi) The temperature $T \geq 0$ and entropy s must obey the Gibbs equation

$$Tds = dw + pd\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon}d\rho - \left(\rho + P\right)\frac{d\varepsilon}{\varepsilon^2} , \tag{10.56}$$

where $w = \frac{\rho}{\varepsilon} - 1$

(vii) In general relativity the relation between heat flow vector q^α and the temperature T is given by (Weinberg-1972):

$$q^\alpha = -K\left(g^{ab} + u^a u^b\right)\left(T_{,b} + T u_{b;c} u^c\right) , \tag{10.57}$$

where K is the thermal conductivity which should be positive everywhere.

7. A PARTICULAR MODEL FOR COLLAPSING STAR

We shall now consider the spherically symmetric non-adiabatic collapse of a fluid distribution on the background of a pseudo spheroidal space-time. So we choose

$$e^\lambda = \frac{1 + 2\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} , \tag{10.58}$$

so that when $e^{\mu(t)} = S^2(t) = 1$, the metric (10.1) is static and t = constant sections of it represent pseudo spheres. With this choice of e^λ, we can express the pressure isotropy equation (10.44) as a second order ordinary differential equation

$$\left(1-u^2\right)\frac{d^2F}{du^2}+u\frac{dF}{du}-F=0\ , \tag{10.59}$$

where the new variables u and F are defined by

$$u=\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} \qquad \text{and} \qquad F=e^{\nu/2}\ . \tag{10.60}$$

Equation (10.59) has already been shown to admit

$$F=e^{\nu/2}=A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right], \tag{10.61}$$

where A and B are arbitrary constants of integration, as its general solution.

The expression for magnitude of heat flux follows from (10.9)

$$8\pi q=-e^{-\lambda}e^{-\nu/2}\left(\frac{\nu'}{2}\right)\left(\frac{1-\sqrt{S}}{S^{3.5}}\right)\ . \tag{10.62}$$

Before the body begins to collapse, *i.e.* $S=1$, there is no heat flux. The metric of the space-time will be static . In this case the interior metric can be matched with Schwarzschild exterior metric across the boundary $r=a$. The values of A and B follow from the boundary conditions as

$$A=\frac{\sqrt{1+\frac{a^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}}+\sqrt{1+2\frac{a^2}{R^2}}\right)+\frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}\ , \tag{10.63}$$

$$B=-\frac{\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}\ . \tag{10.64}$$

Since there is no heat flux , $q=0$ implying $P(a)=0$.

The expressions (10.41) and (10.42) for matter density and fluid pressure after collapse sets in will become

$$8\pi\rho = \frac{8\pi\rho_0}{S^2} + \frac{12\alpha^2\left(1-\sqrt{S}\right)^2}{S^3\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2},$$

(10.65)

$$8\pi P = \frac{8\pi P_0}{S^2} + \frac{4\alpha^2\sqrt{S}\left(1-\sqrt{S}\right)}{S^3\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2},$$

(10.66)

where the matter density ρ_0 and fluid pressure P_0 in the static case are given by

$$8\pi\rho_0 = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}, \qquad (10.67)$$

$$8\pi P_0 = \frac{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right) + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]}{R^2\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}}.$$

(10.68)

On substituting the values of A and B in equation (10.65) and using the fact that ρ_0 and P_0 are positive throughout one can see that the density and pressure of the collapsing distribution are positive.

Also at any instant of time (i.e. when S is fixed)

$$8\pi\frac{d\rho}{dr}=\frac{8\pi}{S^2}\frac{d\rho_0}{dr}-\frac{24\alpha^2\left(1-\sqrt{S}\right)^2\frac{r}{R^2}\left[A+B\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)\right]}{S^3\sqrt{1+\frac{r^2}{R^2}}\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^3},$$

(10.69)

$$8\pi\frac{dP}{dr}=\frac{8\pi}{S^2}\frac{dP_0}{dr}-\frac{8\alpha^2\sqrt{S}\left(1-\sqrt{S}\right)\frac{r}{R^2}\left[A+B\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)\right]}{S^3\sqrt{1+\frac{r^2}{R^2}}\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^3}.$$

(10.70)

Since $\frac{d\rho_0}{dr}$ and $\frac{dP_0}{dr}$ are negative, we can show from equation (10.69) and (10.70) that $\frac{d\rho}{dr}<0$ and $\frac{dP}{dr}<0$, ensuring that the pressure and density are decreasing functions of r.

The quantity $\rho-3P$ and heat flux q have the following expressions

$$8\pi(\rho-3P)=\frac{8\pi}{S^2}(\rho_0-3P_0)+\frac{12\alpha^2\left(1-\sqrt{S}\right)\left(1-2\sqrt{S}\right)}{S^3\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}},$$

(10.71)

$$8\pi q=\frac{4\alpha^2\left(1-\sqrt{S}\right)\sqrt{1+\frac{r^2}{R^2}}\frac{r}{R^2}\left[A+B\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)\right]}{S^{3.5}\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}}.$$

(10.72)

It is not easy to verify the strong-energy condition analytically. However we found by numerical methods that they are satisfied at the centre and on the boundary for specific models.

When the collapsing star becomes a black hole,it is expected that the luminosity L_∞ should become zero. The condition for this is

$$\left[re^{\mu/2}\right]_{(b)} = [2m]_{(b)},$$

$$\text{i.e.} \quad \left[re^{\mu/2}\right]_{(b)} = 2\left[2\alpha^2 r^3 e^{-\nu}\left(1-\sqrt{S_{bh}}\right)^2 + m_0 S_{bh}\right]_{(b)}. \tag{10.73a}$$

This equation can be written as a quadratic equation in $\sqrt{S_{bh}}$, where S_{bh} denotes the black hole time, in the form

$$\left[4\alpha^2 r^3 e^{-\nu} + 2m_0 - r\right]_{(b)} S_{bh} - \left[8\alpha^2 r^3 e^{-\nu}\right]_{(b)}\sqrt{S_{bh}} + \left[4\alpha^2 r^3 e^{-\nu}\right]_{(b)} = 0 \tag{10.73b}$$

Solving equation (10.73b) for $\sqrt{S_{bh}}$, we get

$$\sqrt{S_{bh}} = \left[\frac{2\alpha r e^{-\nu/2}}{2\alpha r e^{-\nu/2} + \sqrt{1-\dfrac{2m_0}{r}}}\right]_{(b)} \tag{10.74}$$

The only surviving component of q^α is $q^1 = q$. Therefore equation (10.57) showing the relation between the heat flow vector q^α and the temperature becomes

$$q = Ke^{-(\mu+\lambda)} e^{\nu/2} \frac{d}{dr}\left(Te^{-\nu/2}\right) \tag{10.75}$$

Comparing equation (10.75) with (10.9) , we get

$$8\pi K \frac{d}{dr}\left(Te^{-\nu/2}\right) = e^{-\nu}\left(-\frac{\nu'}{2}\right)\mu \tag{10.76}$$

Following de Oliveira,Santos and Kolassis (1985) we assume that thermal conductivity K is given by

$$K = \beta T^{\Omega} \geq 0, \tag{10.77}$$

where $\beta > 0$ and $\Omega > 0$ are constants. With this choice of K we can integrate (10.75) and obtain the expression for temparature distribution in the collapsing configuration as

$$T^{\Omega+1} = C(t)e^{(\Omega+1)\nu/2} - \frac{\alpha}{4\pi\beta}\left(\frac{1-\sqrt{S}}{\sqrt{S}}\right)\left(\frac{\Omega+1}{\Omega+2}\right)e^{-\nu/2}, \tag{10.78}$$

where $C(t)$ is an arbitrary function of time t .

The effective surface temperature of a star seen by an external observer can be calculated from the expression (Schwarzschild-1958) as

$$T_{(b)}^{4} = \left[\frac{1}{\pi\delta r^{2}e^{\mu+\lambda}}\right]_{(b)} L_{\infty} \; . \tag{10.79}$$

For photons the constant δ is given by

$$\delta = \frac{\pi^{2}B^{4}}{15h^{3}} \quad , \tag{10.80}$$

where h is the plank's constant and B the Boltzman's constant.

Using the expressions (10.54) and (10.80) in (10.79) we can write it in the form

$$T_{(b)}^{4} = \left\{\left(\frac{15h^{3}}{\pi^{3}B^{4}r^{2}S^{2}}\right)\left(\frac{1+\frac{r^{2}}{R^{2}}}{1+\frac{2r^{2}}{R^{2}}}\right)\left[\frac{\alpha r e^{-\nu/2}\nu'\left(1-\sqrt{S}\right)\left(rS-2m\right)\sqrt{1+\frac{r^{2}}{R^{2}}}}{S\left[\sqrt{1+\frac{r^{2}}{R^{2}}}+2\alpha r\left(\frac{1-\sqrt{S}}{\sqrt{S}}\right)\sqrt{1+2\frac{r^{2}}{R^{2}}}\right]}\right]\right\}_{(b)} . \tag{10.81}$$

Choosing $\Omega = 3$, which represents radiation interacting with matter through diffusive approximation (Misner and Sharp-1965) and equating (10.78) and (10.81) on the boundary $\Sigma_{(b)}$, we get

$$C(t)\left[e^{2\nu}\right]_{(b)} = \left[\frac{\alpha}{5\pi\beta}\frac{1-\sqrt{S}}{\sqrt{S}}e^{-\nu/2}\right]_{(b)} + \left\{\left(\frac{15h^3}{\pi^3 B^4 r^2 S^2}\right)\left(\frac{1+\frac{r^2}{R^2}}{1+2\frac{r^2}{R^2}}\right)\frac{r\alpha e^{-\nu/2}\nu\left(1-\sqrt{S}\right)(rS-2m)\sqrt{1+\frac{r^2}{R^2}}}{S\left[\sqrt{1+\frac{r^2}{R^2}}+2\alpha r\left(\frac{1-\sqrt{S}}{\sqrt{S}}\right)\sqrt{1+\frac{2r^2}{R^2}}\right]}\right\}_{(b)} .$$

(10.82)

The value of $C(t)$ when substituted in equation (10.78) we obtain the temperature on the surface of the star.

Using expressions (10.65) and (10.66) for matter density and fluid pressure the expression for the polytropic index γ takes the form

$$\gamma = \frac{d(\ln P)}{d(\ln \rho)} = \frac{\rho}{P}\frac{dP}{d\rho} = \frac{\rho}{P}\frac{(dP/dt)}{(d\rho/dt)} ,$$

$$= \left\{\frac{S(2+y^2)[X(r)]^2 + 12\alpha^2\left(1-\sqrt{S}\right)^2 y^4 R^2}{Sy^2[x(r)][Y(r)] + 4\alpha^2\sqrt{S}\left(1-\sqrt{S}\right)y^4 R^2}\right\}$$

$$\times\left\{\frac{Sy^2[X(r)][Y(r)] + \alpha^2\left[S + s\sqrt{S}\left(1-\sqrt{S}\right)\right]y^4R^2}{S(2+y^2)[X(r)]^2 + 6\alpha^2\left[\sqrt{S}\left(1-\sqrt{S}\right)+3\left(1-\sqrt{S}\right)^2\right]y^4R^2}\right\} , \qquad (10.83)$$

where $X(r) = Ax + B\left[z - y/\sqrt{2}\right]$,

$$Y(r) = Ax + B\left[z + y/\sqrt{2}\right] ,$$

$$x = \sqrt{1+\frac{r^2}{R^2}} ,$$

$$y = \sqrt{1+2\frac{r^2}{R^2}} ,$$

$$z = x\ln\left(\sqrt{2}x + y\right) .$$

We have calculated the variation of γ at the centre and on the boundary during its evolution from the equilibrium to black hole state for a particular model with $\lambda = 0.1$ of Table-2.1 . This has been shown in Figure-10.1. Also we found by numerical methods that the strong energy condition is satisfied at the centre and on the boundary for this model. These estimates are shown in Table -10.1 and Table -10.2. Also we found that $\frac{dP}{d\rho} < 1$ at the centre and on the boundary.

8. DISCUSSION:

We have discussed here certain aspects of a non-adiabatically collapsing spherical distribution of matter associated with radial heat flow. We found that the pressure and density are positive throughout the distribution during its collapse from equilibrium to black hole. They are also found to be decreasing radially outward. It is difficult to examine analytically the strong energy condition. However we have examined it using numerical methods for a particular model and found that the strong energy condition is satisfied at the centre and on the boundary.The variation of polytropic index γ with respect to the time function $S(t)$ is calculated numerically for a particular model and these variations are shown in Figure -10.1.

The polytropic index at the centre is less than 4/3 and at the boundary it is much larger than 4/3 during the initial stages of collapse. This indicates that the central region is dynamically unstable. During the last stages of collapse the difference between the polytropic indices becomes very small.

The pseudo spheroidal space-time used to investigate the gravitational significance of perfect fluids, charged perfect fluids, Einsteins clusters, anisotropic distributions are also found to be useful in describing the gravitational collapse of fluid distributions accompanied by radial heat flow. Accordingly space-times endowed with a pseudo spheroidal geometry for its physical 3-space are useful in studying the gravitational significance of superdense matter distributions in General Relativity.

TABLE-10.1

The values of $\bar{p} = 8\pi P$, $\bar{\rho} = 8\pi\rho$ and $\bar{\rho} - 3\overline{P}$ at the centre of the collapsing spherical distribution for various values of S between 0 and 1 for the model with $\lambda = 0.1$ of Table-2.1.

S	$\overline{P} = 8\pi P$	$\bar{\rho} = 8\pi\rho$	$\bar{\rho} - 3\overline{P}$
1.00	0.0046	0.0372	0.0233
0.90	0.0059	0.0412	0.0282
0.80	0.0078	0.0583	0.0349
0.70	0.0107	0.0767	0.0446
0.60	0.0154	0.1056	0.0594
0.50	0.0237	0.1154	0.0841
0.40	0.0404	0.2528	0.1314
0.30	0.0806	0.2528	0.1314
0.20	0.2139	1.2954	0:6534

TABLE-10.2

The values of $\bar{p} = 8\pi P$, $\bar{\rho} = 8\pi\rho$ and $\bar{\rho} - 3\bar{P}$ at the boundary surface of the collapsing spherical distribution for various of S between 0 and 1 for the model with $\lambda = 0.1$ of Table-2.1.

S	$\bar{p} = 8\pi P$	$\bar{\rho} = 8\pi\rho$	$\bar{\rho} - 3\bar{P}$
1.00	0.00000	0.0037	0.0037
0.90	0.00011	0.0046	0.0042
0.80	0.00021	0.0059	0.0049
0.70	0.00068	0.0080	0.0059
0.60	0.0013	0.0115	0.0074
0.50	0.0028	0.0184	0.0099
0.40	0.0062	0.0340	0.0154
0.30	0.0156	0.0801	0.0331
0.20	0.0527	0.2885	0.1304

FIGURE-10.1

Variation of γ with respect to S at the centre and on the boundary. The model with λ =0.1 of Table- 2.1 has an equilibrium radius 12.27kms and mass 1.64 solar mass. S=0.15 corresponds to balck hole time.