CHAPTER 9

STATIC SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS IN HIGHER DIMENSIONS ON PSEUDO SPHEROIDAL SPACE-TIMES

1. INTRODUCTION

Developments in superstring theories which require the background space-time manifold to be (1+9)- dimensional have stimulated the study of physics in more than four dimensional space-times. Superstring theories play a key role in understanding the evolution of the universe in its very early stages after its birth. Implications of the theories in higher dimensional space-time in spite of lack of observational evidence supporting their existance have been investigated by several authors. Efforts for studying solutions of Einstein's field equations in higher dimensional space-time are accordingly of interest in this perspective and they do accord further mathematical insights.

Myers and Perry (1986) obtained higher dimensional analogues of Schwarzshild, Reissner-Nordstrom and Kerr metrics. Xu Dianyan (1988) gave higher dimensional Schwarzschild-de Sitter metric and Reissner-Nordstrom - de Sitter metric by solving appropriate set of Einstein's equations. Global regular solutions in higher dimensions have been presented by Shen and Tan (1989) taking the equation of state $\rho + P = 0$, which reduce to Gonzalez-Dianz (1981) solution in four-dimensions.Krori,Borgohain and Das (1989) reported exact solutions of spherically symmetric perfect fluid distributions in higher dimensions. Iyer and Vishveshwara (1989) generalized the Vaidya metric to higher dimensions. Banerjee,Dutta

Choudhury and Chatterjee (1992) have reported non-static prefect fluid solutions in higher dimension.Recently Patel,Mehta and Maharaj (1997) obtained three physically viable interior solutions in D -dimensional space-times.They reduce to Tolman IV (1939) solution,Mehra (1966) solution and Finch - Skea (1989) solution when D=4.

In this chapter we have extended the formalism of 4-dimensional pseudo spheroidal space-time of Chapter-2 to higher dimensions and obtained new class of solutions for Einstein's field equations for prefect fluid in D-dimensional pseudo spheroidal space-times. The hyper surface obtained as t = constant, is assumed to have the geometry of a *(D-1)* - dimensional pseudo spheroid immersed in D-dimensional Euclidean space. The field equations are obtained in Section-2 and a solution of these equations is obtained in Section-3 , for a particular value of the curvature parameter K. The relation between the mass and the size of these configuration is examined in the concluding section.

2. THE FIELD EQUATIONS

A *(D-1)*- dimensional pseudo spheroid Σ immersed in the D-dimensional Euclidean space with metric

$$d\sigma^2 = \sum_{i=1}^{D}\left(dx^i\right)^2, \tag{9.1}$$

will have the Cartesian equation

$$\left(\frac{x^D}{b}\right)^2 - \frac{1}{R^2}\left[\sum_{i=1}^{D-1}\left(x^i\right)^2\right] = 1. \tag{9.2}$$

The sections x^D= constant of Σ are pseudo hyper spheres, while sections x^i=constant for $i= 1, 2,...,D-1$, represent hyperboloids in $(D-1)$ - dimensional space.

We introduce the parametrization

$$x^1 = R \;\; sinh\lambda \;\; \sin\theta_1 \; \sin\theta_2 \; .....\sin\theta_n \;\; \cos\theta_{n+1} \; ,$$

$$x^2 = R \;\; sinh\lambda \;\; \sin\theta_1 \;\; \sin\theta_2 \; .....\sin\theta_n \;\; \sin\theta_{n+1} \; ,$$

$$x^3 = R \;\; sinh\lambda \;\; \sin\theta_1 \;\; \sin\theta_2 \; .....\sin\theta_n \;\; \cos\theta_n \;\; ,$$

(9.3)

$$x^{n+2} = R \;\; sinh\lambda \;\; \cos\theta_1 \; ,$$

$$x^{n+3} = x^D = b\cosh\lambda \; ,$$

where $\lambda, \theta_1, \theta_2, \ldots \theta_{n+1}$ denote natural coordinates on Σ in terms of which (9.1) assumes the form

$$d\sigma^2 = \left(R^2\cosh^2\lambda + b^2\sinh^2\lambda\right)d\lambda^2 + R^2\sinh^2\lambda(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2 + \ldots$$
$$\ldots + \sin^2\theta_1\sin^2\theta_2 \ldots\ldots \sin^2\theta_n d\theta_{n+1}^2),$$

(9.4)

for the metric on Σ. We further choose new radial variable

$$r = R\sinh\lambda \; , \tag{9.5}$$

leading to the expression for the metric $d\sigma^2$ on Σ as

$$d\sigma^2 = \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}} dr^2 + r^2 d\Omega^2 \; , \tag{9.6}$$

where

$$d\Omega^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1\sin^2\theta_2 \ldots \sin^2\theta_n d\theta_{n+1}^2 \tag{9.7}$$

and

$$K = 1 + \frac{b^2}{R^2} > 0. \tag{9.8}$$

The metric (9.6) is regular everywhere and positive definite at all points.

We consider the D-dimensional space-time with metric

$$ds^2 = e^{\nu(r)} dt^2 - \frac{1 + K\dfrac{r^2}{R^2}}{1 + \dfrac{r^2}{R^2}} dr^2 - r^2 d\Omega^2 , \tag{9.9}$$

and label the coordinates as

$$x^0 = t \quad , \; x^1 = r \; , \; x^2 = \theta_1 , \; . \; . \; . \; , x^{D-1} = \theta_{D-2} .$$

The suitability of the space-time with $D = 4$ to describe space-time of well defined sources of gravitational field have already been explored. We have studied here the matter distributions in the form of perfect fluid on D-dimensional pseudo pheroidal space-time, with energy-momentum tensor

$$T_{ij} = (\rho + P) u_i u_j - P g_{ij} , \quad u^i u_i = 1 \tag{9.10}$$

where ρ and P denote the matter density and fluid pressure respectively and

$$u^i = \left(e^{-\nu/2}, 0, \ldots, 0\right), \tag{9.11}$$

denote the D-velocity field.

The Einstein's field equations in D-dimensional space-time

$$\Re_i^j = -8\pi \left(T_i^j - \frac{1}{n+1} \delta_i^j T_K^K \right), \tag{9.12}$$

$i , j = 1 , 2, \ldots D$ and $n = D - 3$, constitute the following system of 3 equations:

$$e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}-\frac{(n+1)\lambda'}{2r}\right)=\frac{8\pi}{n+1}(P-\rho)\ , \tag{9.13}$$

$$e^{-\lambda}\left(\frac{\nu'}{2r}-\frac{\lambda'}{2r}-\frac{n}{r^2}\right)+\frac{n}{r^2}=\frac{8\pi}{n+1}(P-\rho)\ , \tag{9.14}$$

$$e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{(n+1)\nu'}{2r}\right)=\frac{8\pi}{n+1}[n\rho+(n+2)P]\ , \tag{9.15}$$

relating physical variables with metric coefficients.

For the space-time metric (9.9) these equations are equivalent to

$$\frac{16\pi\rho}{n+1}=\frac{(n+2)(K-1)}{R^2}\cdot\frac{\left(1+\frac{nK}{n+2}\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)}\ , \tag{9.16}$$

$$\frac{16\pi P}{n+1}=\frac{\left(1+\frac{r^2}{R^2}\right)\frac{\nu'}{r}}{1+K\frac{r^2}{R^2}}-\frac{n(K-1)}{R^2\left(1+K\frac{r^2}{R^2}\right)}\ , \tag{9.17}$$

$$\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right)-\frac{(K-1)r\nu'}{2R^2}+\frac{nK(K-1)}{R^2}\frac{r^2}{R^2}=0. \tag{9.18}$$

Equation (9.16) is the law for variation of matter density ρ in terms of the parameters R, K and n. Differentiating it with respect to r, we get

$$\frac{16\pi}{n+1}\frac{d\rho}{dr}=-\frac{2(n+2)K(K-1)r}{R^2}\cdot\frac{\left(\frac{n+4}{n+2}+\frac{nK}{n+2}\frac{r^2}{R^2}\right)}{R^2\left(1+K\frac{r^2}{R^2}\right)^2}. \tag{9.19}$$

Since for pseudo spheroidal space-times $K > 1$, the gradient of matter density is negative showing that ρ is a decreasing function of r throughout. It attains its maximum value

$$8\pi\rho(0) = \frac{(n+2)(K-1)}{R^2} \tag{9.20}$$

at the centre . At the boundary $r=a$ it attains the value $\rho(a)$, given by

$$8\pi\rho(a) = \frac{(n+2)(K-1)}{R^2} \cdot \frac{1+\dfrac{nK}{n+2}\dfrac{a^2}{R^2}}{\left(1+K\dfrac{a^2}{R^2}\right)^2}. \tag{9.21}$$

Variation of pressure in the distribution is known only when ν is furnished in equation (9.17) by solving equation (9.18) for ν.

3. A SOLUTION OF FIELD EQUATIONS

Adopting new variables F and u defined by

$$F = e^{\nu/2}, \tag{9.22}$$

and

$$u = \sqrt{\frac{K}{K-1}}\sqrt{1+\frac{r^2}{R^2}}, \tag{9.23}$$

the equation (9.18) can be couched in a convenient form

$$\left(1-u^2\right)\frac{d^2F}{du^2} + u\frac{dF}{du} - n(K-1)F = 0 \tag{9.24}$$

Let $K = \dfrac{n+1}{n}$, so that the equation (9.24) becomes

$$\left(1-u^2\right)\frac{d^2F}{du^2} + u\frac{dF}{du} - F = 0 . \tag{9.25}$$

Equation (9.25) is identical with the differential equation (2.27) when $K = 2$ which was solved in Chapter-2.

Equation (9.25) admits the general solution

$$e^{\nu/2} = F = C_1 u + C_2\left[u\ln\left(u+\sqrt{u^2-1}\right)-\sqrt{u^2-1}\right] \tag{9.26}$$

where $u = \sqrt{n+1}\sqrt{1+\frac{r^2}{R^2}}$

The space-time metric for D-dimensional solution reads

$$ds^2 = \left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{n+1}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{n}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}\right)-\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}\right]\right\}^2 dt^2$$

$$-\frac{1+\frac{n+1}{n}\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2 d\Omega^2 \tag{9.27}$$

where $A=\sqrt{n+1}\,C_1$ and $B=\sqrt{n+1}\,C_2$ denote arbitrary constants of integration.

The matter density and fluid pressure of the distribution describing the physical content of the space-time of (9.27) , have expressions:

$$\frac{16n\pi}{(n+1)(n+2)}\rho = \frac{1+\frac{n+1}{n+2}\frac{r^2}{R^2}}{R^2\left(1+\frac{n+1}{n}\frac{r^2}{R^2}\right)^2}, \tag{9.28}$$

$$\frac{16\pi}{n+1}P = \frac{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}.L(r)+\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}\right]}{R^2\left(1+\frac{n+1}{n}\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}}.+B\left[\sqrt{1+\frac{r^2}{R^2}}L(r)-\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}\right]\right\}},$$

(9.29)

where

$$L(r) = \ln\left(\sqrt{n+1}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{n}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}\right). \tag{9.30}$$

If the matter distribution extends upto a finite radius $r=a$, the space-time in the exterior region $r \geq a$ will be that of Schwarzschild exterior metric in D-dimensions given by Myers and Perry (1986):

$$ds^2 = \left(1-\frac{2m}{nr^n}\right)dt^2 - \left(1-\frac{2m}{nr^n}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{9.31}$$

where m denotes the total mass of the configuration. We join (9.27) with (9.31) across the boundary of separation $r=a$ by stipulating the following boundary conditions :

(i) The metric coefficients of (9.27) should be continuous with those of (9.31) across $r=a$.

(ii) The fluid pressure should vanish at $r=a$.

These conditions lead to the following equations

$$\frac{1+\frac{a^2}{R^2}}{1+\frac{n+1}{n}\frac{a^2}{R^2}} = 1-\frac{2m}{na^n}, \tag{9.32}$$

$$A\sqrt{1+\frac{a^2}{R^2}}+B\left[\sqrt{1+\frac{a^2}{R^2}}L(a)-\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{a^2}{R^2}}\right]=\sqrt{1-\frac{2m}{na^n}}, \tag{9.33}$$

$$A\sqrt{1+\frac{a^2}{R^2}}+B\left[\sqrt{1+\frac{a^2}{R^2}}L(a)+\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{a^2}{R^2}}\right]=0, \tag{9.34}$$

which determine the arbitrary constants A , B and the total mass m of the configuration as

$$A=\frac{\sqrt{\frac{n+1}{n}}\left[\sqrt{1+\frac{a^2}{R^2}}L(a)+\sqrt{\frac{n}{n+1}}\sqrt{1+\frac{n+1}{n}\frac{a^2}{R^2}}\right]}{2\left(1+\frac{n+1}{n}\frac{a^2}{R^2}\right)}, \tag{9.35}$$

$$B=-\frac{\sqrt{\frac{n+1}{n}}\sqrt{1+\frac{a^2}{R^2}}}{2\left(1+\frac{n+1}{n}\frac{a^2}{R^2}\right)}, \tag{9.36}$$

$$m=\frac{\frac{a^2}{R^2}.a^n}{2\left(1+\frac{n+1}{n}\frac{a^2}{R^2}\right)}. \tag{9.37}$$

Equation (9.28) shows that density is positive throughout the distribution . Using the values of A and B in (9.29), the expression for pressure becomes

$$\frac{16\pi}{n+1}P=\frac{\sqrt{\frac{n+1}{n}}\sqrt{1+\frac{r^2}{R^2}}\sqrt{1+\frac{a^2}{R^2}}[L(a)-L(r)]+H(r)}{R^2\left(1+\frac{n+1}{n}\frac{r^2}{R^2}\right)\left\{\sqrt{\frac{n+1}{n}}\sqrt{1+\frac{r^2}{R^2}}\sqrt{1+\frac{a^2}{R^2}}[L(a)-L(r)]+G(r)\right\}},$$

(9.38a)

where

$$H(r) = \sqrt{1+\frac{r^2}{R^2}}\sqrt{1+\frac{n+1}{n}\frac{a^2}{R^2}} - \sqrt{1+\frac{a^2}{R^2}}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}, \qquad (9.38b)$$

and

$$G(r) = \sqrt{1+\frac{r^2}{R^2}}\sqrt{1+\frac{n+1}{n}\frac{a^2}{R^2}} + \sqrt{1+\frac{a^2}{R^2}}\sqrt{1+\frac{n+1}{n}\frac{r^2}{R^2}}. \qquad (9.38c)$$

Equation (9.38) shows that the pressure is positive throughout the distribution since $L(a) \geq L(r)$ .The complexity of expression (9.38) for pressure makes the task of verification of the fulfillment of strong energy condition highly formidable. However we find that this condition is satisfied at the centre.

4. DISCUSSION

We extend the method used in Chapter-2 , to determine the mass and size of the configuration , in higher dimensions.

The density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)}$, has the expression

$$\lambda = \frac{1+\frac{n+1}{n+2}\frac{a^2}{R^2}}{\left(1+\frac{n+1}{n}\frac{a^2}{R^2}\right)^2} . \qquad (9.39)$$

Equation (9.39) is a quadratic equation in $\frac{a^2}{R^2}$. Its value is obtained in terms of λ and n as

$$\frac{a^2}{R^2}=\frac{n^2-2\lambda n(n+2)+\sqrt{n^4+8\lambda n^2(n+2)}}{2\lambda(n+1)(n+2)}. \qquad (9.40)$$

If the values of $\rho(a)$, λ and the dimension parameter n are specified a priori, the values of R, a and m can be determined respectively from equations (9.20), (9.40) and (9.37).

The solution (9.27) discussed here is the *D*-dimensional generalization of the solution (2.29) obtained in Chapter-2.